CHAOTIC RESONANCE: HOPPING RATES, SPECTRA AND SIGNAL-TO-NOISE RATIOS

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Chaotic Resonance: Hopping Rates, Spectra and Signal-to-Noise Ratios

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Abstract. We consider a noise-free bistable system with a low frequency signal and a secondary harmonic excitation that causes the system to experience chaotic motion with a broadband portion of the output spectrum. The signal-to-noise ratio (SNR) is defined on the basis of this braodband spectrum. We present the theoretical background for approximate calculation of the hopping rate, the output spectra and SNR of the system. It is shown that, under a proper choice of the secondary excitation, the SNR can be enhanced. This phenomenon is referred to as *chaotic resonance.*We show similarities between results obtained for chaotic resonance on the one hand and classical stochastic resonance induced by random perturbations on the other. As an example, chaotic resonance in the Holmes-Brunsden oscillator is studied.

INTRODUCTION

Stochastic resonance (SR) is the phenomenon wherein, for bystable systems subjected to noise and a weak periodic signal, the output signal-to-noise ration (SNR) can under certain conditions be improved by increasing the noise [1,2,3]. It has recently been shown [4,5,6] that a similar effect can be achieved for deterministic (i.e., noise-free) systems with a periodic signal and a secondary periodic excitation that causes the system to experience chaotic motion. Owing to this, the system has a broadband portion of the output spectrum and, therefore, a finite SNR. Under a proper choice of the secondary forcing the SNR can be enhanced. We refer to this phenomenon, first noted in [6], as *chaotic resonance.*

We consider the system

$$\frac{d^2x}{dt^2} + U'(x,t,\mu) = -\epsilon b\frac{dx}{dt} + \epsilon\gamma\cos\omega t, \tag{1}$$

with a modulated potential

$$U(x,t,\mu) = U_0(x) - s(t,\mu)x = U_0(x) - \mu x A\cos\lambda t, \tag{2}$$

where $U_0(x)$ is a two-well potential of the Duffing-Holmes type. We assume $\epsilon < \mu \ll 1$, $\lambda \ll \omega$. We also assume that the primary low-frequency signal $s(t,\mu)$ is unable to induce chaotic transition across the potential barrier, and that the seconaary forcing helps to bring about such transitions and can lead to SR. Formally this implies that the Melnikov necessary condition for chaos [7,8] is not satisfied for $\gamma = 0$, i.e., in the absence of the secondary forcing irregular escapes cannot occur.

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Denote the right and the left potential well by V_{+1} and V_{-1}, respectively. Identify the paths in each of the wells by $x_{\pm 1}$. Then $x_i(t \mid t_0, j)$ is a process $x(t) = x_i$ at t provided that $x(t_0) = x_j$ at t_0, and $p_i(t \mid t_0, j)$ is the conditional probability that $x = x_i$ at time t given that $x = x_j$ at time t_0.

Since $x_i(t)$ is interpreted as a chaotic oscillation in the domain within a loop of the separatrix, we can write $x_i(t) = x_i^0(t) + c_i$, where c_i is the mean value of $x_i(t)$ and is taken to coincide with the position of the focus. For the system with a symmetrical potential we have $c_{+1} = c_{-1} = c$.

At a given time t a point can be either in the right or left well of the potential (2), or in the flight region between the wells (i.e., outside the cores defined by the unperturbed systems homoclinic orbits). Numerical simulations show that the time of flight motion is negligible in relation to the time spent in the wells. We interpret short time flights between the wells as a series of instantaneous impulses with independent increments. This implies that the system state after each jump is independent of its previous state before the jump, and the motions separated by a jump can be taken to be uncorrelated.

Let $T_{\pm 1}$ be the mean residence time in the domains $V_{\pm 1}$, respectively. The mean escape rate from the domain is defined as $W_{\pm 1} = 1/T_{\pm 1}$. The rate equations that govern the presence of a particle in one of the two states are written in the form [2]

$$\frac{dp_{+1}}{dt} = W_{-1}(t) - [W_{-1}(t) + W_{+1}(t)]\,p_{+1}, \quad p_{-1} = 1 - p_{+1}, \tag{3}$$

with initial conditions $p_i(t \mid t_0, j) = \delta_{ij} = \{1, i = j; 0, i \neq j\}$. If $\mu \ll 1$, the weakly perturbed escape rates can be written in the form [2]

$$W_{\pm 1}(t) = \alpha/2 \pm \mu\beta \cos \lambda t, \tag{4}$$

where higher order terms are neglected. From (3), (4) we obtain

$$p_{+1}(t \mid t_0, x_0) = \frac{1}{2}[e^{-\alpha(t-t_0)} N_{+1,j} + 1] + \mu\beta Z(\lambda)\cos(\lambda t - \gamma), \tag{5}$$

in which $N_{+1,j} = j,\ j = \pm 1, Z(\lambda) = (\lambda^2 + \alpha^2)^{-1/2}, \tan\gamma = \lambda/\alpha$. Equation (5) is obtained by noting that terms of order μ are substantial only for the periodic part of $p_{+1}(t \mid t_0, x_0)$ and can be neglected in the aperiodic part.

The autocorrelation function can be written in the form

$$K(t, s \mid t_0, j) = \sum_{k,r=\pm 1} < x_k(t + s \mid t, x_r) x_r(t \mid t_0, j) > \tag{6}$$

Theoretical investigations as well as experimental results prove that the contribution of the broadband portion of the spectrum of the intrawell chaos may be assumed to be negligible compared to the contribution of the spectrum of irregular jumps between wells (see [9] for references and discussion). We can therefore neglect the autocorrelation terms $\langle x_k^0(t+s \mid t, x_k)x_k^0(t \mid t_0, j)\rangle$ in (6) and consider the broadband spectrum as consisting, in general, of the spectrum associated with the intermittent jumps. On the other hand, since the motions in the regions V_{+1} and V_{-1} are mutually uncorrelated, we can exclude the cross-correlations $\langle x_k^0(t+s \mid t, x_r)x_r^0(t \mid t_0, j)\rangle$, $k \neq r$, from (6). Thus the autocorrelation function is reduced to four terms and corresponds to the two-state model [2]

$$K(t, s \mid t_0, j) = \sum_{k,r=\pm 1} c_k c_j p_k(t+s \mid t, c_r) p_r(t \mid t_0, j). \tag{7}$$

The power spectrum corresponding to the autocorrelation function (7) is [2]

$$\Phi(\Omega) = 2c^2\alpha Z^2(\Omega) + 2\pi\mu^2(\beta c)^2 Z^2(\lambda)[\delta(\Omega-\lambda) + \delta(\Omega+\lambda)], \tag{8}$$

where δ denotes the Dirac delta-function. The SNR at the frequency λ can be found from (8)

$$R = \pi(\mu\beta)^2/\alpha. \tag{9}$$

Equation (9) is similar to the result obtained for classical SR in [2], [6].

To calculate the hopping rate, we need the noise distribution parameters of the intrawell fictitious noise. In general, a quantitative model of the noise distribution is not available. Following [6], we assume that it is Gaussian. In this case we can use the threshold-crossing theory for calculation of the escape rate [10].

Consider a potential barrier as a threshold to be crossed. Let $x = x^*$ be the top of the potential well, and $x = c$ be the equilibrium position corresponding to the bottom of the well. For the equilibrium we have $U'(c) = 0$. Then we can write $U(x^*) \approx U(c) + \frac{1}{2}U''(c)(x^* - c)^2$, or $\frac{1}{2}(x^* - c)^2 = [U(x^*) - U(c)]/U''(c) = \Delta V$. Thus the escape rates from the right and left wells can be given in the form [10]

$$W_i = \frac{l_i}{2\pi}\exp\left(-\frac{\Delta V_i}{D_i}\right), \; i = \pm 1, \tag{10}$$

where the variance of chaotic noise D_i and the parameter l_i are defined by relations

$$D_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_i(\Omega)\, d\Omega, \;\; l_i^2 = \frac{1}{2\pi D_i}\int_{-\infty}^{\infty} \Omega^2 F_i(\Omega)\, d\Omega \tag{11}$$

Here $F_i(\Omega)$ is the power spectrum of the fictitious noise generated by the process $x_i(t)$.

For a system with weak asymmetry we assume

$$\Delta V_{\pm 1} = \Delta \pm \mu u \cos\lambda t, \;\; F_{\pm 1}(\Omega) = F(\Omega) \pm \mu f(\Omega), D_{\pm 1} = \sigma^2 \pm \mu d, \tag{12}$$

where $\Delta = -U_0(c)/U_0''(c)$, $u = Ac/U_0''(c)$, and

$$\sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\Omega)\, d\Omega, \quad l^2 = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \Omega^2 F(\Omega)\, d\Omega. \tag{13}$$

As indicated earlier, nonoscillatory terms of order μ are negligible in the calculation of the spectral density and the SNR. Thus relations (10), (11), (12) yield

$$W_{\pm 1} = \frac{\alpha}{2}(1 \pm \mu \frac{u}{\sigma^2} \cos \lambda t) \pm \mu(\textit{nonoscillatory terms}) \tag{14}$$

where α is the hopping rate in the absence of the signal, given by

$$\alpha = \frac{l}{\pi} \exp\left(-\frac{\Delta}{\sigma^2}\right) \tag{15}$$

¿From (4), (14) we find

$$\beta = \alpha u / 2\sigma^2$$

By introducing (15), (16) into (9), we obtain the SNR in the form

$$R(\sigma) = \frac{\mu^2 l}{4} \frac{u^2}{\sigma^4} \exp\left(-\frac{\Delta}{\sigma^2}\right) \tag{16}$$

As a function of the noise intensity, $R(\sigma)$ has a bell-shape, as predicted by the SR theory [2], that is $R(\sigma) \to 0$ for $\sigma \to 0$, or $\sigma \to \infty$. In other words, an increase of the excitation can result in an improvement of the output SNR. This result for chaotic resonance was suggested in [6] and found with computer simulation in [4]. As was shown in [4], the dependence of R on the forcing amplitude γ is similar to its dependence on the noise intensity D in a system with random noise. Equation (17) is consistent with results of [4].

CHAOTIC RESONANCE IN A SYSTEM WITH NEAR-HOMOCLINIC CHAOS

In general, the quantitative model of the spectrum $F(\Omega)$ is not available. However, in the special case, when the unperturbed system has a homoclinic attractor and perturbed orbits pass through the chaotic layer near the homoclinic separatrix, the power spectrum can be computed analytically by employing the Melnikov theory [11]. This homoclinic structure exists in (1) if $b = 0$. When $b \neq 0$, the homoclinic separatrix is no longer attracting. However, there exists a set of parameters for which the most of the perturbed orbits remain in a narrow layer in the neighborhood of the separatrix. In this case, analytical estimation remains valid.

As an example, we consider the system (1) with the Duffing-Holmes potential $U_0(x) = -x^2/2 + x^4/4$. In this case the broadband part of the spectrum of the motion within a well can be written in the form (see details in [11])

$$F(\Omega) = F_0(\Omega)/T, \;\; F_0(\Omega) = 2\pi^2 \operatorname{sech}^2(\pi\Omega/2), \tag{17}$$

where T is the mean time of passage through the stochastic layer in the system with the unperturbed separatrix. To the leading order term, the parameter T is given by [11]

$$T \sim T_0 = -\ln(\epsilon M^*), \qquad (18)$$

where M^* is the maximum of the Melnikov function [8]

$$M^*(\omega) = \gamma S_M(\omega) - 4b/3, \qquad (19)$$

the Melnikov scale factor for the Duffing-Holmes system is $S_M(\omega) = \sqrt{2}\pi\omega \operatorname{sech} \pi\omega$.

By using a simple transformation, we write $\sigma^2 = D_0/T_0$,where the variance D_0 corresponds to the dimensionless spectral density $F_0(\Omega)$. As shown earlier, for small σ an increase in the noise intensity leads to an improvement of the SNR. As seen from (17), (18), D_0 is a fixed parameter and the SNR maximum corresponds to the minimum of T_0. For given γ and b, this minimum depends on the excitation frequency and corresponds to the maximum of the Melnikov scale factor $S_M(\omega)$. This effect was obtained in [4] with computer simulation

CONCLUSIONS

We considered a noise-free bystable system with a low frequency signal and a secondary harmonic excitation. The effect of the signal and of the secondary excitation is to cause hopping motions associated with chaos, as well as a spectral density with a broadband portion. We obtained the expression for the output signal-to-noise ratio (SNR). We referred to the enhancement of the SNR as chaotic resonance. Our derivations show similarities between results obtained for chaotic resonance on the one hand and classical stochastic resonance induced by random perturbations on the other.

The special case of a system with a nearly-homoclinic chaotic attractor is studied in detail. In this case, the spectral density of chaotic noise can be expressed explicitly via the Melnikov function. This allows direct calculation of the SNR and shows that the maximum SNR occurs when the excitation frequency corresponds to the peak frequency of the Melnikov scale factor.

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