A UNIFIED THEORY OF DETERMINISTIC AND NOISE-INDUCED TRANSITIONS: MELNIKOV PROCESSES AND THEIR APPLICATION IN ENGINEERING, PHYSICS AND NEUROSCIENCE

by

Emil Simiu
Building and Fire Research Laboratory
National Institute of Standards and Technology
Gaithersburg, MD 20899 USA

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A Unified Theory of Deterministic and Noise-Induced Transitions: Melnikov Processes and Their Application in Engineering, Physics and Neuroscience

Emil Simiu

National Inst. of Standards and Technology, Gaithersburg, MD 20899-8611, USA
emil.simiu@nist.gov

Abstract. For a class of deterministic systems chaotic dynamics entails irregular transitions between motions within a potential well (librations) and motions across a potential barrier (rotations). The necessary condition for the occurrence of chaos – and transitions – is that the system's Melnikov function have simple zeros. The behavior of those systems' stochastic counterparts, including their chaotic behavior, is similarly characterized by their Melnikov processes. The application of the Melnikov method shows that deterministic and stochastic excitations play similar roles in the promotion of chaos, meaning that stochastic systems exhibiting transitions between librations and rotations have chaotic behavior, including sensitivity to initial conditions, just like their deterministic counterparts. We briefly review the Melnikov method and its use to obtain: criteria guaranteeing the non-occurrence of transitions in systems excited by bounded processes; upper bounds for the probability that transitions can occur during a specified time interval in systems excited by unbounded processes; and assessments of the influence of the excitation's spectral density on the transition rate. We also briefly review applications of Melnikov processes.

INTRODUCTION

Chaotic dynamics theory and its applications were until recently concerned primarily with deterministic systems. A technique developed within the framework of chaotic dynamics, the Melnikov method, is applicable to a wide class of stochastic dynamical systems as well. Originally, the Melnikov method was developed for multistable planar systems with periodic or quasiperiodic excitation. The extension of the method for stochastically-excited systems is based on the approximation of physically realizable stochastic processes by processes consisting of finite sums of harmonics with random parameters. For each individual realization of the process those parameters take on fixed values. The Melnikov method thus becomes

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applicable to systems with stochastic excitation [1]. In this paper we briefly review Melnikov theory and applications that illustrate its usefulness.

OVERVIEW OF MELNIKOV THEORY FOR DETERMINISTIC AND STOCHASTIC SYSTEMS

A wide class of frictionless and unforced planar multistable systems have one or more saddle points from which nongeneric orbits – homoclinic or heteroclinic orbits – emanate in forward and reverse time. These orbits form a separatrix between regions of the phase plane. Motions starting inside (outside) the separatrix stay inside (outside) the separatrix for all time. If the system is subjected to a perturbation of order $\epsilon \ll 1$, and the perturbation is bounded and sufficiently smooth, it follows from the persistence theorem that the saddle points persist in planes of section $t = \text{const.}$, where t denotes time. Owing to the perturbation, however, the homoclinic orbits, which may be viewed as coinciding stable and unstable manifolds, become divided into a stable manifold and a separate, unstable manifold. The *Melnikov distance* is the distance between the separated stable and unstable manifolds, measured along a line normal to the unperturbed manifolds. We now consider the system

$$\ddot{x} = V'(x) + \epsilon[\gamma G(t) - \beta\dot{x}] \qquad (1)$$

where $V(x)$ is a multi-well potential. It can be shown that, to first order, the Melnikov distance is proportional to the *Melnikov function*, which has the expression

$$M(t) = \int_{-\infty}^{\infty} \gamma h(\zeta)G(t-\zeta)d\zeta - \beta c \qquad (2)$$

where the constant c depends on $h(t) = \dot{x}(-\zeta)$, and $\dot{x}(\zeta)$ is the ordinate of the system's homoclinic orbit [1]. Assume the excitation is

$$G(t) = \sum_{i=1}^{N} \gamma_i \cos(\omega_i t) \qquad (i = 1, 2, ..., N) \qquad (3)$$

The Melnikov function can then be written

$$M(t) = -\beta c + \gamma \sum_{i=1}^{N} \gamma_i S(\omega_i) \sin[\omega_i t + \theta_{0i}]. \qquad (4)$$

where the constant c and the *Melnikov scale factor* $S(\omega)$ depend on the system's homoclinic orbits. For sufficiently small ϵ, if the Melnikov function has simple zeros, the stable and unstable manifolds of the perturbed system intersect transversely. The Melnikov condition for transverse intersection is

$$\gamma \sum_{i=1}^{N} \gamma_i S(\omega_i) > \beta c \qquad (5)$$

The intersections with a plane of section of intersecting stable and unstable manifolds form a homoclinic tangle that exhibits lobes defined by segments of the tangle and a curve, called *homoclinic tangle*, consisting of the union of segments that most closely approximates the homoclinic orbit of the unperturbed system. Transport from the inside to the outside of the region enclosed by the pseudoseparatrix and vice-versa is effected by lobes within the homoclinic tangle. The mapping of certain areas of the phase plane by the nonlinear dynamical system entails expansion, contraction and folding leading to geometrical structures that may be studied by using symbolic dynamics techniques. Associated with such structures are trajectories that may be sensitive to initial conditions, and may therefore be imply the existence of a positive Lyapounov exponent. The Smale-Birkhoff theorem states that for a periodically or quasiperiodically excited planar system to be chaotic its Melnikov function must have simple zeros (*Melnikov necessary condition for chaos*). Unlike the separatrix of an unperturbed system, which is impermeable in the sense that it cannot be crossed by any orbit, a pseudoseparatrix can be permeable. This allows the occurrence of chaotic motions with irregular transitions between librational and rotational motions.

The extension to stochastically excited systems of results obtained for the quasiperiodic excitation case follows from the fact that physically realizable stochastic processes can be approximated as closely as desired by quasiperiodic sumns with random parameters. For example, a Gaussian process $G(t)$ with uniform distribution over the interval $[0,2\pi]$ may be approximated (or simulated) by the finite sum

$$G_N(t) = \sum_{i=1}^{N} \cos(\omega_i t + \phi_i) \tag{6}$$

where $\{\phi_i = 1, 2, ..., N\}$ are independent identically distributed random variables with uniform distribution over the interval $[0, 2\pi]$ and $a_i = [\Psi_0(\omega)\Delta\omega/\pi]^{1/2}$, $\omega_i = i\Delta\omega$, $N\Delta\omega = \omega_{cutoff}$, and ω_{cutoff} is the cutoff frequency.

The ensemble of Melnikov functions induced by a stochastic process $G(t)$ with spectral density function Ψ_0) is referred to as a *Melnikov process* [2]. The expectation and spectral density of the Melnikov process can be shown [1] to be, respectively,

$$E[M(t)] = -\beta c \tag{7}$$

$$\Psi_M(\omega) = \gamma^2 S^2(\omega)\Psi_0(\omega) \tag{8}$$

Knowledge of the spectral density allows the estimation of the mean time between consecutive zero upcrossings for the Melnikov process, τ_u.Simple chaotic transport considerations can then be used to show that τ_u is a lower bound for the system's escape time τ_c. If τ_c is large, use of the Poisson distribution yields lower bounds for the probability that no escapes will occur during a specified time interval [2].

It follows from the Melnikov condition for chaos that the larger the ordinates of the Melnikov process, the larger is the effectiveness of the excitation in inducing transitions, and conversely. The expression for the spectrum of the Melnikov process, $\Psi_M(\omega)$, is useful insofar as its provides information on the extent to which the Melnikov scale factor, $S(\omega)$, and the spectral density of the excitation, $\Psi_0(\omega)$, are matched to yield large – or small – ordinates of $\Psi_M(\omega)$.

Melnikov processes induced by other types of stochastic processes can be defined by similar approximations. In particular, consider the case of dichotomous noise

$$G(t) = a_n \tag{9}$$

$$[\alpha + (n-1)t_1 < t \leq (\alpha + n)t_1, \tag{10}$$

where n is the set of integers, the random varioable α is uniformly distributed between 0 and 1, the independent random variables a_n take on the values -1 and 1 with probabilities 1/2 and 1/2, respectively, and t_1 is a parameter of $G(t)$. From the Melnikov necessary condition for chaos if follows that transitions induced by the dichotomous excitation cannot occur if [3]

$$\gamma/\beta < 0.471. \tag{11}$$

If the system is subjected to a *multiplicative excitation* $\gamma(x, \dot{x})G(t)$, then in the expression of the Melnikov function the function $h(t)$ in the Melnikov integral is simply replaced by the product $\gamma(x, \dot{x})h(t)$.

APPLICATIONS

Along-Shore Currents Induced by Randomly Fluctuating Wind Over a Corrugated Ocean Floor. This application entails a simple model of mesoscale wind-induced alongshore ocean flow over a continental margin with variable bottom topography. The model was originally developed for the case of forcing by surface stresses fluctuating harmonically in time. It was extended in [2] for the case of stochastic forcing. The unperturbed system has homoclinic orbits due to th e presence of the sea bottom corrugations. The wind-induced flow can cause the motion to be chaotic. The Melnikov method yields parameters for which the flow is chaotic if the excitation is assumed to be harmonic, and upper bounds for the probability that chaos can occur during any specified time if the excitation is stochastic.

Snap-Through of Continuous Buckled Column With Distributed Transverse Random Load. In this application the Melnikov method is used to obtain criteria on the occurrence of noise-induced jumps in a spatially-extended dynamical system. The system chosen for illustration is a vertical column. Snap-through occurs if the column motion makes excursions beyond the column's unstable, vertical equilibrium position [4].

Melnikov-Based Open-Loop Control of Noise-Induced Escapes. We consider a one-degree-of-freedom multistable system subjected to a stochastic excitation. We can reduce that system's escape rate by applying to it a control force that, in a trivial control case, is proportional at all times to the excitation, to within some short time lag. A far more efficient control can be applied by using the information inherent in the Melnikov scale factor, and apply to the system a control force that has negligible frequency content in the frequency range for which that factor is small. By doing so we do not spend energy that would be ineffective from the control standpoint, since its contributions to the controlled system's Melnikov process – which determines the escape rate – would be negligible [5,6].

Stochastic Resonance. Melnikov theory yields qualitative results on the basis of which useful inferences can be made on the behavior of systems exhibiting stochastic resonance. Basically, the Melnikov method uses the fact that the deterministic and stochastic excitations play qualitatively equivalent roles in the promotion of chaos and escapes over a potential barrier, the motions being in both cases topologically conjugate to a shift map [7]. This fact suggested the extension of SR approaches beyond classical SR. It was shown in [7] that the SNR can alternatively be improved by keeping the noise unchanged and adding a deterministic excitation selected in accordance with Melnikov theory, rather than by increasing the noise. Also, since Melnikov theory provides information on excitation frequencies that are effective in increasing a system's characteristic rate, the chaotic dynamics approach makes it possible to assess the role of the excitation's spectral density in the enhancement of the SNR, a problem of interest in classical SR for which other available approaches can be unwieldy.

Modeling of Auditory Nerve Fiber. [8]. Experiments have established two basic features of auditory nerve fiber dynamics. First, mean firing rates produced by harmonic excitation in the presence of weak noise are largest for excitation frequencies contained in a relatively narrow "best" interval; for frequencies outside that interval mean firing rates decrease and, for both low and high frequencies, become vanishingly small. Second, white or nearly white noise excitation results in multimodal interspike interval histograms (ISIH's) with modes approximately equal to integer multiples of the period corresponding to the fiber's best frequency. (For a given experiment an ISIH represents the number of occurrences of firings as a function of the time interval separating them.) The Fitzhugh-Nagumo (FHN) model appears to be unable to reproduce these two dynamical features. In the presence of noise the disagreement between typical FHN model predictions and experimental results appears to be even stronger. In contrast, the Melnikov method leads to a strikingly successful modeling of the behavior of the auditory nerve fiber as a bistable dynamical system experiencing chaotic behavior under periodic, quasiperiodic, and stochastic excitation [8].

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