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RESEARCH ANNOUNCEMENT: BOUNDEDNESS OF ORBITS FOR TRAPEZOIDAL OUTER BILLIARDS

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Abstract. We state and briefly sketch the proof of a result on boundedness of outer billiard orbits for trapezoidal outer billiards.

1. Introduction

Outer billiards were introduced by B.H. Neumann in about 1960 [6] and later popularized by J. Moser in [5]. Moser used outer billiards as an example of a problem amenable to the methods of KAM. In [5] it was shown that for outer billiards with a six times differentiable boundary all orbits must be bounded. Moser then went on to ask whether unbounded orbits may exist for outer billiards with less smooth boundaries. This question has now been answered for a number of non-differentiable outer billiards.

In the search for outer billiards with unbounded orbits, polygons are a natural candidates since they are in some sense furthest from being integrable. In the late 80's several authors [8], [4], [3] independently proved that for a certain class of polygons, extending the class of lattice polygons with non-parallel sides, the orbits remain bounded. This larger class, named the *quasi-rational polygons*, also includes all regular polygons. At this point the progress on the problem stalled for a number of years. While numerical studies suggested that unbounded orbits existed for some polygonal outer billiards as well as for some other non-smooth shapes rigorous proofs could not be obtained.

In the last few years there has been a resurgence of interest in the problem and a positive answer has been discovered in at least two cases. Specifically, R. Schwartz proved that a large class of kite-shaped quadrilaterals, including the kite-shaped quadrilateral of the Penrose tiling do indeed have unbounded orbits [7]. Schwartz's work is deep and far reaching, and is likely to be only the beginning of a very fruitful line of research. Shortly thereafter D. Dolgopyat and B. Fayad proved existence of unbounded orbits for a semicircular outer billiard [1], which were first observed experimentally by S. Tabachnikov. Thus based on this admittedly small and biased sample unboundedness appears to prevail for non-smooth billiards.

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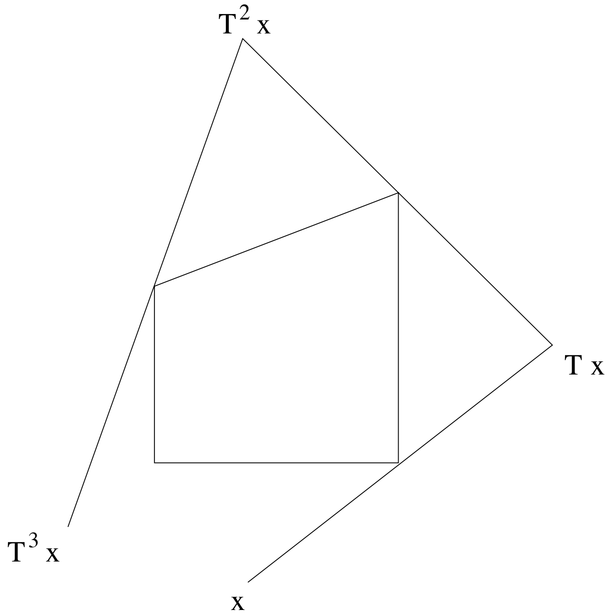


FIGURE 1. Definition of the outer billiard map.

Surprisingly, for a large subset of trapezoids, which are among the simplest non-quasi-rational polygons, all outer billiard orbits *are* bounded. This is the main result of this announcement.

2. Outer billiards

Let $B \subset \mathbb{R}^2$ be a convex set with oriented boundary. The *outer billiard map* is defined on $\mathbb{R}^2 \setminus B$ by taking the image of $p \in \mathbb{R}^2 \setminus B$ to be the point along the ray tangent to the boundary of B, at the same distance from the point of tangency as p (see Figure 1). The resulting map has many remarkable properties (see [9] for a good survey).

We will consider outer billiards in which B is a trapezoid whose vertices are (-1/2,0), (1/2,0), (1/2,1), (-1/2,1-α), where $\alpha \in (0, 2/3)$. Let this set of quadrilaterals be denoted by $\mathcal{Q}$. The main result is

Theorem 1. *All orbits of an outer billiard map about a trapezoid in $\mathcal{Q}$ are bounded.*

The result is proved by showing that the first return map to a special set of lines, forming a Poincaré section of T, is symbolically conjugate to a simple dynamical system related to a circle rotation by 2α. In fact, after the conjugacy is established much more can be said about the orbit structure of the outer billiard map. For example, periods of all periodic orbits can be easily computed as well as estimates on the growth of orbit complexity [2].

We now briefly describe the construction leading to the proof of Theorem 1.

3. First-return map

We begin by defining a Poincaré section of T. The square of the outer billiard map T^2 is a translation by twice the vector joining the the vertices "visited" by the

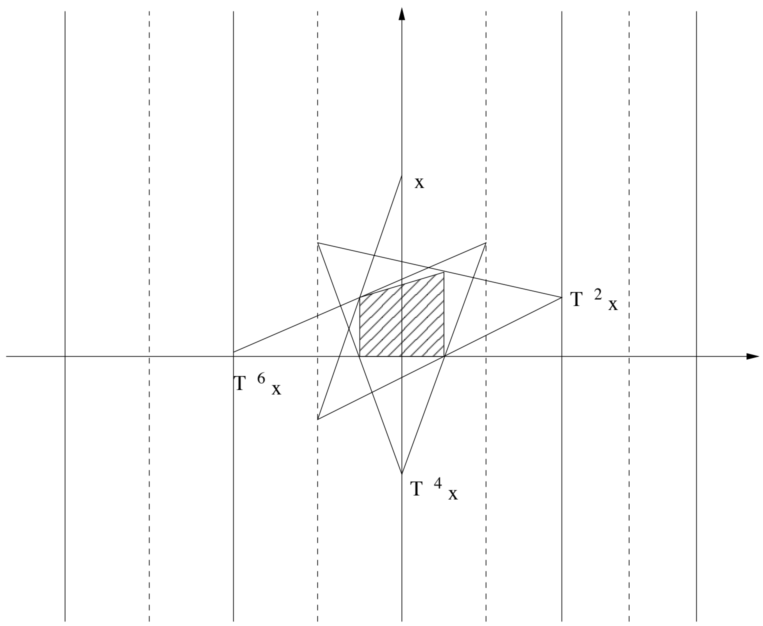


FIGURE 2. Two families of invariant line $\{x = 2k\}$ and $\{x = 2k+1\}$ preserved by T^2.

orbit segment. Considering all the possible translation vectors for T^2 it is easy to see that all of them have a horizontal component either 0 or 2. Thus the set of lines $\{x = 2k + t | k \in \mathbb{Z}\}$ $t \in [-1, 1)$ is preserved by T^2 (Figure 2). Furthermore, because the orbit winds around the table in the same direction it can be shown that the first-return map on every line in this family is well-defined.

By a symmetry argument, the attention can be further restricted to half-lines $L_t = \{x = t, y > 1 - t\alpha/2\}$, $t \in (0, 1)$. The strip formed by the union of the half-lines $\cup_{t \in (0,1)} L_t$ gives a Poincaré section of T. For simplicity we present the argument for the half-line $L_0 = \{x = 0, y > 1 - \alpha/2\}$, since computations for L_t, $t \neq 0$, are virtually identical.

Let the first-return map to L_0 be denoted by F. Since T is a piecewise isometry, F is an interval exchange on infinitely many intervals. A direct but lengthy inspection yields and explicit formula for F.

$$F(x) = \begin{cases} x + 2\alpha & \text{if } x \in I_{k,1} \\ x - 2 + 2\alpha & \text{if } x \in I_{k,2} \\ x + 2 & \text{if } x \in I_{k,3} \\ x & \text{if } x \in I_{k,4} \end{cases} \tag{1}$$

where

$$\begin{aligned} I_{k,1} &= \{x \in I_k | [\![x + 2\alpha(k(x)+1)]\!] > 1 - \alpha/2, x < 2(k(x)+1) - 2\alpha\} \\ I_{k,2} &= \{x \in I_k | [\![x + 2\alpha(k(x)+1)]\!] > 1 - \alpha/2, x > 2(k(x)+1) - 2\alpha\} \\ I_{k,3} &= \{x \in I_k | [\![x + 2\alpha(k(x)+1)]\!] < 1 - \alpha/2, x < 2(k(x)+1)\} \\ I_{k,4} &= \{x \in I_k | [\![x + 2\alpha(k(x)+1)]\!] < 1 - \alpha/2, x > 2(k(x)+1)\} \\ I_k &= (2k + 1 - \alpha/2, 2(k+1) + 1 - \alpha/2) \end{aligned} \tag{2}$$

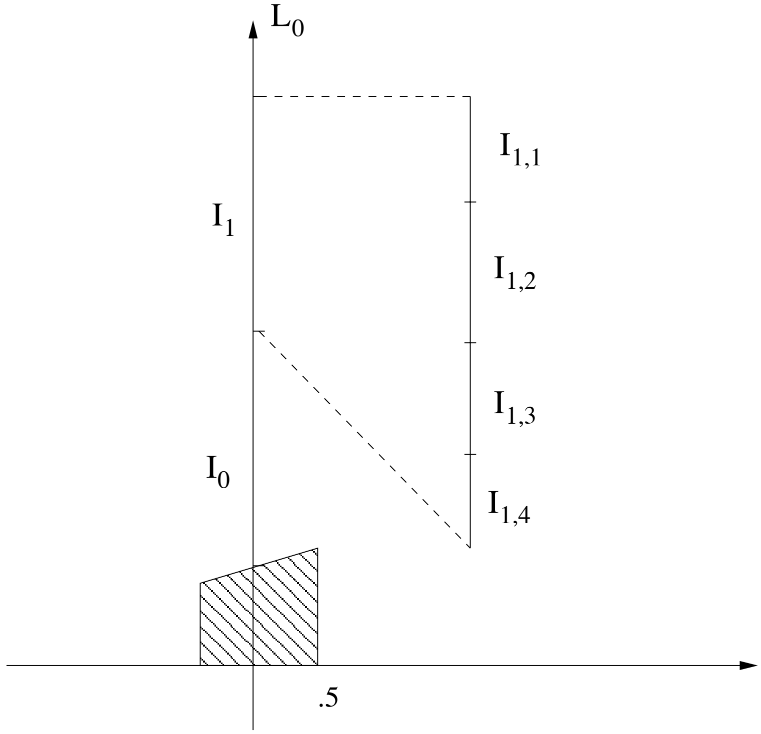


FIGURE 3. Interval exchange partition.

and

$$\begin{aligned} k(x) &= \lfloor x - (1 - \frac{\alpha}{2}) \rfloor \\ [\![x]\!] &= x - 2\lfloor x/2 \rfloor \end{aligned}$$

The intervals I_k partition L_0 and are in turn partitioned into subintervals $I_{k,j}$ which are exchanged by F (Figure 3).

For rational α the arrangements of subintervals $I_{k,j}$ are clearly periodic. Figure 4 shows one cycle of intervals I_k and their corresponding subintervals $I_{k,j}$ laid out horizontally to show the relationship between partitions of adjacent intervals. The subinterval highlighted in blue is defined by the first of the two inequalities defining $I_{k,j}$ in (2). The partition defined by the second set of inequalities in (2) (indicated in the figure with red vertical lines) is fixed relative to I_k, while the former interval is shifted by 2α for each successive interval I_k. Thus, for $\alpha = p/q \in \mathbb{Q}$ the partition $I_{k,j}$ repeats with period $2q$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, on the other hand, the partition $I_{k,j}$ is quasi-periodic.

Figure 4 also shows why for $\alpha \in \mathbb{Q}$ all orbits must be bounded: every cycle of the partition $I_{k,j}$ is capped above and below by "impassible" intervals, such that orbits from below cannot go up and orbits from above cannot go down. The reason for this is that the interval at the top of the cycle has $I_{k,3} = \emptyset$ and the interval at the bottom of the cycle has $I_{k,2} = (2(k+1)+1-5\alpha/2, 2(k+1)+1-\alpha/2)$, which under F transforms to $(2k+1-\alpha/2, 2k+1-3\alpha/2) \subset I_{k,1}$. Simple as it is, this is a new observation because parallel sides exclude the trapezoids from the

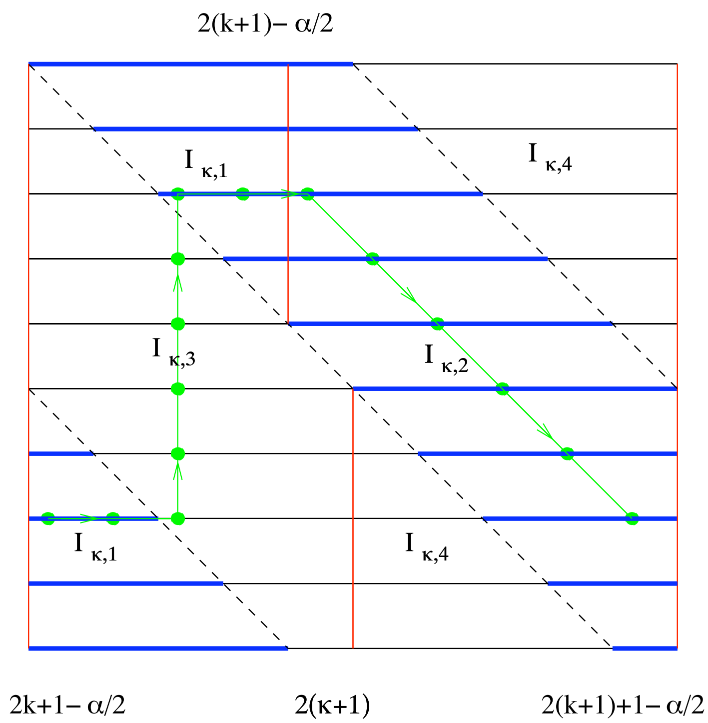


FIGURE 4. Intervals I_k rotated to horizontal position and stacked vertically, and a representative orbit (green) for $\alpha = 1/10$.

set of quasi-rational polygons to which all of the known boundedness results are restricted. Combining this with discreteness of orbits we can conclude that

Lemma 1. *If $\alpha \in \mathbb{Q}$ then all orbits are bounded and periodic.*

Really, almost any question (e.g. orbit periods, orbit diameters) one might care to ask about a lattice trapezoid can be answered by examining a diagram like Figure 4.

A simple consequence of (1) is that an F-orbit projects to an orbit of rotation by 2α when taken modulo 2. This observation, which is key to the proof of orbit boundedness, immediately gives

Lemma 2. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then every non-fixed point of F is non-periodic.*

That is for irrational α only the simplest periodic orbits, that close after one trip around the table, exist.

4. Symbolic coding and the conjugate circle rotation model

The orbit of a point under iterates of F can be coded as a sequence of 0's, 1's and -1's by taking the n-th symbol to be $\omega_n(x) = k(F^{n+1}(x)) - k(F^n(x))$. This encodes how the F-orbit of x moves between the intervals I_k. In spite of the three symbols appearing in $\omega(x)$ the sequence actually turns out to be determined by a binary coding of a circle rotation by 2α.

Definition 1. Let $R : S^1 \to S^1$ be a rotation and $J \subset S^1$ a subinterval then the *rotation sequence* corresponding to $x \in S^1$ is $w_n(x) = \chi_J(R^n(x))$, where χ_J is the characteristic function of J.

Following the observation made at the end of Section 3 we can relate the rotation sequence $w_n(x)$ and the F-orbit coding sequence $\omega_n(x)$.

Lemma 3. *Let $\pi_1(x) = [\![x + 2k(x)\alpha + (1 + \alpha/2)]\!]$, $\pi_{-1}(x) = [\![x - (1 - \alpha/2)]\!]$, and let R be a rotation by 2α on a circle of length 2 with $J = [1 + \alpha/2 - 2\alpha, 2 - 2\alpha]$. Then the sequence obtained by deleting 1's from $\omega_n(x)$ is the same as the sequence $-w_n(\pi_1(x))$, and the sequence obtained by deleting -1's from $\omega_n(x)$ is the same as $w_n(\pi_{-1}(x))$.*

Note that whether end points of J are included or excluded is irrelevant because they correspond to discontinuity points of F.

The two maps π_1 and π_{-1} define a pair of projections from L_0 to S^1 (of length 2). As Figure 5 illustrates, F-orbits project to rotation orbits by 2α via π_1 and π_{-1} in two different ways. (Notice that the figure also shows how the orbit reenters the lower "impassible" interval of the cycle, as described above.) The "union" of the corresponding rotation sequences is the coding sequence of the F-orbit. How these sequences combine to make ω is determined by the following scheme. Let S be the circle of length 2, $J = [1 + \alpha/2 - 2\alpha, 2 - 2\alpha]$, $t_1(0) = \pi_1(x)$ and $t_{-1}(0) = \pi_{-1}(x)$

$$\begin{array}{rcl} \text{if } t_1(i) \in J & : & \omega_i(x) = 1,\ t_1(i+1) = t_1(i) + 2\alpha \\ \text{if } t_1(i) \notin J, t_{-1}(i) \in J & : & \omega_i(x) = -1,\ t_{-1}(i+1) = t_{-1}(i) + 2\alpha \\ \text{if } t_1(i), t_{-1}(i) \in J & : & \omega_i(x) = 0,\ t_j(i+1) = t_j(i) + 2\alpha,\ j = 1, -1 \end{array} \tag{3}$$

In words, we move the two points t_1 and t_{-1} around the circle by iterates of R until one or the other enters the distinguished interval J (the blue interval in Figure 4) at which point the point that is outside of J stops while the point inside J continues to move by R. Once the point that was inside J exits the interval, the joint motion resumes. If both points enter the distinguished segment simultaneously t_1 has "the right of way". Of course, the two points t_1 and t_{-1} are images of $F(x)$ under π_1 and π_{-1} respectively. Consequently, the coding sequence $\omega(x)$ can be reconstructed by writing 1's and -1's, according to the order in which the points t_1 and t_{-1} visit the interval J.

5. Orbit boundedness

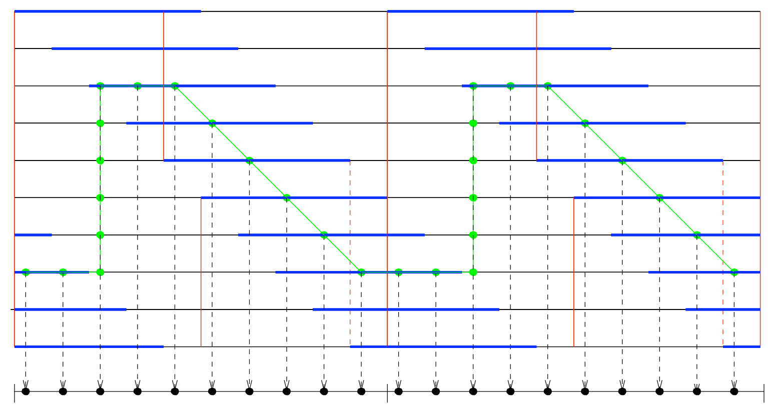
It is now easy to show that all orbits are bounded. It suffices to check that

$$\sup_N \sum_{n=0}^{N} \omega_n(x) < \infty. \tag{4}$$

The proof of this bound proceeds in two steps.

First, we show that the two rotation sequences of Lemma 3 comprising $\omega_n(x)$ differ only by a shift. This follows easily because the points t_1 and t_{-1} in (3) lie along the same rotation orbit.

Next, we need to show that the interleaving of the two rotation sequences comprising $\omega(x)$ cannot create arbitrarily large deviations in the sum (4). Suppose first that the elements of the two rotation sequences simply alternated in $\omega(x)$, then clearly the supremum in (4) would be at most 1 because 1's and -1's cancel

(a)


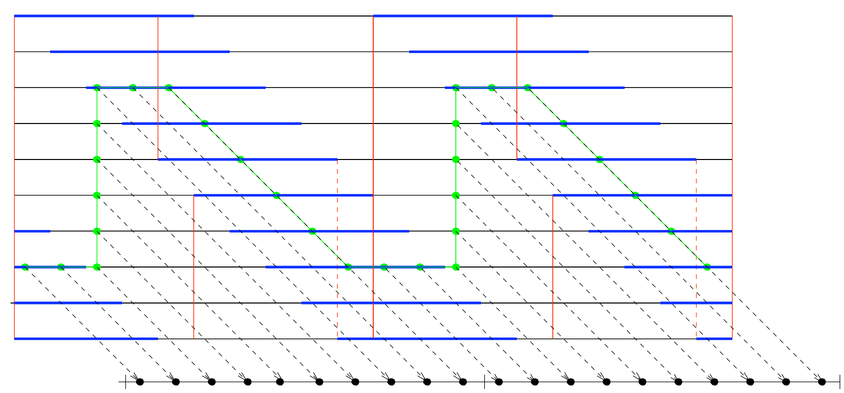
(b)


FIGURE 5. Projections π_1 (a) and π_{-1} (b).

almost exactly. This, of course, is not very likely to happen but if we can show that the corresponding members of the two sequences are never too far apart the result would clearly follow. A simple analysis of scheme (3) shows this to be the case and we have

$$\sup_N \sum_{n=0}^{N} \omega_n(x) \leq C(x) + \sup_N \sum_{n=0}^{N} w_{n+k(x)}(x) - w_n(x), \tag{5}$$

where $C(x)$ is a constant depending only on x.

To complete the proof we show that the right side of (5) is in turn bounded above by

$$C(x) + \sup_N \sum_{n=0}^{N} w_{n+k(x)}(x) - w_n(x) \leq C(x) + \sup_N \sum_{n=0}^{k(x)} w_{n+N} \leq C(x) + k(x).$$

The left hand side is a constant depending only on x and Theorem 1 follows. Note that the bound is not uniform in x and so it is possible for orbits to have arbitrarily large excursions.

6. Conclusion

A natural extension of this work would be to tackle the case of more general polygons with families of invariant lines, whose dynamics can be reduced to an infinite interval exchange on a finite union of lines. In [7], Schwartz succeeded in doing exactly this for kite-shaped quadrilaterals. Already in this case, however, the methods required are considerably more complicated because the first return map is more complex. The reward for the hard work are deep and beautiful results (and pictures). One can only guess what fascinating new challenges and unexpected gems Moser's problem will produce.

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