

## Vortices in Attractive Bose-Einstein Condensates in Two Dimensions

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The form and stability of quantum vortices in Bose-Einstein condensates with attractive atomic interactions is elucidated. They appear as ring bright solitons, and are a generalization of the Townes soliton to nonzero winding number  $m$ . An infinite sequence of radially excited stationary states appear for each value of  $m$ , which are characterized by concentric matter-wave rings separated by nodes, in contrast to repulsive condensates, where no such set of states exists. It is shown that robustly stable as well as unstable regimes may be achieved in confined geometries, thereby suggesting that vortices and their radial excited states can be observed in experiments on attractive condensates in two dimensions.

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The study of vortices has a long and illustrious scientific history reaching back to Helmholtz and Lord Kelvin in the 19th century [1]. Vortices associated with quantized circulation are a central feature of superfluidity [2]. Singly [3,4] and multiply quantized [5] vortices have been observed in Bose-Einstein condensates (BECs) with repulsive atomic interactions. Complex vortex structures have been shown to be stable in repulsive BEC's, including vortex dipoles [6,7] and vortex rings [8]. The nonlinear Schrödinger equation (NLSE), which provides an excellent description of BECs at the mean-field level [9], supports vortex solutions, which have been studied extensively in the case of repulsive atomic interactions [10]. The main goal of this Letter is to clarify the meaning and nature of single vortices and their excited states in BECs with *attractive* interactions, and thus encourage experimental exploration of stable and unstable two-dimensional (2D) BECs.

Solutions to the NLSE with attractive, or focusing nonlinearity, in contrast to repulsive, or defocusing nonlinearity, are unstable in free space in three dimensions (3D) and are stable in one dimension (1D) [11]. The imposition of an external potential in 3D can produce a long-lived metastable regime [12,13]. In the metastable and unstable regimes, growth and collapse cycles [14] and implosion [15] have been studied in 3D. In the stable regime, bright soliton propagation [16] and interactions [17] in a waveguide have been investigated in 1D [18].

The critical dimensionality for the NLSE is 2D [11,20]. We will show that quantum vortices and their radially excited states can be made robustly stable in confined, attractive 2D condensates and are a generalization of the Townes soliton [21] to nonzero winding number  $m$ . The Townes soliton is fundamental to understanding the self-similar collapse of solutions to the 2D NLSE [22]. Its generalization to winding number  $|m| \geq 1$  has been studied in the context of optics, where such solutions are called “ring-profile solitary waves” or “spinning bright solitons” [23]. This is in fact the attractive analog of the well-known single-vortex solution in repulsive conden-

sates [2], as we will show. An example contrasting the form of vortices in attractive and repulsive BECs is illustrated in Fig. 1(a) and 1(b).

In previous studies of quantum vortices in condensed matter [2] and optical systems [23] with attractive nonlinearity, the phase variation derived wholly from circulation of matter about the central vortex core. In this work, we investigate the most general type of single-vortex stationary solutions in attractive BECs for which the order parameter also alternates sign along radial lines. For each winding number  $m$ , we find a denumerably infinite set of radially excited states characterized by the successive formation of nodes at  $r = \infty$ . An example of such an excited state is illustrated in Fig. 1(c). In contrast, Fig. 1(d) shows how extremely different is the repulsive case.

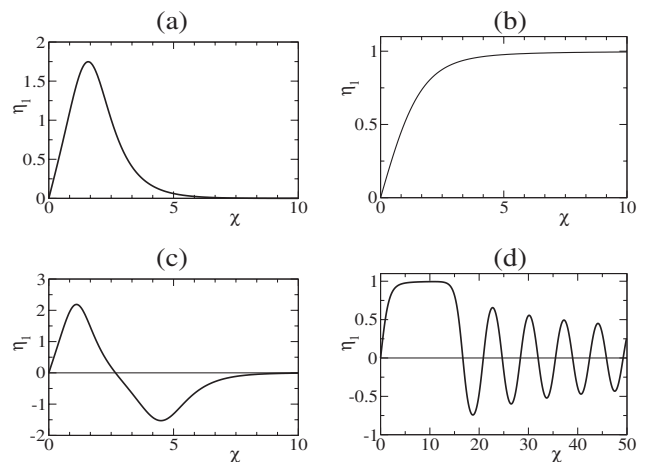


FIG. 1. A quantum vortex of winding number  $m = 1$  in free space: (a) attractive case; (b) repulsive case. A radially excited state: (c) the first excited state in the attractive case; (d) in the repulsive case, a radially excited state requires an infinite number of nodes and asymptotically resembles the Coulomb function [26]. The radial dependence of the order parameter of an infinitely extended condensate is depicted. Note that all axes are dimensionless.

Consider an order parameter of the form

$$\psi(r, \phi, t) = f_m(r) \exp(im\phi) \exp(-i\mu t/\hbar) \exp(i\theta_0), \quad (1)$$

which solves the 2D NLSE

$$[-(\hbar^2/2M)\nabla_{r,\phi}^2 + g_{2D}|\psi|^2 + V(r)]\psi = i\hbar\partial_t\psi, \quad (2)$$

with  $V(r)$  a central potential in two dimensions,  $m$  the winding number,  $M$  the atomic mass,  $g_{2D} \equiv (4\pi\hbar^2 a_s/M)\sqrt{M\omega_z/2\pi\hbar}$  the 2D atomic interaction strength,  $a_s < 0$  the  $s$ -wave scattering length, and  $\mu$  the chemical potential or eigenvalue. In Eq. (2) it was assumed that the BEC remains in the ground state in a harmonic trap of angular frequency  $\omega_z$  in the  $z$  direction [24], so that  $r \equiv \sqrt{x^2 + y^2}$ . Then, taking  $\theta_0 = 0$  and defining  $\eta_m(\chi) \equiv \sqrt{|g_{2D}/\mu|} f_m(r)$ ,  $\chi \equiv (\sqrt{2M|\mu|/\hbar})r$ , one obtains a rescaled 2D NLSE of the form

$$\eta_m'' + \chi^{-1}\eta_m' - m^2\chi^{-2}\eta_m + \eta_m^3 - V(\chi)\eta_m + \sigma_\mu\eta_m = 0, \quad (3)$$

where  $\sigma_\mu = \text{sgn}(\mu) = \pm 1$ . The solutions to this nonlinear second order ordinary differential equation describe quantum vortices and their radially excited states in an attractive BEC in an external potential  $V(\chi)$ . Examples for  $\sigma_\mu = -1$  and  $V = 0$  are shown in Figs. 1(a) and 1(c).

The form of the radial wave function  $\eta_m$  can be obtained from Eq. (3) by numerical shooting methods [25]. This requires the initial conditions  $\eta_m(\chi_0)$ ,  $\eta_m'(\chi_0)$ , which can be obtained via a power series around  $\chi = 0$ :

$$\eta_m(\chi) = \sum_{j=0}^{\infty} a_j \chi^{2j+m}, \quad (4)$$

where the  $a_j$  are coefficients. Note that  $\eta_m(\chi) \rightarrow a_0\chi^m$  as  $\chi \rightarrow 0$ . Upon substitution into Eq. (3) and solution of the resulting simultaneous equations, one finds that all coefficients  $a_j$  for  $j \neq 0$  can be expressed as a polynomial in powers of  $a_0$ . Thus the power series shows that  $a_0$ , the coefficient of the first nonzero term in the series, together with the winding number  $m$ , is sufficient to determine the entire solution. Note that we take  $a_0 \geq 0$  and  $m \geq 0$ ; solutions identical in form exist for  $a_0 < 0$  and/or  $m \leq 0$ .

We first consider the case of no external potential,  $V(\chi) = 0$ . By following the form of the wave function as a function of  $a_0$ , one observes its entire development. The case of  $m = 1$  and  $\sigma_\mu = -1$  is illustrated in Fig. 2. For very small  $a_0$ , the linear, divergent Bessel function solution  $K_m(\chi)$  is recovered for small  $\eta_m(\chi)$ . However, the non-linearity regulates the divergence as  $\eta_m$  becomes on the order of unity. It subsequently oscillates around unity, with the oscillations damping away as  $\chi \rightarrow \infty$  [Fig. 2(a)]. As  $a_0$  approaches a critical value  $a_0^{\text{vortex}}$ , the oscillations are pushed out towards  $\chi = \infty$  and a localized central peak appears in the wave function near the origin [Fig. 2(b)]. For  $a_0 = a_0^{\text{vortex}}$  precisely [26], a node forms at  $\chi = \infty$  and a quantum vortex is obtained. As  $a_0$  is increased, the node

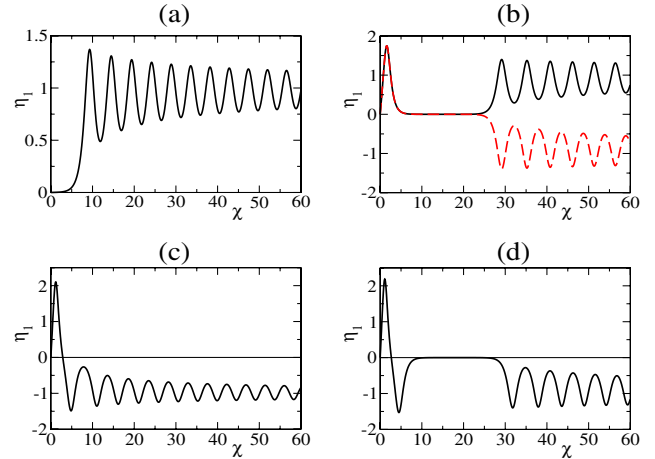


FIG. 2 (color online). The development of excited states of a quantum vortex in an attractive BEC. (a) For small  $a_0$ , no quantum vortex is formed. (b) As  $a_0 \rightarrow a_0^{\text{vortex}}$  from below (solid black curve) or above (dashed red curve), the oscillations are pushed out towards  $\chi = \infty$ . For  $a_0 = a_0^{\text{vortex}}$  a node appears at  $\chi = \infty$ , and a true vortex appears. (c) For still larger  $a_0$ , the node moves inwards. (d) As  $a_0$  approaches a second critical value, the oscillations are again pushed out, until a second node appears at  $\chi = \infty$ . In this way one obtains the first excited state. An infinite sequence of excited states can be constructed in this way. Note that the case of an infinitely extended condensate of winding number  $m = 1$  is depicted and all axes are dimensionless.

moves inwards and oscillations resume beyond it [Fig. 2(c)]. To move the oscillations towards infinity again, one must increase  $a_0$  towards a second critical value [Fig. 2(d)]. When this value is reached precisely, one obtains the first excited state, with no oscillations past the first node, and a second node appears at  $\chi = \infty$ . In this way one can construct an infinite set of excited states by increasing  $a_0$  to successive critical points. These critical points are always characterized by the formation of an additional node at  $\chi = \infty$ .

The appearance of a denumerably infinite set of critical points is in strong contrast to the case of a vortex in a repulsive condensate. As we have shown elsewhere [26], for a repulsive BEC there is only *one* critical value of  $a_0$ , i.e.,  $a_0^{\text{vortex}}$ . Larger values of  $a_0$  result in an infinite number of nodes and the wave function asymptotically resembles the Coulomb function, as illustrated in Fig. 1(d). We called these *ring solitons*, in contrast to vortex solutions, as they have a different analytic structure and asymptotic behavior. Thus, in an infinitely extended system, vortices in repulsive condensates cannot be radially excited in a stationary way, whereas in attractive condensates they have a denumerably infinite set of excited states. We note that, for positive chemical potential, i.e.,  $\sigma_\mu = +1$ , one can find ring solitons in attractive condensates which have a similar form to those of repulsive condensates.

For winding number  $m = 0$  and  $a_0 = 2.206\,200\,86\dots$ , one obtains the Townes soliton. Increasing  $a_0$  results in the formation of successive nodes at  $\chi = \infty$ , just as for  $m = 1$ ,

i.e., one finds the radially excited states of the Townes soliton. In this special case, it has been formally proven that an infinite set of radially excited states corresponding to the formation of nodes exists, and that the Townes soliton is the unique “ground state” in that sequence [11]. It is in this sense that the vortex solutions we have described are generalizations of the Townes soliton to nonzero winding number. All such solutions wherein a node has formed at infinity are normalizable.

In order to study attractive BECs in experiments, it is vital to consider stability properties in confined systems. The special stability properties of two dimensions can be illustrated by a simple variational study. Consider the variational ansatz

$$\psi(r, \phi, t) = Ar^m e^{-r^2/2r_0^2} e^{im\phi} e^{-i\mu t/\hbar}, \quad (5)$$

where  $r_0$  and  $A$  are variational parameters, subject to the power law potential  $V(r) = V_0 r^j$ ,  $j > 0$ . Integrating Eq. (2) over  $\psi^*(r, \phi, t)$ , one finds simultaneous equations for the chemical potential and  $g_{2D}$ . Then, using the normalization  $\int d^2r |\psi|^2 = N$  to eliminate  $A$ , where  $N$  is the total number of atoms, one obtains  $\mu(\mathcal{N})$  parametrically in  $r_0$ :

$$\mu = \frac{-\hbar^2 \Gamma(m+2)}{2Mr_0^2 \Gamma(m+2)} + \frac{V_0(1+j)\Gamma(m+\frac{j}{2}+1)r_0^j}{\Gamma(m+2)}, \quad (6)$$

$$\mathcal{N} = \frac{\Gamma(m+2)}{\pi^{-1/2}\Gamma(m+\frac{1}{2})} - \frac{MV_0 j \Gamma(m+\frac{j}{2}+1)r_0^{j+2}}{\pi^{-1/2}\hbar^2 \Gamma(m+\frac{1}{2})}, \quad (7)$$

where  $\mathcal{N} \equiv MN|g_{2D}|/2\pi\hbar^2 = 2|a_s|N\sqrt{M\omega_z}/2\pi\hbar$  is the dimensionless nonlinearity. The solution is radially stable when the Vakhitov-Kolokolov (VK) criterion [27]  $d\mu/d\mathcal{N} \leq 0$  is satisfied. One finds that the VK criterion always holds for  $\mathcal{N} < \mathcal{N}_c$ , where

$$\mathcal{N}_c = 2\sqrt{\pi}\Gamma(m+2)/\Gamma(m+1/2). \quad (8)$$

Equation (8) is *independent* of both  $V_0$  and  $j$ , i.e., radial stability does not depend on the details of any positive power law potential. In the limit in which  $j \rightarrow \infty$ , one obtains the same result for a cylindrical hard-wall potential. When  $\mathcal{N} \geq \mathcal{N}_c$ , the total energy  $E[\psi] = N[\mu - (g_{2D}/2) \int d^2r |\psi|^4] \rightarrow -\infty$  as  $r_0 \rightarrow 0$ , meaning that the wave function is unstable and implodes. When  $\mathcal{N} < \mathcal{N}_c$ , the energy has a *global* minimum at some finite  $r_0$ . In contrast, in three dimensions, the minimum, when it exists, is always local, so that, at best, one obtains metastability [13,28].

We now consider in detail the most experimentally relevant case, i.e.,  $j = 2$  and  $V_0 = M\omega^2/2$ , a harmonic trap. Note that typical trapping frequencies are on the order of  $\omega = 2\pi \times 100$  Hz, which gives a time scale of  $T \equiv 2\pi/\omega = 10$  ms. For no external potential,  $V(r) = 0$ , it is well known that the Townes soliton ( $m = 0$ ) is radially unstable, while the quantum vortex of winding number  $m = 1$  [Fig. 1(a)] is azimuthally unstable. The variational analysis of Eqs. (5)–(8) suggests that the addition of a

confining potential can stabilize the solutions radially for general  $m$ . However, this simple variational study did not consider azimuthal stability. Both radial and azimuthal stability can be determined by linear stability analysis, i.e., the Bogoliubov–de Gennes equations (BDGE) [10], a standard method which we do not reproduce here, for the sake of brevity. The case of general  $m$  *without* radial excitations has been studied previously [29–33]. It was found that for sufficiently small  $\mathcal{N}$  the vortex with  $m = 1$  is stable, while for  $m \geq 2$  the solution is unstable to quadrupole oscillations, though instability times can be much greater than  $T$ . We can therefore say that for small  $\mathcal{N}$  vortices of winding number  $m \geq 2$  are *experimentally stable*.

Applying the BDGE’s, we studied the azimuthal and radial stability of the radially excited states of the Townes soliton. The results for the first excited state, which has a matter-wave ring separated from a soliton core by a radial node, are illustrated in Figs. 3(a) and 3(b). The winding number of each Bogoliubov mode is denoted by  $q$ . The frequency of each mode is denoted by  $\Omega_q$  with instability time  $t_q \equiv 2\pi/\text{Im}(\Omega_q)$ . From Fig. 3(b), it is apparent that a radial instability ( $q = 0$ ) occurs for all  $\mathcal{N}$ , since  $\text{Im}(\Omega_0) \neq 0$ . However, one finds experimental stability for small  $\mathcal{N}$ , since  $t_q/T \gg 1$ . Other modes become unstable at higher  $\mathcal{N}$ , starting with the dipole,  $q = 1$ . For large  $\mathcal{N}$  the dominant instability occurs in the quadrupole mode.

In Figs. 3(c) and 3(d) is shown the same stability analysis for the first excited state of the  $m = 1$  vortex, which has two concentric matter-wave rings separated by a radial

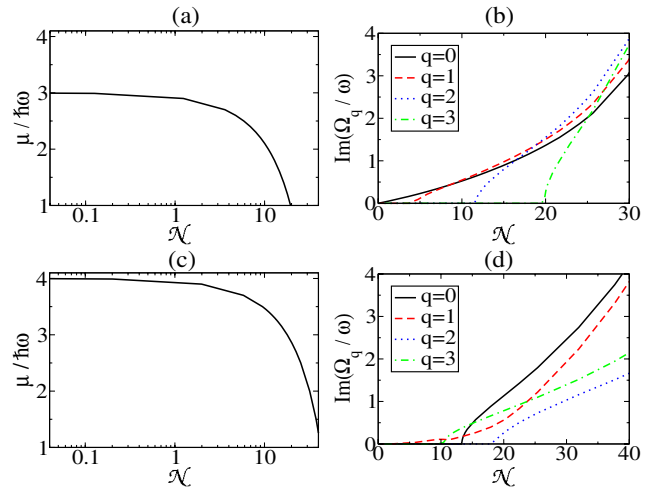


FIG. 3 (color online). Stable regimes of quantum vortices in confined attractive BECs. Shown are the chemical potential spectra and linear instability frequencies as a function of the nonlinearity for (a)–(b) the first excited state of the Townes soliton, and (c)–(d) the first excited state of a singly quantized vortex  $m = 1$ , all in a harmonic potential. In (b),(d) the winding number of the instability mode is denoted by  $q$  and the instability time given by  $2\pi/\text{Im}(\Omega_q)$ . Note that all axes are scaled to harmonic oscillator units.

node. Unlike the excited states of the Townes soliton, here the solution is linearly stable for  $\mathcal{N} < 5.5$ . The solution continues to be experimentally stable up to  $\mathcal{N} \sim 10$ , although a dipole instability occurs for  $t_1 \gg T$ . For large  $\mathcal{N}$  the dominant instability is radial. For the radially excited states of both the Townes soliton and the  $m = 1$  vortex, the VK criterion, which has never been proven formally, fails, as is apparent in Figs. 3(a) and 3(c). For winding number  $m = 2$  we find that the first excited state is most unstable to quadrupolar excitations, although for  $\mathcal{N} \lesssim 10$  it is experimentally stable.

In experiments, one may access the stable regime of quantum vortices and their excited states via a Feshbach resonance, which allows for extremely small scattering lengths on the order of the Bohr radius; this technique was used to successfully create bright solitons [16,17]. Vortices may be created by rotating the condensate; alternatively, one may start with a single vortex in a repulsive condensate and adiabatically switch the scattering length from positive to negative. Excited states may be made by phase imprinting through a pinhole mask, by passing a tightly focused laser pulse through the condensate center, or by utilizing a doughnut mode of a laser. It seems likely that the most useful approach will use these methods to first create dark ring solitons [34] on a repulsive condensate with a central vortex [26], and then tuning the scattering length to be negative. An important point in the assumption of the two-dimensional regime is that, in order for our stability criterion to be valid, the condensate must remain oblate, so that excitations in  $z$  do not occur; this may be achieved by a correct choice of experimental parameters.

In conclusion, we have shown that quantum vortices and their radially excited states in attractive BECs can be created stably in confined systems. We contrasted vortices in attractive BECs, which can be thought of as ring bright solitons or spinning Townes solitons, to their counterparts in repulsive BECs. We showed that there exists a denumerably infinite set of excited states which, in an infinitely extended condensate, correspond to the creation of nodes at  $r = \infty$ . In a harmonic trap, these are stable or experimentally stable for sufficiently small nonlinearity. In contrast to 3D, there is no metastability. We note that, in an optics context, vortices, or “spinning bright solitons,” can also be stabilized by competing nonlinearities, i.e., a defocusing quintic nonlinearity with a focusing cubic nonlinearity [35].

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