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Determination of the Uncertainties for ITS-90 Realization by SPRTs Between Fixed Points

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Abstract: Calibrated Standard Platinum Resistance Thermometers (SPRTs) are used to realize the International Temperature Scale of 1990 (ITS-90) from 13.8033 K to 1234.93 K. The SPRTs are calibrated at a series of fixed points, each assigned a temperature on the ITS-90, by measuring the ratios of the SPRT resistances at those temperatures to that at the triple point of water (TPW). For realizing the scale with a calibrated SPRT, a user measures the resistance ratio at the unknown temperature and uses ITS-90-defined equations to interpolate between fixed points. The uncertainty of the SPRT temperature is therefore largely influenced by the propagation of fixed-point resistance-ratio uncertainties. In this paper, we rigorously derive the equations for calculating these uncertainties for a variety of circumstances and we use software tools written by us to perform these calculations using realistic uncertainties for fixed points and other input parameters. For properly calculating the standard uncertainty for SPRT realization of the ITS-90, correlations between the input quantities must be considered, in particular those involving measurement of the TPW resistance. The proper calculation depends on three factors involving SPRT use and calibration. The different combinations of these factors result in six different equations for calculating the realization uncertainty. We derive these six equations, specify the conditions of their use, and discuss the relevant uncertainty components for each of them. We also compare the results of these equations with those of two approximations that may be used for calculating the standard uncertainty and explain the conditions under which the

simpler approximations agree with the more detailed calculations. Because these calculations are complicated, we are making our software tools available upon request to the user community.

1. Introduction

When performing a realization of the temperature T_{90} on the International Temperature Scale of 1990 (ITS-90) [1] with a calibrated standard platinum resistance thermometer (SPRT), a user determines the temperature from the SPRT resistance ratio

$$W(T_{90}) \equiv R(T_{90}) / R_{\mathrm{TPW}} , \tag{1}$$

where $R(T_{90})$ is the resistance of the SPRT at T_{90}. Also, R_{TPW} is the resistance of the SPRT at the the triple point of water (TPW) temperature, defined as $T_{\mathrm{TPW}} \equiv 273.16$ K. Here, it is assumed that the SPRT is in the same physical state for both R and R_{TPW} (the "physical state" of an SPRT will be discussed later in this introduction). For calculating T_{90} from W, a calibration-dependent deviation function D (commonly denoted as ΔW) is used to relate W to an ITS-90 reference function $W_{\mathrm{r}}(T_{90})$ [1] such that

$$W_{\mathrm{r}}(T_{90}) = W - D . \tag{2}$$

The equations that mathematically define $W_{\mathrm{r}}(T_{90})$ and D are provided in Section 2. The temperature T_{90} is subsequently obtained by inverting $W_{\mathrm{r}}(T_{90})$ through an iterative process. The ITS-90 specifies 11 subranges over which SPRTs may be calibrated, each with its own deviation function. Each deviation function is constructed using a subrange-specified functional form, which is also provided in Section 2; the coefficients of this function are determined using measurements of W at a subrange-specified set of fixed points with ITS-90-defined temperatures. The value of W at the i^{th} fixed point is denoted as $W_{\text{FP},i}$ and defined as

$$W_{\text{FP},i} \equiv R_{\text{FP},i} / R_{\text{TPW},i} . \tag{3}$$

Here, it is again assumed that the SPRT is the same physical state during both resistance measurements. Ultimately, when the SPRT measures W at an unknown temperature, W_{r} (and hence T_{90}) is calculated from W and all the $W_{\text{FP},i}$ values for the desired subrange.

During an SPRT calibration, the calibrator determines values of $W_{\text{FP},i}$ with uncertainty $u(W_{\text{FP},i})$ for all fixed points used in the subrange (this paper uses the notation $u(x)$ as the standard uncertainty for the quantity x). For each measurement of $W_{\text{FP},i}$ a measurement of R is made while the SPRT is inserted in the fixed point cell. Afterwards the appropriate corrections (e.g. pressure head) are applied to it to obtain its value at the fixed-point temperature; this value is denoted here as $R_{\text{FP},i}$. Often a similar measurement of R is made in a calibration-lab TPW cell immediately afterwards to obtain the TPW resistance (denoted here as $R^{\text{cal}}_{\text{TPW},i}$). For calibrations of capsule-type SPRTs at low temperatures, it is cumbersome and unnecessary to measure R_{TPW} after every fixed point, so only one measurement of R_{TPW} (denoted here as $R^{\text{cal}}_{\text{TPW}}$) is used for calculating $W_{\text{FP},i}$. The calibration report provides coefficients used for determining D for the requested subrange. The report also provides a value for R_{TPW} (denoted here as $R^{\text{rep}}_{\text{TPW}}$) that is related to the values of R_{TPW} measured during the calibration. After calibration of the SPRT, the user of the SPRT often measures the SPRT resistance in the user laboratory, using a different TPW cell and different resistance-measurement equipment, obtaining $R_{\mathrm{TPW}}^{\mathrm{user}}$.

Proper calculation of T_{90} is complicated because of detrimental changes in the physical state of an SPRT that cause shifts in $R(T_{90})$. There are reversible changes in state that occur naturally when the SPRT temperature is changed and which do not cause resistance shifts when the SPRT temperature is returned to T_{90} (examples are changes in the density of lattice vacancies and changes in the electron transport properties); these changes are of no concern because they cause no irreproducibility in the empirical W versus temperature relation. The detrimental changes that lead to irreproducibility of the W relation are caused by:

1) mechanical shock on the SPRT (which causes strain on the platinum),
2) changes in the oxidation state of the platinum at elevated temperatures
3) changes in moisture distribution in the SPRT
4) thermal shock
5) grain growth
6) contamination
7) loss of the hermetic seal of the SPRT
8) other unknown causes.

In this paper, when we refer to an SPRT at T_{TPW} as being in the same physical state as at temperature T_{90}, we mean that there are no changes in state that would cause the resistance shifts described above.

For thermometers used at temperatures above 234 K, shifts in R_{TPW} over the course of a calibration are of the order of several tenths of a millikelvin, even for the most careful calibrations. By comparison, the uncertainty of the TPW realization itself is often an order of magnitude better. Provided that any resistance shift occurring between the measurements of $R(T_{90})$ and R_{TPW} is very small or nonexistent, the previously-accumulated shifts of these two quantities are highly correlated. In this case, these shifts in resistance measurements cause virtually no shift in resistance ratio measurements. Because of this, SPRT calibration uncertainties will be lower if $R_{\mathrm{TPW},i}^{\mathrm{cal}}$ is measured after every $R_{\mathrm{FP},i}$ for use in calculating $W_{\mathrm{FP},i}$ when calibration fixed-point temperatures are above 234 K; this assures that these two resistances are measured while the SPRT is in the same physical state. Also, because of resistance-shift effects, SPRT users measuring temperatures above 234 K are strongly advised to calculate W using a measurement of $R_{\mathrm{TPW}}^{\mathrm{user}}$ made soon after making resistance measurements at the unknown temperature in order to minimize temperature-measurement uncertainties. Nevertheless, users that do not have access to a TPW cell may have no choice but to use $R_{\mathrm{TPW}}^{\mathrm{rep}}$ for calculating W.

Calculations of SPRT uncertainties are generally based on the ISO Guide to the Expression of Uncertainty in Measurement (GUM) [2]. A considerable amount of literature exists on GUM-compliant calculation of SPRT uncertainties between fixed-points when using the ITS-90. [3] Most recently, work has been done to use Lagrangian and non-Lagrangian analytical methods for determining $u(T_{90})$ in terms of W and the calibration values of $W_{\mathrm{FP},i}$ measured for the SPRT subrange [3-5]. In [4], White showed that the proper equation for calculating $u(T_{90})$ depends on whether the user employs $R_{\mathrm{TPW}}^{\mathrm{user}}$ or $R_{\mathrm{TPW}}^{\mathrm{rep}}$ when calculating T_{90}. He also showed that the equation for calculating $u(T_{90})$ depends on whether the calibration laboratory made one measurement of $R_{\mathrm{TPW}}^{\mathrm{cal}}$ for the entire calibration or made separate measurements of $R_{\mathrm{TPW},i}^{\mathrm{cal}}$ for each fixed point. White presented three equations for calculating $u(T_{90})$. However, some important uncertainty components, such as the uncertainty from resistance shifts mentioned above, were not included in the equations. Also, all equations expressed $u(T_{90})$ in terms of the uncertainties of $R_{\mathrm{FP},i}$. In the ITS-90, however, temperature is expressed in terms of W and $W_{\mathrm{FP},i} = R_{\mathrm{FP},i} / R_{\mathrm{TPW},i}$, and therefore certain resistance-related measurement errors (e.g., resistance-standard errors) may cancel in the resistance ratio and provide no contribution to $u(T_{90})$. Because of this, it is preferable to express SPRT measurement uncertainties in terms of W and $W_{\mathrm{FP},i}$ whenever possible. When this is not possible, it is best to express the uncertainties in terms of the quantities $\Re(T_{90})$, $\Re_{\mathrm{TPW}}$, and $\Re_{\mathrm{FP},i}$ (the ratios of $R(T_{90})$, R_{TPW}, and $R_{\mathrm{FP},i}$, respectively, relative to a resistance standard R_{stnd}), since these are the quantities which are directly measured using a resistance bridge.

In this paper we describe a method for calculating $u(T_{90})$ and use software tools to perform sample calculations using realistic values for fixed-point uncertainties and other input parameters. We rigorously derive six general equations for performing the calculations and explain the circumstances under which each equation should be used. The proper equation depends on three factors involving SPRT use and calibration: 1) whether the user determines $W(T_{90})$ using $R_{\mathrm{TPW}}^{\mathrm{user}}$ or $R_{\mathrm{TPW}}^{\mathrm{rep}}$, 2) whether the unknown temperature is measured inside the original calibration laboratory (using the same TPW cell and resistance-measurement equipment) or externally (using a different TPW cell and resistance-measurement system) and 3) whether, during calibration, the $W_{FP,i}$ are determined by using TPW resistance measurements performed after each fixed-point resistance measurement or by using one TPW resistance value for the entire calibration. All equations describe $u(T_{90})$ in terms of resistance-ratio uncertainties when appropriate. The equations include a formalism for including uncertainties for ITS-90 non-uniqueness and for shifts in the values of $W_{FP,i}$ after calibration. Sample plots are shown to illustrate the difference in the results of the uncertainty calculations for the different equations. We also compare these six equations with two approximations to the GUM uncertainty which may be used for calculating $u(T_{90})$ (see Eq. 52 and Eq. 54) and show the conditions under which these simpler approximations agree well with the more detailed calculations.

Recognizing that it may be difficult for many in the user community to implement this method, we are making our software tools available to those wishing to use them for their own input parameters. Information on how to obtain the tools is given at the end of the manuscript.

This paper addresses only the uncertainties of temperature measurement due to uncertainties of SPRT calibration and use. It does not include other temperature-measurement uncertainties that may be encountered by the SPRT user, such as temperature gradients, fluctuations and drifts.

2. ITS-90 Reference Function, and Deviation Functions and Fixed Points for the Subranges

The ITS-90 reference function for SPRTs is [1]

$$\ln[W_r(T_{90})] = A_0 + \sum_{i=1}^{12} A_i \left[\frac{\ln(T_{90}/273.16\ \mathrm{K}) + 1.5}{1.5}\right]^i, \qquad 13.8033\ \mathrm{K} \le T_{90} \le 273.16\ \mathrm{K}$$

$$W_r(T_{90}) = C_0 + \sum_{i=1}^{9} C_i \left[\frac{T_{90}/\mathrm{K} - 754.15}{481}\right]^i, \qquad 273.15\ \mathrm{K} \le T_{90} \le 1234.93\ \mathrm{K} \qquad 4)$$

The ITS-90 deviation functions for the different subranges are listed below. In order to address uncertainties from ITS-90 non-uniqueness, we introduce here the quantity $N(T_{90})$ which, if known, would correct for this non-uniqueness. Since $N(T_{90})$ is not described in the ITS-90 documents, it is assumed to be zero and ignored when determining the coefficients used for calculating D. However, the uncertainty $u(N)$ provides a non-uniqueness component to $u(T_{90})$. The deviation functions are as follows [1,6]:

Subranges 1-3: $$D = a[W(T_{90}) - 1] + b[W(T_{90}) - 1]^2 + \sum_{i=1}^{5} c_i [\ln W(T_{90})]^{i+n} + N(T_{90}),$$

Subrange 1: $n = 2$

Subrange 2: $c_4 = c_5 = n = 0$

Subrange 3: $c_2 = c_3 = c_4 = c_5 = 0$, $n = 1$ 5)

$A_0 =$	−2.135 347 29	$C_0 =$	2.781 572 54
$A_1 =$	3.183 247 20	$C_1 =$	1.646 509 16
$A_2 =$	−1.801 435 97	$C_2 =$	−0.137 143 90
$A_3 =$	0.717 272 04	$C_3 =$	−0.006 497 67
$A_4 =$	0.503 440 27	$C_4 =$	−0.002 344 44
$A_5 =$	−0.618 993 95	$C_5 =$	0.005 118 68
$A_6 =$	−0.053 323 22	$C_6 =$	0.001 879 82
$A_7 =$	0.280 213 62	$C_7 =$	−0.002 044 72
$A_8 =$	0.107 152 24	$C_8 =$	−0.000 461 22
$A_9 =$	−0.293 028 65	$C_9 =$	0.000 457 24
$A_{10} =$	0.044 598 72		
$A_{11} =$	0.118 686 32		
$A_{12} =$	−0.052 481 34		

Subrange 4: $$D = a[W(T_{90}) - 1] + b[W(T_{90}) - 1]\ln W(T_{90}) + N(T_{90}), \tag{6}$$

Subranges 5-11: $$D = a[W(T_{90}) - 1] + b[W(T_{90}) - 1]^2 + c[W(T_{90}) - 1]^3$$
$$+ d[W(T_{90}) - W(660.323\ ^\circ\mathrm{C})]^2 + N(T_{90}),$$

Subrange 5, 8, 9: $c = d = 0$

Subrange 6: all coefficients used.

Subrange 7: $d = 0$

Subrange 10, 11: $b = c = d = 0$ 7)

Here, a, b, and c_i are calibration coefficients determined by solving the equation at the fixed-point temperatures using W_r and the set of $W_{FP,i}$ values measured for the subrange; as a result, D is implicitly a function of this set of $W_{FP,i}$ values.

The fixed points used by the ITS-90 are the triple point of equilibrium hydrogen (e-H_2 TP), the vapor pressure of equilibrium hydrogen near 17.035 K (e-H_2 VP1), the vapor pressure of equilibrium hydrogen near 20.27 K (e-H_2 VP2), the triple point of neon (Ne TP), the triple point of oxygen (O_2 TP), the triple point of argon (Ar TP), the triple point of mercury (Hg TP), the melting point of gallium (Ga MP), the freezing point of indium (In FrP), the freezing point of tin (Sn FrP), the freezing point of zinc (Zn FrP), the freezing point of aluminum (Al FrP), and the freezing point of silver (Ag FrP). The use of these fixed points by the 11 respective ITS-90 subranges is provided in Table 1.

3. Additional Definitions and Notations

For describing uncertainties in T_{90}, it is necessary to refine old definitions and introduce new definitions and notation for this paper. First, we refine the definition of R_{TPW} to be the resistance of the SPRT *in its physical state at the time of its measurement* at temperature $T_{\mathrm{TPW}} \equiv 273.16$ K. We similarly refine the definition of $R_{\mathrm{FP},i}$ to be the resistance of the SPRT *in its physical state at the time of its measurement* at the i^{th} fixed point with ITS-90 defined temperature $T_{\mathrm{FP},i}$. Because the physical state of the SPRT may have changed slightly, we introduce the change-of-state correction factor $S_{\mathrm{FP},i}^{\mathrm{TPW}}$ such that

$$S_{\mathrm{FP},i}^{\mathrm{TPW}} = R_{\mathrm{TPW},i} / \mathcal{R}_{\mathrm{TPW},i} \quad \text{(if an } R_{\mathrm{TPW}} \text{ measurement is made after each fixed point)}$$

$$S_{\mathrm{FP},i}^{\mathrm{TPW}} = R_{\mathrm{TPW}} / \mathcal{R}_{\mathrm{TPW},i} \quad \text{(if one } R_{\mathrm{TPW}} \text{ value is used for entire calibration).} \qquad 8)$$

where $\mathcal{R}_{\mathrm{TPW},i}$ is the resistance that the SPRT would have at temperature T_{TPW} if it were in the same physical state (see discussion in Section 1) as when $R_{\mathrm{FP},i}$ was measured. In this case, Eq. 3 becomes

$$W_{\mathrm{FP},i} \equiv \frac{R_{\mathrm{FP},i}}{R_{\mathrm{TPW},i}} S_{\mathrm{FP},i}^{\mathrm{TPW}} \quad \text{(if an } R_{\mathrm{TPW}} \text{ measurement is made after each fixed point)}$$

$$W_{\mathrm{FP},i} \equiv \frac{R_{\mathrm{FP},i}}{R_{\mathrm{TPW}}} S_{\mathrm{FP},i}^{\mathrm{TPW}} \quad \text{(if one } R_{\mathrm{TPW}} \text{ value is used for entire calibration).} \qquad 9)$$

For calculation of $W_{\mathrm{FP},i}$, $S_{\mathrm{FP},i}^{\mathrm{TPW}}$ is assumed to have a value of unity, but its uncertainty will be used to determine the uncertainty in T_{90} from the SPRT change of state.

Similarly, for use of the calibrated SPRT to determine an unknown temperature T_{90} by measuring $R(T_{90})$ and R_{TPW}, we define $S_{T_{90}}^{\mathrm{TPW}}$ as the SPRT change-of-state correction factor

$$S_{T_{90}}^{\mathrm{TPW}} = R_{\mathrm{TPW}} / \mathsf{R}_{\mathrm{TPW}} \, . \tag{10}$$

Here, $\mathsf{R}_{\mathrm{TPW}}$ is the resistance that the SPRT would have at T_{TPW} if it were in the same physical state as when $R(T_{90})$ was measured. Using Eq. 10, we account for changes in state in the SPRT by modifying Eq. 1 to give

$$W(T_{90}) \equiv \frac{R(T_{90})}{R_{\mathrm{TPW}}} S_{T_{90}}^{\mathrm{TPW}} \tag{11}$$

For calculation of $W(T_{90})$, $S_{T_{90}}^{\mathrm{TPW}}$ is assumed to have a value of unity, but its uncertainty will be used to determine the uncertainty in T_{90} from the SPRT change of state.

We also define $T_{\mathrm{TPW}}^{\mathrm{cal}}$ and $T_{\mathrm{FP},i}^{\mathrm{cal}}$ as the temperatures realized, after application of all known corrections, in the calibration laboratory's TPW cell and the i^{th} fixed-point cell, respectively. For the case where an R_{TPW} measurement is made after each fixed point, we define $R_{\mathrm{FP},i}^{\mathrm{cal}}$ as the resistance measured at $T_{\mathrm{FP},i}^{\mathrm{cal}}$ and $R_{\mathrm{TPW},i}^{\mathrm{cal}}$ as the resistance measured at $T_{\mathrm{TPW}}^{\mathrm{cal}}$ after the measurement of $R_{\mathrm{FP},i}^{\mathrm{cal}}$. For the case where one R_{TPW} value is used for entire calibration, we define $R_{\mathrm{TPW}}^{\mathrm{cal}}$ as the resistance measured at $T_{\mathrm{TPW}}^{\mathrm{cal}}$. Using these definitions, we may then define the realization correction factors

$$C_{\mathrm{TPW}}^{\mathrm{cal}} \equiv R_{\mathrm{TPW},i}^{\mathrm{cal}} / R_{\mathrm{TPW}} \cong 1 + \frac{dW_{\mathrm{r}}}{dT_{90}}\left(T_{\mathrm{TPW}}^{\mathrm{cal}} - 273.16\ \mathrm{K}\right) \qquad 12)$$

and

$$C_{\mathrm{FP},i}^{\mathrm{cal}} \equiv R_{\mathrm{FP},i}^{\mathrm{cal}} / R_{\mathrm{FP},i} \cong 1 + \frac{dW_{\mathrm{r}}}{dT_{90}}\left(T_{\mathrm{FP},i}^{\mathrm{cal}} - T_{\mathrm{FP},i}\right) \qquad 13)$$

to account for unknown systematic errors in the realizations performed in real TPW cells and other fixed-point cells. The values of $C_{\mathrm{TPW}}^{\mathrm{cal}}$ and $C_{\mathrm{FP},i}^{\mathrm{cal}}$ may be assumed to be equal to unity for good-quality cells. The uncertainties $u(C_{\mathrm{TPW}}^{\mathrm{cal}})$ and $u(C_{\mathrm{FP},i}^{\mathrm{cal}})$ involve the uncertainties of the TPW and other fixed-point realizations but not the uncertainties of the resistance measurements or those due to SPRT resistance shifts.

For the fixed points measured in the calibration laboratory, we define

$$W_{\mathrm{FP},i}^{\mathrm{cal}} \equiv \frac{R_{\mathrm{FP},i}^{\mathrm{cal}}}{R_{\mathrm{TPW},i}^{\mathrm{cal}}} S_{\mathrm{FP},i}^{\mathrm{TPW}} = \frac{\mathfrak{R}_{\mathrm{FP},i}^{\mathrm{cal}}}{\mathfrak{R}_{\mathrm{TPW},i}^{\mathrm{cal}}} S_{\mathrm{FP},i}^{\mathrm{TPW}}, \text{ (if one } R_{\mathrm{TPW}} \text{ measurement is made after each fixed point)}$$

$$W_{\mathrm{FP},i}^{\mathrm{cal}} \equiv \frac{R_{\mathrm{FP},i}^{\mathrm{cal}}}{R_{\mathrm{TPW}}^{\mathrm{cal}}} S_{\mathrm{FP},i}^{\mathrm{TPW}} = \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} S_{\mathrm{FP},i}^{\mathrm{TPW}}, \quad \text{(if one } R_{\mathrm{TPW}} \text{ value is used for the entire calibration).} \qquad 14)$$

Here, $\Re_{\mathrm{FP},i}^{\mathrm{cal}}$, $\Re_{\mathrm{TPW},i}^{\mathrm{cal}}$, and $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$ are the resistance ratios relative to a resistance standard $R_{\mathrm{stnd}}^{\mathrm{cal}}$ for the resistances $R_{\mathrm{FP},i}^{\mathrm{cal}}$, $R_{\mathrm{TPW},i}^{\mathrm{cal}}$, and $R_{\mathrm{TPW}}^{\mathrm{cal}}$, respectively. For an uncertainty analysis, it is important to consider the resistance ratios because they are the quantities that are measured directly in the calibration laboratory rather than the resistances themselves. When determining $W_{\mathrm{FP},i}^{\mathrm{cal}}$, the effect of the resistance standard cancels out and is therefore irrelevant in the uncertainty analysis for $W_{\mathrm{FP},i}^{\mathrm{cal}}$.

Combining Eqs. 9, 12, 13 and 14 results in the relation

$$W_{\mathrm{FP},i} = W_{\mathrm{FP},i}^{\mathrm{cal}} \frac{C_{\mathrm{TPW}}^{\mathrm{cal}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}}. \qquad 15)$$

4. Derivation of Uncertainty Equations

Using the general law of uncertainty propagation [2], the uncertainty of SPRT realizations of T_{90} between fixed points is given by

$$u(T_{90})^2 = \sum_{i=1}^{n} \left(\frac{\partial T_{90}}{\partial x_i} \right) u(x_i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{i,j} \frac{\partial T_{90}}{\partial x_i} \frac{\partial T_{90}}{\partial x_j} u(x_i)\, u(x_j) \qquad 16)$$

where the x_i are the n input quantities required to obtain T_{90}. The relevant quantities and the derivatives $\partial T_{90}/\partial x_i$ may be found by expanding the differential dT_{90}:

$$dT_{90} = \sum_{i=1}^{n} \frac{\partial T_{90}}{\partial x_i} dx_i \ . \tag{17)}$$

Performing the expansion,

$$\begin{aligned} dT_{90} &= \frac{\partial T_{90}}{\partial W_{\mathrm{r}}} dW_{\mathrm{r}} = \frac{\partial T_{90}}{\partial W_r} \left[dW - dD\left(W, W_{\mathrm{FP},i}, N\right) \right] \\ &= \frac{\partial T_{90}}{\partial W_{\mathrm{r}}} \left[\left(1 - \frac{\partial D}{\partial W}\right) dW - \sum_{i=1}^{n} \left(\frac{\partial D}{\partial W_{\mathrm{FP},i}} dW_{\mathrm{FP},i} \right) - dN \right] \\ &\cong \frac{\partial T_{90}}{\partial W_{\mathrm{r}}} \left[dW - \sum_{i=1}^{n} \left(\frac{\partial D}{\partial W_{\mathrm{FP},i}} dW_{\mathrm{FP},i} \right) - dN \right]. \end{aligned} \tag{18)}$$

Here, the relevant quantities are W, $W_{\mathrm{FP},i}$ and N. In Eq. 18, W will be correlated with the $W_{\mathrm{FP},i}$ if the value of R_{TPW} used for calculating W is obtained from the calibration report (rather than from a measurement by the user). In addition, W will be correlated with the $W_{\mathrm{FP},i}$ if the SPRT measurement is "internal" (using the same TPW cell and resistance-measurement equipment as used during the calibration) rather than "external" (using a different TPW cell and resistance measurement system). Also, the different $W_{\mathrm{FP},i}$ will be correlated if they all share the same value for R_{TPW}. These factors result in six cases for calculating T_{90} and $u(T_{90})$, which are summarized in Table 2 and discussed in the following sections.

Because calculation of correlation coefficients can be cumbersome, the methodology used by this paper involves expanding out the differentials dW and $dW_{\mathrm{FP},i}$, expressing them in terms of independent x_i (hence $r_{i,j} = 0$ for all i and j in Eq. 14). The differences for the six different cases described above due to correlations then result in different choices of x_i and/or different expressions for $\partial T_{90}/\partial x_i$.

Case 1: External User determines W using $R_{\mathrm{TPW}}^{\mathrm{user}}$, one $R_{\mathrm{TPW}}^{\mathrm{cal}}$ value used for each fixed point

For this arrangement, the user is external to the calibration laboratory and W is calculated using a value of R_{TPW} measured by the user (denoted as $R_{\mathrm{TPW}}^{\mathrm{user}}$), and one measurement of R_{TPW} is made for each fixed point during the calibration. The measurement of $R_{\mathrm{TPW}}^{\mathrm{user}}$ is performed preferably immediately after the measurement at the unknown temperature to minimize SPRT change-of-state uncertainties. For this case, we define $T_{\mathrm{TPW}}^{\mathrm{user}}$ as the actual temperature realized in the user TPW cell and then define the realization correction factor

$$C_{\mathrm{TPW}}^{\mathrm{user}} \equiv R_{\mathrm{TPW}}^{\mathrm{user}} / R_{\mathrm{TPW}} \cong 1 + \frac{dW_{\mathrm{r}}}{dT_{90}}\left(T_{\mathrm{TPW}}^{\mathrm{user}} - 273.16\ \mathrm{K}\right). \qquad 19)$$

Here, it is assumed that the physical state of the SPRT is the same for both $R_{\mathrm{TPW}}^{\mathrm{user}}$ and R_{TPW}, so that $C_{\mathrm{TPW}}^{\mathrm{user}}$ is determined only by deviations of $T_{\mathrm{TPW}}^{\mathrm{user}}$ from 273.16 K and not by SPRT changes of state. We furthermore define

$$W^{\mathrm{user}}(T_{90}) \equiv \frac{R^{\mathrm{user}}(T_{90})}{R^{\mathrm{user}}_{\mathrm{TPW}}} S^{\mathrm{TPW}}_{T_{90}} = \frac{\Re^{\mathrm{user}}(T_{90})}{\Re^{\mathrm{user}}_{\mathrm{TPW}}} S^{\mathrm{TPW}}_{T_{90}} \qquad 20)$$

to describe the value of *W* that is actually measured by the user. Here, $\Re^{\mathrm{user}}(T_{90})$ and $\Re^{\mathrm{user}}_{\mathrm{TPW}}$ are the resistance ratios relative to the user resistance standard $R^{\mathrm{user}}_{\mathrm{stnd}}$ for $R^{\mathrm{user}}(T_{90})$ and $R^{\mathrm{user}}_{\mathrm{TPW}}$, respectively. Using Eqs. 11, 19, and 20 we obtain

$$W(T_{90}) = W^{\mathrm{user}}(T_{90}) \cdot C^{\mathrm{user}}_{\mathrm{TPW}} . \qquad 21)$$

Equation 21 separates out the resistance-ratio-measurement and TPW-realization parts of *W*, which is useful for separating out the uncertainty components of this quantity.

Differentiating Eq. 21 yields

$$dW = d\left(W^{\mathrm{user}} \cdot C^{\mathrm{user}}_{\mathrm{TPW}}\right) = dW^{\mathrm{user}} + W^{\mathrm{user}} dC^{\mathrm{user}}_{\mathrm{TPW}} , \qquad 22)$$

and differentiating Eq. 15 results in

$$dW_{\mathrm{FP},i} = d\left(\frac{W^{\mathrm{cal}}_{\mathrm{FP},i} \cdot C^{\mathrm{cal}}_{\mathrm{TPW}}}{C^{\mathrm{cal}}_{\mathrm{FP},i}}\right) = dW^{\mathrm{cal}}_{\mathrm{FP},i} + \frac{W^{\mathrm{cal}}_{\mathrm{FP},i}}{C^{\mathrm{cal}}_{\mathrm{FP},i}} dC^{\mathrm{cal}}_{\mathrm{TPW}} - \frac{W^{\mathrm{cal}}_{\mathrm{FP},i} C^{\mathrm{user}}_{\mathrm{TPW}}}{\left(C^{\mathrm{cal}}_{\mathrm{FP},i}\right)^2} dC^{\mathrm{cal}}_{\mathrm{FP},i}$$

$$\cong dW^{\mathrm{cal}}_{\mathrm{FP},i} + W^{\mathrm{cal}}_{\mathrm{FP},i} dC^{\mathrm{cal}}_{\mathrm{TPW}} - W^{\mathrm{cal}}_{\mathrm{FP},i} dC^{\mathrm{cal}}_{\mathrm{FP},i} . \qquad 23)$$

Combining the results of Eqs. 22 and 23 into Eq. 18 provides

$$dT_{90} = \frac{\partial T_{90}}{\partial W_r}\left(dW^{\text{user}} + W^{\text{user}} dC_{\text{TPW}}^{\text{user}} - \sum_{i=1}^{n}\left[\frac{\partial D}{\partial W_{\text{FP},i}}\left(dW_{\text{FP},i}^{\text{cal}} + W_{\text{FP},i}^{\text{cal}} dC_{\text{TPW}}^{\text{cal}} - W_{\text{FP},i}^{\text{cal}} dC_{\text{FP},i}^{\text{cal}} \right)\right] - dN \right). \quad 24)$$

The relevant quantities for $u(T_{90})$ are therefore W^{user}, $C_{\text{TPW}}^{\text{user}}$, $W_{\text{FP},i}^{\text{cal}}$, $C_{\text{TPW}}^{\text{cal}}$, $C_{\text{FP},i}^{\text{cal}}$, and N. These quantities and their uncertainty components are summarized in Table 3. The total uncertainty for T_{90} is then

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_{\text{r}}}\right)^2 \left(u\left(W^{\text{user}}\right)^2 + W^{\text{user}\,2} u\left(C_{\text{TPW}}^{\text{user}}\right)^2 + \left(\sum_{i=1}^{n}\frac{\partial D}{\partial W_{\text{FP},i}} W_{\text{FP},i}^{\text{cal}}\right)^2 u\left(C_{\text{TPW}}^{\text{cal}}\right)^2 \right.$$

$$\left. + \sum_{i=1}^{n}\left[\left(\frac{\partial D}{\partial W_{\text{FP},i}}\right)^2 \left(u\left(W_{\text{FP},i}^{\text{cal}}\right)^2 + W_{\text{FP},i}^{\text{cal}\,2} u\left(C_{\text{FP},i}^{\text{cal}}\right)^2\right)\right] + u(N)^2 \right). \quad 25)$$

Case 2: Measurements performed in calibration lab; an $R_{\mathrm{TPW}}^{\mathrm{cal}}$ measurement for each fixed point

For this arrangement, the temperature is measured in the laboratory where the SPRT was calibrated (using the same TPW cell and resistance measurement equipment), and a measurement of R_{TPW} is made for each fixed point during the calibration. The user still makes measurements of R_{TPW} to minimize uncertainties due to SPRT resistance shifts. For this case, we note that

$$C_{\mathrm{TPW}}^{\mathrm{user}} = C_{\mathrm{TPW}}^{\mathrm{cal}} . \qquad 26)$$

Therefore, Eq. 21 then may be expressed as

$$W(T_{90}) = W^{\mathrm{user}}(T_{90}) \cdot C_{\mathrm{TPW}}^{\mathrm{cal}} . \qquad 27)$$

Differentiating W yields

$$dW = d\left(W^{\mathrm{user}} \cdot C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = dW^{\mathrm{user}} + W^{\mathrm{user}} dC_{\mathrm{TPW}}^{\mathrm{cal}} . \qquad 28)$$

Inserting the results of Eq. 28 and Eq. 23 into Eq. 18 provides

$$dT_{90} = \frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\left[dW^{\mathrm{user}} + \left(W^{\mathrm{user}} - \sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}}W_{\mathrm{FP},i}^{\mathrm{cal}}\right)dC_{\mathrm{TPW}}^{\mathrm{cal}} - \sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}}\left(dW_{\mathrm{FP},i}^{\mathrm{cal}} - W_{\mathrm{FP},i}^{\mathrm{cal}}dC_{\mathrm{FP},i}^{\mathrm{cal}}\right) - dN\right]. \qquad 29)$$

The relevant quantities for $u(T_{90})$ are therefore W^{user}, $W_{\mathrm{FP},i}^{\mathrm{cal}}$, $C_{\mathrm{TPW}}^{\mathrm{cal}}$, $C_{\mathrm{FP},i}^{\mathrm{cal}}$, and N. These quantities and their uncertainty components are summarized in Table 4. The total uncertainty for T_{90} is then

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right)^2\left(u\left(W^{\mathrm{user}}\right)^2 + \left[W^{\mathrm{user}} - \sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}}W_{\mathrm{FP},i}^{\mathrm{cal}}\right]^2 u\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)^2 + \sum_{i=1}^{n}\left[\left(\frac{\partial D}{\partial W_{\mathrm{FP},i}}\right)^2\left(u\left(W_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2 + W_{\mathrm{FP},i}^{\mathrm{cal}\,2}u\left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2\right)\right] + u(N)^2\right). \qquad 30)$$

<u>Case 3: External user determines W using $R_{\mathrm{TPW}}^{\mathrm{rep}}$; an $R_{\mathrm{TPW}}^{\mathrm{cal}}$ measurement for each fixed point</u>

In this arrangement, the user is external to the calibration laboratory and calculates W using a value of R_{TPW} provided in the calibration report. Also, a measurement of R_{TPW} is made for each fixed point during the calibration. The value of W determined by the user is then defined by:

$$W^{\mathrm{user}}(T_{90}) \equiv \frac{R^{\mathrm{user}}(T_{90})}{R_{\mathrm{TPW}}^{\mathrm{rep}}}S_{T_{90}}^{\mathrm{TPW}}, \qquad 31)$$

where $R_{\text{TPW}}^{\text{rep}}$ is the TPW resistance provided in the calibration report and is a weighted mean of $R_{\text{TPW},i}^{\text{cal}}$:

$$R_{\text{TPW}}^{\text{rep}} = \sum_{i=1}^{n} f_i R_{\text{TPW},i}^{\text{cal}} , \quad \sum_{i=1}^{n} f_i = 1 , \tag{32)}$$

where the weighting factor f_i is arbitrarily decided on by the calibration laboratory. For example,

$$f_i = \frac{1}{n} \tag{33)}$$

and

$$\begin{aligned} f_i &= 0 \qquad i \neq n \\ f_i &= 1 \qquad i = n \end{aligned} \tag{34)}$$

(where n is the last fixed point in the calibration sequence) are possible definitions for f_i. Also, $S_{T_{90}}^{\text{TPW}}$ is the change in state of the SPRT defined by Eq. 10. Replacing $R_{\text{TPW},i}^{\text{cal}}$ with $R_{\text{TPW}}^{\text{rep}}$ in Eq. 12, and substituting Eq. 12 in Eq. 10 gives

$$S_{T_{90}}^{\text{TPW}} \equiv R_{\text{TPW}} / \mathcal{R}_{\text{TPW}} = \frac{R_{\text{TPW}}^{\text{rep}}}{C_{\text{TPW}}^{\text{cal}} \mathcal{R}_{\text{TPW}}} . \tag{35)}$$

Then, using the above equation with Eq. 1 yields

$$W = \frac{R^{\text{user}}}{R^{\text{rep}}_{\text{TPW}}} C^{\text{cal}}_{\text{TPW}} S^{\text{TPW}}_{T_{90}} \,. \qquad 36)$$

Using $R^{\text{rep}}_{\text{TPW}}$ for calculating W^{user} is generally not recommended because of additional uncertainty introduced. First, there is an uncertainty due to resistance standards of the user and calibration laboratory. Secondly, the uncertainty in $S^{\text{TPW}}_{T_{90}}$ can be very large and difficult to estimate.

To aid in the handling of correlations between $R^{\text{rep}}_{\text{TPW}}$, $R^{\text{cal}}_{\text{TPW},i}$, and $R^{\text{cal}}_{\text{FP},i}$, we express $R^{\text{rep}}_{\text{TPW}}$ as

$$R^{\text{rep}}_{\text{TPW}} = \Re^{\text{rep}}_{\text{TPW}} R^{\text{cal}}_{\text{stnd}} = \sum_{i=1}^{n} f_i \Re^{\text{cal}}_{\text{TPW},i} R^{\text{cal}}_{\text{stnd}} \,, \qquad 37)$$

where $\Re^{\text{rep}}_{\text{TPW}}$ is the resistance ratio measured between $R^{\text{rep}}_{\text{TPW}}$ and the calibration laboratory resistance standard. Also we insert Eq. 14 into Eq. 15 to obtain

$$W_{\text{FP},i} = \frac{\Re^{\text{cal}}_{\text{FP},i}}{\Re^{\text{cal}}_{\text{TPW},i}} \frac{C^{\text{cal}}_{\text{TPW}}}{C^{\text{cal}}_{\text{FP},i}} S^{\text{TPW}}_{\text{FP},i} \,. \qquad 38)$$

Differentiating W provides

$$dW = \frac{C_{\mathrm{TPW}}^{\mathrm{cal}} S_{T_{90}}^{\mathrm{TPW}}}{R_{\mathrm{TPW}}^{\mathrm{rep}}} dR^{\mathrm{user}} - \frac{R^{\mathrm{user}} C_{\mathrm{TPW}}^{\mathrm{cal}} S_{T_{90}}^{\mathrm{TPW}}}{\left(R_{\mathrm{TPW}}^{\mathrm{rep}}\right)^2} \left[\sum_{i=1}^{n} f_i \left(R_{\mathrm{stnd}}^{\mathrm{cal}} d\Re_{\mathrm{TPW},i}^{\mathrm{cal}} + \Re_{\mathrm{TPW},i}^{\mathrm{cal}} dR_{\mathrm{stnd}}^{\mathrm{cal}} \right) \right]$$
$$+ \frac{R^{\mathrm{user}} S_{T_{90}}^{\mathrm{TPW}}}{R_{\mathrm{TPW}}^{\mathrm{rep}}} dC_{\mathrm{TPW}}^{\mathrm{cal}} + \frac{R^{\mathrm{user}} C_{\mathrm{TPW}}^{\mathrm{cal}}}{R_{\mathrm{TPW}}^{\mathrm{rep}}} dS_{T_{90}}^{\mathrm{TPW}}$$

$$\cong W^{user} \left[\frac{dR^{\mathrm{user}}}{R^{\mathrm{user}}} + dC_{\mathrm{TPW}}^{\mathrm{cal}} + dS_{T_{90}}^{\mathrm{TPW}} - \sum_{i=1}^{n} f_i \left(\frac{d\Re_{\mathrm{TPW},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{rep}}} + \frac{dR_{\mathrm{stnd}}^{\mathrm{cal}}}{R_{\mathrm{stnd}}^{\mathrm{cal}}} \right) \right], \qquad 39)$$

and differentiating $W_{\mathrm{FP},i}$ yields

$$dW_{\mathrm{FP},i} = \frac{C_{\mathrm{TPW}}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}} \frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} - \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}} C_{\mathrm{TPW}}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{\left(\Re_{\mathrm{TPW},i}^{\mathrm{cal}}\right)^2 C_{\mathrm{FP},i}^{\mathrm{cal}}} d\Re_{\mathrm{TPW},i}^{\mathrm{cal}} + \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} \frac{dC_{\mathrm{TPW}}^{\mathrm{cal}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}}$$
$$- \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}} C_{\mathrm{TPW}}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}} \left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2} dC_{\mathrm{FP},i}^{\mathrm{cal}} + \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} \frac{C_{\mathrm{TPW}}^{\mathrm{cal}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}} dS_{\mathrm{FP},i}^{\mathrm{TPW}}$$

$$\cong \frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} - W_{\mathrm{FP},i}^{\mathrm{cal}} \frac{d\Re_{\mathrm{TPW}}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{TPW}}^{\mathrm{cal}} - W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{FP},i}^{\mathrm{cal}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dS_{\mathrm{FP},i}^{\mathrm{TPW}}. \qquad 40)$$

Substituting the above two equations into Eq. 18 results in

$$dT_{90} = \frac{\partial T_{90}}{\partial W_{\mathrm{r}}} \left(W^{\mathrm{user}} \cdot \left[\frac{dR^{\mathrm{user}}}{R^{\mathrm{user}}} + dC_{\mathrm{TPW}}^{\mathrm{cal}} + dS_{T_{90}}^{\mathrm{TPW}} - \sum_{i=1}^{n} f_i \left(\frac{d\Re_{\mathrm{TPW},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{rep}}} + \frac{dR_{\mathrm{stnd}}^{\mathrm{cal}}}{R_{\mathrm{stnd}}^{\mathrm{cal}}} \right) \right] \right.$$
$$\left. - \sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} \left[\frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} - W_{\mathrm{FP},i}^{\mathrm{cal}} \left(\frac{d\Re_{\mathrm{TPW},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} - dC_{\mathrm{TPW}}^{\mathrm{cal}} + dC_{\mathrm{FP},i}^{\mathrm{cal}} - dS_{\mathrm{FP},i}^{\mathrm{TPW}} \right) \right] - dN \right). \qquad 41)$$

Noting from Eq. 32 that

$$\sum_{i=1}^{n} f_i = 1, \tag{42}$$

and regrouping to combine terms with the same differential, Eq. 41 becomes

$$dT_{90} = \frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\left(W^{\mathrm{user}} \cdot \left[\frac{dR^{\mathrm{user}}}{R^{\mathrm{user}}} + dS_{T_{90}}^{\mathrm{TPW}} - \frac{dR_{\mathrm{stnd}}^{\mathrm{cal}}}{R_{\mathrm{stnd}}^{\mathrm{cal}}}\right] + \left[W^{\mathrm{user}} - \sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}}\right] \cdot dC_{\mathrm{TPW}}^{\mathrm{cal}} - dN \right.$$
$$\left. - \left[\sum_{i=1}^{n}\left(W^{\mathrm{user}} f_i - \frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}}\right) \cdot \frac{d\Re_{\mathrm{TPW},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} + \frac{\partial D}{\partial W_{\mathrm{FP},i}} \cdot \left(\frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}}} - W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{FP},i}^{\mathrm{cal}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dS_{\mathrm{FP},i}^{\mathrm{TPW}}\right)\right]\right). \tag{43}$$

The relevant quantities for $u(T_{90})$ are therefore R^{user}, $S_{T_{90}}^{\mathrm{TPW}}$, $C_{\mathrm{TPW}}^{\mathrm{cal}}$, $\Re_{\mathrm{TPW},i}^{\mathrm{cal}}$, $R_{\mathrm{stnd}}^{\mathrm{cal}}$, $\Re_{\mathrm{FP},i}^{\mathrm{cal}}$, $C_{\mathrm{FP},i}^{\mathrm{cal}}$, $S_{\mathrm{FP},i}^{\mathrm{TPW}}$, and N. Note that $R_{\mathrm{TPW}}^{\mathrm{rep}}$ is a function of $R_{\mathrm{TPW},i}^{\mathrm{cal}}$ and is therefore not an independent relevant quantity itself. The relevant quantities and their uncertainty components are summarized in Table 5. The total uncertainty is therefore

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right)^2 \left(W^{\mathrm{user}\,2} \cdot \left[\frac{u(R^{\mathrm{user}})^2}{R^{\mathrm{user}\,2}} + u(S_{T_{90}}^{\mathrm{TPW}})^2 + \frac{u(R_{\mathrm{stnd}}^{\mathrm{cal}})^2}{(R_{\mathrm{stnd}}^{\mathrm{cal}})^2}\right] + \left[W^{\mathrm{user}} - \sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}}\right]^2 \cdot u(C_{\mathrm{TPW}}^{\mathrm{cal}})^2 \right.$$
$$+ \sum_{i=1}^{n}\left[\left(W^{\mathrm{user}} f_i - \frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2 \cdot \frac{u(\Re_{\mathrm{TPW},i}^{\mathrm{cal}})^2}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}\;2}}\right.$$
$$\left.\left. + \left(\frac{\partial D}{\partial W_{\mathrm{FP},i}}\right)^2 \cdot \left(\frac{u(\Re_{\mathrm{FP},i}^{\mathrm{cal}})^2}{\Re_{\mathrm{TPW},i}^{\mathrm{cal}\;2}} + W_{\mathrm{FP},i}^{\mathrm{cal}\;2} u(C_{\mathrm{FP},i}^{\mathrm{cal}})^2 + W_{\mathrm{FP},i}^{\mathrm{cal}\;2} u(S_{\mathrm{FP},i}^{\mathrm{TPW}})^2\right)\right] + u(N)^2\right). \tag{44}$$

It should be noted that if $\Re^{\mathrm{cal}}_{\mathrm{TPW},i}$ is measured immediately after $\Re^{\mathrm{cal}}_{\mathrm{FP},i}$ and care is taken not to knock the SPRT between measurements, the value of $u\left(S^{\mathrm{TPW}}_{\mathrm{FP},i}\right)$ will probably be negligible. This quantity is nevertheless included for the sake of completeness.

Case 4: External User determines W using $R^{\mathrm{user}}_{\mathrm{TPW}}$, one $R^{\mathrm{cal}}_{\mathrm{TPW}}$ value for entire calibration

For this arrangement, the user is external to the calibration laboratory, W is calculated using a value of R_{TPW} measured by the user, and one value of R_{TPW} is used for the entire calibration. The uncertainty for this case is derived similarly to that in Case 1. However, since only one $R^{\mathrm{cal}}_{\mathrm{TPW}}$ (and thus $\Re^{\mathrm{cal}}_{\mathrm{TPW}}$) is used, $dW^{\mathrm{cal}}_{\mathrm{FP},i}$ must be expressed in terms of $\Re^{\mathrm{cal}}_{\mathrm{TPW}}$ and $\Re^{\mathrm{cal}}_{\mathrm{FP},i}$ using Eq. 14 in order that $u(T_{90})$ be shown in terms of uncorrelated relevant quantities. In addition, additional uncertainties $S^{\mathrm{TPW}}_{\mathrm{FP},i}$ for changes in state of the SPRT between measurements of $\Re^{\mathrm{cal}}_{\mathrm{TPW}}$ and $\Re^{\mathrm{cal}}_{\mathrm{FP},i}$ must be included. Then

$$\sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}}dW^{\mathrm{cal}}_{\mathrm{FP},i}=\sum_{i=1}^{n}\frac{\partial D}{\partial W_{\mathrm{FP},i}}\left(\frac{d\Re^{\mathrm{cal}}_{\mathrm{FP},i}}{\Re^{\mathrm{cal}}_{\mathrm{TPW}}}-W^{\mathrm{cal}}_{\mathrm{FP},i}\frac{d\Re^{\mathrm{cal}}_{\mathrm{TPW}}}{\Re^{\mathrm{cal}}_{\mathrm{TPW}}}+W^{\mathrm{cal}}_{\mathrm{FP},i}dS^{\mathrm{TPW}}_{\mathrm{FP},i}\right). \qquad 45)$$

Substituting the rhs of the above equation into Eq. 24 yields

$$dT_{90} = \frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\left(dW^{\mathrm{user}} + W^{\mathrm{user}} dC_{\mathrm{TPW}}^{\mathrm{user}} + \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}}\right] \cdot \left[\frac{d\Re_{\mathrm{TPW}}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} - dC_{\mathrm{TPW}}^{\mathrm{cal}}\right] \right.$$
$$\left. - \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} \left(\frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} - W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{FP},i}^{\mathrm{cal}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dS_{\mathrm{FP},i}^{\mathrm{TPW}} \right)\right] - dN \right) \qquad 46)$$

The relevant quantities for $u(T_{90})$ are therefore W^{user}, $C_{\mathrm{TPW}}^{\mathrm{user}}$, $\Re_{\mathrm{FP},i}^{\mathrm{cal}}$, $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$, $C_{\mathrm{TPW}}^{\mathrm{cal}}$, $C_{\mathrm{FP},i}^{\mathrm{cal}}$, and N. These quantities and their uncertainty components are summarized in Table 6. Note that for this case the uncertainty of $S_{\mathrm{FP},i}^{\mathrm{TPW}}$ is likely to be larger than for case 1, because measurement of the latter is not made immediately afterwards. The total uncertainty for T_{90} is then

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right)^2 \left(u(W^{\mathrm{user}})^2 + W^{\mathrm{user}\,2} u(C_{\mathrm{TPW}}^{\mathrm{user}})^2 + \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}}\right]^2 \cdot \left[\frac{u(\Re_{\mathrm{TPW}}^{\mathrm{cal}})^2}{\Re_{\mathrm{TPW}}^{\mathrm{cal}\,2}} + u(C_{\mathrm{TPW}}^{\mathrm{cal}})^2\right] \right.$$
$$\left. + \left[\sum_{i=1}^{n} \left(\frac{\partial D}{\partial W_{\mathrm{FP},i}}\right)^2 \left(\frac{u(\Re_{\mathrm{FP},i}^{\mathrm{cal}})^2}{\Re_{\mathrm{TPW}}^{\mathrm{cal}\,2}} + W_{\mathrm{FP},i}^{\mathrm{cal}\,2} \left[u(C_{\mathrm{FP},i}^{\mathrm{cal}})^2 + u(S_{\mathrm{FP},i}^{\mathrm{TPW}})^2 \right] \right)\right] + u(N)^2 \right). \qquad 47)$$

<u>Case 5: Measurements performed in calibration lab; one $R_{\mathrm{TPW}}^{\mathrm{cal}}$ value for entire calibration</u>

For this arrangement, the temperature is measured in the laboratory where the SPRT was calibrated (using the same TPW cell and resistance measurement equipment), and one value of R_{TPW} is used for the entire calibration. The uncertainty for this case is derived similarly to that in Case 2. However, since only one $R_{\mathrm{TPW}}^{\mathrm{cal}}$ (and thus $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$) is used, $dW_{\mathrm{FP},i}^{\mathrm{cal}}$ must be expressed in terms of $\Re^{\mathrm{cal}}_{\mathrm{TPW}}$,$\Re^{\mathrm{cal}}_{\mathrm{FP},i}$, and $S^{\mathrm{TPW}}_{\mathrm{FP},i}$ using Eq. 14 in order that $u(T_{90})$ be shown in terms of uncorrelated relevant quantities. Substituting the rhs of Eq. 45 into Eq. 29 provides

$$dT_{90} = \frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\left(dW^{\mathrm{user}} + \left[W^{\mathrm{user}} - \left(\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W^{\mathrm{cal}}_{\mathrm{FP},i} \right) \right] \cdot dC^{\mathrm{cal}}_{\mathrm{TPW}} + \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W^{\mathrm{cal}}_{\mathrm{FP},i} \right] \cdot \frac{d\Re^{\mathrm{cal}}_{\mathrm{TPW}}}{\Re^{\mathrm{cal}}_{\mathrm{TPW}}} \right.$$
$$\left. - \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W^{\mathrm{cal}}_{\mathrm{FP},i}} \left(\frac{d\Re^{\mathrm{cal}}_{\mathrm{FP},i}}{\Re^{\mathrm{cal}}_{\mathrm{TPW}}} - W^{\mathrm{cal}}_{\mathrm{FP},i} dC^{\mathrm{cal}}_{\mathrm{FP},i} + W^{\mathrm{cal}}_{\mathrm{FP},i} dS^{\mathrm{TPW}}_{\mathrm{FP},i} \right) \right] - dN \right). \qquad 48)$$

The relevant quantities for $u(T_{90})$ are therefore W^{user}, $\Re^{\mathrm{cal}}_{\mathrm{FP},i}$, $\Re^{\mathrm{cal}}_{\mathrm{TPW}}$, $C^{\mathrm{cal}}_{\mathrm{TPW}}$, $C^{\mathrm{cal}}_{\mathrm{FP},i}$, $S^{\mathrm{TPW}}_{\mathrm{FP},i}$, and N. These quantities and their uncertainty components are summarized in Table 7. Note that for this case the uncertainty of $S^{\mathrm{TPW}}_{\mathrm{FP},i}$ is likely to be larger than for Case 2, because measurement of the latter is not made immediately afterwards. The total uncertainty for T_{90} is then

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_{\mathrm{r}}} \right)^2 \left(u\left(W^{\mathrm{user}}\right)^2 + \left[W^{\mathrm{user}} - \sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W^{\mathrm{cal}}_{\mathrm{FP},i} \right]^2 \cdot u\left(C^{\mathrm{cal}}_{\mathrm{TPW}}\right)^2 \right.$$
$$+ \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W^{\mathrm{cal}}_{\mathrm{FP},i} \right]^2 \cdot \frac{u\left(\Re^{\mathrm{cal}}_{\mathrm{TPW}}\right)^2}{\Re^{\mathrm{cal}\,2}_{\mathrm{TPW}}}$$
$$\left. + \left[\sum_{i=1}^{n} \left(\frac{\partial D}{\partial W_{\mathrm{FP},i}} \right)^2 \cdot \left(\frac{u\left(\Re^{\mathrm{cal}}_{\mathrm{FP},i}\right)^2}{\Re^{\mathrm{cal}\,2}_{\mathrm{TPW}}} + W^{\mathrm{cal}\,2}_{\mathrm{FP},i} \left[u\left(C^{\mathrm{cal}}_{\mathrm{FP},i}\right)^2 + u\left(S^{\mathrm{TPW}}_{\mathrm{FP},i}\right)^2 \right] \right) \right] + u(N)^2 \right). \qquad 49)$$

Case 6: External user determines W using $R_{\mathrm{TPW}}^{\mathrm{rep}}$; one $R_{\mathrm{TPW}}^{\mathrm{cal}}$ value for entire calibration

In this arrangement, the user is external to the calibration laboratory and calculates W using a value of R_{TPW} provided in the calibration report. Also, one value of R_{TPW} is used for the entire calibration. This case is similar to Case 3, so the value of W determined by the user is defined by Eq. 33. However, here $R_{\mathrm{TPW}}^{\mathrm{rep}}$ is defined by

$$R_{\mathrm{TPW}}^{\mathrm{rep}} = R_{\mathrm{TPW}}^{\mathrm{cal}} \qquad 50)$$

As with Case 3, this arrangement is not recommended because of the large contribution to the uncertainty due to the change of state of the SPRT.

The expression for dW is given by Eq. 39. Since only one $R_{\mathrm{TPW}}^{\mathrm{cal}}$ (and thus $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$) is used,

$$W_{\mathrm{FP},i} = \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} \frac{C_{\mathrm{TPW}}^{\mathrm{cal}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}} S_{\mathrm{FP},i}^{\mathrm{TPW}} . \qquad 51)$$

Differentiating the above equation gives

$$dW_{\mathrm{FP},i} = \frac{C_{\mathrm{TPW}}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}} \frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} - \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}} C_{\mathrm{TPW}}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)^2 C_{\mathrm{FP},i}^{\mathrm{cal}}} d\Re_{\mathrm{TPW}}^{\mathrm{cal}} + \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} \frac{dC_{\mathrm{TPW}}^{\mathrm{cal}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}}$$

$$- \frac{C_{\mathrm{TPW}}^{\mathrm{cal}} \Re_{\mathrm{FP},i}^{\mathrm{cal}} S_{\mathrm{FP},i}^{\mathrm{TPW}}}{\left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2 \Re_{\mathrm{TPW}}^{\mathrm{cal}}} dC_{\mathrm{FP},i}^{\mathrm{cal}} + \frac{\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} \frac{C_{\mathrm{TPW}}^{\mathrm{cal}}}{C_{\mathrm{FP},i}^{\mathrm{cal}}} dS_{\mathrm{FP},i}^{\mathrm{TPW}}$$

$$\cong \frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} - W_{\mathrm{FP},i}^{\mathrm{cal}} \frac{d\Re_{\mathrm{TPW}}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{TPW}}^{\mathrm{cal}} - W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{FP},i}^{\mathrm{cal}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dS_{\mathrm{FP},i}^{\mathrm{TPW}} . \qquad 52)$$

Substituting the rhs of Eq. 39 and Eq. 52 into Eq. 18 provides

$$dT_{90} = \frac{\partial T_{90}}{\partial W_{\mathrm{r}}} \left(W^{user} \cdot \left[\frac{dR^{\mathrm{user}}}{R^{\mathrm{user}}} + dS_{T_{90}}^{\mathrm{TPW}} - \frac{dR_{\mathrm{stnd}}^{\mathrm{cal}}}{R_{\mathrm{stnd}}^{\mathrm{cal}}} \right] \right.$$

$$+ \left[W^{\mathrm{user}} - \sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}} \right] \cdot \left[dC_{\mathrm{TPW}}^{\mathrm{cal}} - \frac{d\Re_{\mathrm{TPW}}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} \right]$$

$$\left. - \left[\sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} \left(\frac{d\Re_{\mathrm{FP},i}^{\mathrm{cal}}}{\Re_{\mathrm{TPW}}^{\mathrm{cal}}} - W_{\mathrm{FP},i}^{\mathrm{cal}} dC_{\mathrm{FP},i}^{\mathrm{cal}} + W_{\mathrm{FP},i}^{\mathrm{cal}} dS_{\mathrm{FP},i}^{\mathrm{TPW}} \right) \right] - dN \right). \qquad 53)$$

The relevant quantities for $u(T_{90})$ are therefore R^{user}, $S_{\mathrm{use}}^{\mathrm{cal}}$, $C_{\mathrm{TPW}}^{\mathrm{cal}}$, $\Re_{\mathrm{TPW},i}^{\mathrm{cal}}$, $R_{\mathrm{stnd}}^{\mathrm{cal}}$, $\Re_{\mathrm{FP},i}^{\mathrm{cal}}$, $S_{\mathrm{FP},i}^{\mathrm{TPW}}$, $C_{\mathrm{FP},i}^{\mathrm{cal}}$, and N. The relevant quantities and their uncertainty components are summarized in Table 8. Note that for this case the uncertainty of $\Re_{\mathrm{FP},i}^{\mathrm{cal}}$ due to SPRT change-of-state between measurement of $\Re_{\mathrm{FP},i}^{\mathrm{cal}}$ and $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$ is likely to be larger than for case 3, because measurement of the latter is not made immediately afterwards. The total uncertainty is then given by

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_\mathrm{r}}\right)^2 \left(W^{\mathrm{user}\,2} \cdot \left[\frac{u(R^{\mathrm{user}})^2}{R^{\mathrm{user}\,2}} + u\left(S_{T_{90}}^{\mathrm{TPW}}\right)^2 + \frac{u\left(R_{\mathrm{stnd}}^{\mathrm{cal}}\right)^2}{R_{\mathrm{stnd}}^{\mathrm{cal}\;2}} \right] \right.$$
$$+ \left[W^{\mathrm{user}} - \sum_{i=1}^{n} \frac{\partial D}{\partial W_{\mathrm{FP},i}} W_{\mathrm{FP},i}^{\mathrm{cal}} \right]^2 \cdot \left[u\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)^2 + \frac{u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)^2}{\Re_{\mathrm{TPW}}^{\mathrm{cal}\;2}} \right]$$
$$\left. + \left[\sum_{i=1}^{n} \left(\frac{\partial D}{\partial W_{\mathrm{FP},i}}\right)^2 \cdot \left(\frac{u\left(\Re_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2}{\Re_{\mathrm{TPW}}^{\mathrm{cal}\;2}} + W_{\mathrm{FP},i}^{\mathrm{cal}\;2} \left[u\left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2 + u\left(S_{\mathrm{FP},i}^{\mathrm{TPW}}\right)^2 \right] \right) \right] + u(N)^2 \right). \qquad 54)$$

Approximations to the GUM Uncertainty for Cases 1, 3, 4, and 6

An approximation has been frequently used for simplifying the GUM uncertainties for Cases 1 and 4 (Eq. 25 and Eq. 47, respectively). For both cases, this approximation assumes uncorrelated values of $C_{\mathrm{TPW}}^{\mathrm{cal}}$ when it is used for calculating the different $W_{\mathrm{FP},i}^{\mathrm{cal}}$, as if a different TPW cell were used for each TPW realization during the calibration. Therefore, $dC_{\mathrm{TPW}}^{\mathrm{cal}}$ is replaced by $dC_{\mathrm{TPW},i}^{\mathrm{cal}}$ in Eq. 24 and Eq. 46. For Case 4, the approximation also assumes uncorrelated values for $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$ when it is used for calculating the different $W_{\mathrm{FP},i}^{\mathrm{cal}}$, and so $d\Re_{\mathrm{TPW}}^{\mathrm{cal}}$ is replaced by $d\Re_{\mathrm{TPW},i}^{\mathrm{cal}}$. The latter is then combined with $d\Re_{\mathrm{FP},i}^{\mathrm{cal}}$ to make $dW_{\mathrm{FP},i}^{\mathrm{cal}}$ using Eq. 45, making this case identical to Case 1. With these assumptions, Eq. 25 and Eq. 47 are changed to

$$u(T_{90})^2 = \left(\frac{\partial T_{90}}{\partial W_\mathrm{r}}\right)^2 \left(u\left(W^{\mathrm{user}}\right)^2 + W^{\mathrm{user}\,2} u\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)^2 + \sum_{i=1}^{n} \left[\left(\frac{\partial D}{\partial W_{\mathrm{FP},i}}\right)^2 u\left(W_{\mathrm{FP},i}\right)^2 \right] + u(N)^2 \right) \qquad 55)$$

where

$$u\left(W_{\mathrm{FP},i}\right)^2 = u\left(W_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2 + W_{\mathrm{FP},i}^{\mathrm{cal}\,2}u\left(C_{\mathrm{TPW},i}^{\mathrm{cal}}\right)^2 + W_{\mathrm{FP},i}^{\mathrm{cal}\,2}u\left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2 \qquad 56)$$

The above approximation may be extended to apply to Case 3 and Case 6 to provide the uncertainty using simpler equations than Eq. 44 and Eq. 54. For these cases the uncertainty from the change of SPRT change of state must still be included, since it is often a relatively large uncertainty. Also, the uncertainty contributed by the user is $u(R^{\mathrm{user}})$ (as in Eq. 44) rather than the combination of $u(W^{\mathrm{user}})$ and $C_{\mathrm{TPW}}^{\mathrm{user}}$. These modifications to Eq. 55 provide

$$u\left(T_{90}\right)^2 = \left(\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right)^2 \left(W^{\mathrm{user}\,2} \cdot \left[\frac{u\left(R^{\mathrm{user}}\right)^2}{R^{\mathrm{user}\,2}} + u\left(S_{T_{90}}^{\mathrm{TPW}}\right)^2 \right] + \sum_{i=1}^{n} \left[\left(\frac{\partial D}{\partial W_{\mathrm{FP},i}} \right)^2 u\left(W_{\mathrm{FP},i}\right)^2 \right] + u(N)^2 \right) \qquad 57)$$

where $u(W_{\mathrm{FP},i})$ is provided by Eq. 27. For the remainder of the paper, we will refer to Eq. 55 as Approximation 1 and to Eq. 57 as Approximation 2.

Calculation of the Uncertainties

The total uncertainty as provided for Cases 1-6 involve calculations of $\partial D/\partial W_{\mathrm{FP},i}$. These calculations may be performed using numerical analysis for determining the coefficients in the appropriate equation for D in Eqs. 5-7. [7] The calculations may also be performed using algebraic expressions derived by White for each of the 11 ITS-90 subranges. [8] When using the algebraic expressions of [8], it should be noted that $\partial D/\partial W_{\mathrm{FP},i}$ is equivalent to the function F_{i+1}

used in that work. In addition, the function $F_1 \equiv F_{H_2O}$ from [8] is not directly used in the uncertainty equations presented here, since we have used a mathematical identity (equivalent to Eq. 20 in [8]) relating F_1 to the other F_i values.

5. Sample Calculations

Shown in Figure 1 is the total standard uncertainty $u(T_{90})$ for SPRT measurement in subrange 6 (0 °C to 961.78 °C) using the calculations for a) Case 1 (Eq. 25), b) Case 2 (Eq. 30), and c) Case 3 (Eq. 44). Plots of the uncertainties for Cases 4, 5, and 6 are not shown because the results are virtually identical to those of 1, 2, and 3, respectively, for the parameters used. The coefficients for D were determined using numerical analysis with LU decomposition [7]. The total standard uncertainty is represented by a thick black curve. The contributions from the individual components are represented by colored curves. The values and legend captions for these components are given in Table 9, with one exception: for Case 3 the curves representing the contributions from the Sn, Zn, Al and Ag fixed-point uncertainties are the combined uncertainties for $\Re^{\text{cal}}_{\text{FP},i}$ and $C^{\text{cal}}_{\text{FP},i}$ rather than the combined uncertainties for $W^{\text{cal}}_{\text{FP},i}$ and $C^{\text{cal}}_{\text{FP},i}$. In Table 9, the combined uncertainties of $W^{\text{cal}}_{\text{FP},i}$ and $C^{\text{cal}}_{\text{FP},i}$ are quantified in units of temperature in the following manner. The uncertainty

$$u\left(W_{\mathrm{FP},i}^{\mathrm{cal}},C_{\mathrm{FP},i}^{\mathrm{cal}}\right)\equiv\left[u\left(W_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2+W_{\mathrm{FP},i}^{\mathrm{cal}\;2}u\left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2\right]^{1/2}$$

$$=\left[\frac{u\left(\Re_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2}{\Re_{\mathrm{TPW}}^{\mathrm{cal}\;2}}+W_{\mathrm{FP},i}^{\mathrm{cal}\;2}\frac{u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)^2}{\Re_{\mathrm{TPW}}^{\mathrm{cal}\;2}}+W_{\mathrm{FP},i}^{\mathrm{cal}\;2}u\left(C_{\mathrm{FP},i}^{\mathrm{cal}}\right)^2\right]^{1/2} \qquad 58)$$

is defined as the combined uncertainty in dimensionless units and

$$u_T\left(W_{\mathrm{FP},i}^{\mathrm{cal}},C_{\mathrm{FP},i}^{\mathrm{cal}}\right)\equiv u\left(W_{\mathrm{FP},i}^{\mathrm{cal}},C_{\mathrm{FP},i}^{\mathrm{cal}}\right)\cdot\left.\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right|_{T_{90}=T_{\mathrm{FP},i}} \qquad 59)$$

is defined as this uncertainty in units of temperature. Likewise,

$$u_T\left(C_{TPW}^{\mathrm{cal}}\right)\equiv u\left(C_{TPW}^{\mathrm{cal}}\right)\cdot\left.\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right|_{T_{90}=273.16K} \qquad 60)$$

and

$$u_T\left(C_{TPW}^{\mathrm{user}}\right)\equiv u\left(C_{TPW}^{\mathrm{user}}\right)\cdot\left.\frac{\partial T_{90}}{\partial W_{\mathrm{r}}}\right|_{T_{90}=273.16K} \qquad 61)$$

are defined as the TPW cell uncertainties in units of temperature.

So that the sample calculations represent typical experimental conditions, the values of $u_T\left(W_{\mathrm{FP},i}^{\mathrm{cal}},C_{\mathrm{FP},i}^{\mathrm{cal}}\right)$ listed in Table 9 are the median standard realization uncertainty values provided by those national standards laboratories participating in CIPM or CCT key comparisons [9]. For simplicity the non-uniqueness uncertainty $u(N)$ is assumed to be zero. The values of $u\left(W^{\text{user}}\right)$ and $u\left(\Re_{\text{TPW}}^{\text{cal}}\right)$ are both assumed to be 5×10^{-8}, a value based upon recent studies of resistance bridge uncertainties [10,11]. For Cases 1, 2, 4, and 5, the uncertainty $u\left(S_{T_{90}}^{\text{TPW}}\right)$ is assumed to be zero; in these cases it is assumed that the TPW resistance measurement is made immediately after the T_{90} resistance measurement is made, so the SPRT change of state is minimal. For Cases 3 and 6, where long periods of time may elapse between the TPW and T_{90} measurements, $u\left(S_{T_{90}}^{\text{TPW}}\right)$ is estimated to be 1.0 μΩ/Ω, a value consistent with studies made of SPRT stability [12]. The uncertainties $u\left(R^{\text{user}}\right)$ and $u\left(R_{\text{stnd}}^{\text{cal}}\right)$ are both 0.25 μΩ/Ω, a value based upon the stability of current resistance standards. Finally, for plot c), the values for the f_i in Eq. 44 have been arbitrarily chosen to be $f_1 = f_2 = f_3 = 0$ and $f_4 = 1$.

Figure 2 compares the calculations of Cases 1−6. All uncertainty component values except $u_T\left(C_{\text{TPW}}^{\text{user}}\right)$ and $u_T\left(C_{\text{TPW}}^{\text{cal}}\right)$ are given in Table 9. The values for these two uncertainty components are equivalent and are a) 0.06 mK, b) 0.15 mK, and c) 0.25 mK. The values of a) and b) are the median and highest values, respectively, for those standards laboratories involved with key comparisons. The value of c) is large but well within the TPW uncertainty claimed by many national laboratories.

The results of Fig. 2a show that if $u_T\left(C_{\text{TPW}}^{\text{user}}\right)<0.06$ mK and $u_T\left(C_{\text{TPW}}^{\text{cal}}\right)<0.06$ mK, the total uncertainty for Cases 1, 2, 4, and 5 yield nearly identical results. Cases 3 and 6, where the user calculates W with the calibration-report TPW resistance, give the largest uncertainty; this is due to the large contribution from $u\left(S_{T_{90}}^{\text{TPW}}\right)$. From b) and c), it is evident that when $u_T\left(C_{\text{TPW}}^{\text{user}}\right)$ and

$u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ are significant compared to the other fixed-point uncertainties, there is a noticeable difference between Cases 1 and 2 and between Cases 4 and 5. Cases 2 and 5 give the smallest T_{90} uncertainty; this is due to the correlation of the input values for the TPW temperature, since the user TPW cell is the same as that used during the SPRT calibration. In all plots of this figure, it can be seen that the uncertainties for Cases 1, 2, and 3 are virtually identical to those of Cases 4, 5, and 6, respectively, showing that the correlation of the input values for $\Re_{\mathrm{TPW}}^{\mathrm{cal}}$ does not have much influence on the total uncertainty for the value of $u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ used in this figure.

The differences between the results of the six different cases can be seen for larger values of $u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)$, as shown in Figure 3. Here, $u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 4 \times 10^{-7}$, which corresponds to a value of 0.1 mK and is nearly ten times larger than the value used for Figures 1-2. All other uncertainty components are as listed in Table 9. Even with such a large value for $u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)$, the difference between the total uncertainty for Case 1 and 4, for Case 2 and 5, and for Case 3 and 6 are usually less than 0.1 mK.

Figure 4 shows the differences between the values of $u(T_{90})$ given by Approximation 1 and those from Case 1 and Case 4. The figure also shows the differences between the values of $u(T_{90})$ given by Approximation 2 and those from Case 3 and Case 6. The parameter combinations are the same as those in Fig. 2. In a), where $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 0.06$ mK, the figure shows that the differences are all within 10%. In b), where $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 0.15$ mK, the differences are within 20% and for most of the temperature range they are within 10%. Finally, in c), where $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 0.25$ mK, the differences can be as large as 40% and they are clearly larger for Approximation 2 than for Approximation 1. These results show that for those standards laboratories with TPW and fixed-point uncertainties comparable to the key comparison median values, the approximations should be quite satisfactory for estimating total uncertainties for SPRT realizations.

6. Summary

We have derived the uncertainties for SPRT realization of the ITS-90 between fixed points for six different cases of SPRT calibration/use. These cases are based on three factors: 1) whether the value of R_{TPW} used for determining W is measured by the user or taken from the calibration report, 2) whether the SPRT is employed by an external user or by an internal user employing the same TPW cell and resistance-measurement equipment used in the calibration, and 3) whether, during the calibration, a measurement of R_{TPW} was made for each determination of $W_{\mathrm{FP},i}$ or if only one value of R_{TPW} was used for all determinations of $W_{\mathrm{FP},i}$. The results for all cases were compared to each other and to an equation that has been frequently used to approximate the GUM uncertainty for T_{90}. The comparisons found that a) $u(T_{90})$ is a few tenths of a millikelvin larger when the value of R_{TPW} is taken from the calibration report, b) for those cases where R_{TPW} is user-determined, the values of $u(T_{90})$ are nearly identical when $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$ and $u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ are less than 0.06 mK, and c) the correlation of the R_{TPW} values used for determining the $W_{\mathrm{FP},i}$ values in the calibration has very little influence on $u(T_{90})$. The comparisons also showed that for those cases where R_{TPW} is user-determined, the approximation gives a value of $u(T_{90})$ that is nearly equal to that of the exact calculation when $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$ and $u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ are less than 0.06 mK.

The software tools used in this paper for calculating the SPRT uncertainty using the equations derived above are capable of calculating the total uncertainty for all values of uncertainty components. These software tools are available to the user community by sending a request via email to christopher.meyer@nist.gov.

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Table Captions

1. Subranges for the SPRT definition of the ITS-90, displaying their calibration fixed points FP,*i*. Here, "Subrange" refers to the subrange number and TP, MP, and FrP refer to triple point, melting point, and freezing point, respectively.

2. Six cases affecting the equation for calculating SPRT uncertainties between fixed-points.

3. Relevant quantities for determining the uncertainty for T_{90}, for the case where the user is external to the calibration lab and determines W with a user-obtained measurement of R_{TPW}.

4. Relevant quantities for determining the uncertainty for T_{90}, for the case where T_{90} is measured by the calibration laboratory, using the same TPW cell and resistance equipment as that used in calibration.

5. Relevant quantities for determining the uncertainty for T_{90}, for the case where the user is external to the calibration laboratory and determines W using R_{TPW} from calibration report.

6. Relevant quantities for determining the uncertainty for T_{90}, for the case where the user is external to calibration lab and determines W with a user-obtained measurement of R_{TPW}.

7. Relevant quantities for determining the uncertainty for T_{90}, for the case where T_{90} is measured by the calibration laboratory, using the same TPW cell and resistance equipment as that used in calibration.

8. Relevant quantities for determining the uncertainty for T_{90}, for the case where the user is external to the calibration laboratory and determines W using R_{TPW} from calibration report.

9. Values and legend captions for the uncertainty components used for the plots in Figures 1-3 (unless specified otherwise) a) Fixed point and resistance-measurement uncertainties. b) SPRT change-of-state uncertainties. The reader is referred to Eqs. 58-61 for definitions of some of the uncertainties listed in the table. The reader is also referred to Tables 3-8 for determining which of the uncertainties in (a) is relevant for each case.

Figure Captions

1. Standard SPRT uncertainties for subrange 6 (0 °C to 961.78 °C) for a) Case 1, b) Case 2, and c) Case 3. The results for Cases 4, 5, and 6 are not shown because they are virtually identical to those of Case 1, 2 and 3, respectively, for the parameters used. Uncertainties propagated from individual uncertainty components (whose values are listed in Table 9) are shown as well as the total uncertainty (see text for explanation of legend labels). In a) the curves representing the propagation of the calibration-lab and user TPW uncertainties are identical above 200 °C. In c) the curves representing the propagation of the calibration-lab resistance standard and the user resistance-measurement system are identical. The total uncertainty in c) is considerably larger than in a) and b) due to the large uncertainty component from SPRT change of state.

2. Effect of TPW realization uncertainties $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$ and $u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ on total SPRT uncertainties for subrange 6 (0 °C to 961.78 °C) for the six cases described in this paper. For these plots, the results of Case 1 and Case 4 are identical, as is the case with the results of Case 2 and Case 5 and the results of Case 3 and Case 6. The values of $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$ and $u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ are listed in the plots, and the uncertainties of the Sn, Zn, Al, and Ag fixed points and other parameters are listed in Table 9. Note that $u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$ is not used to calculate the total uncertainty for Cases 3 and 6.

3. Total SPRT uncertainties for subrange 6 (0 °C to 961.78 °C) for the six cases described in this paper for $u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ = 4.0 $\times10^{-7}$, which corresponds to 0.1 mK in temperature units. All other uncertainty components are listed in Table 9.

4. Comparison of Approximation 1 (Eq. 55) with the exact calculations of the GUM uncertainties for Cases 1 and 4 (which are identical in this figure), and comparison of Approximation 2 (Eq. 57) with the exact calculations of the GUM uncertainties for Cases 3 and 6 (which are identical here). The calculations are made for subrange 6 (0 °C to 961.78 °C). The parameter combinations are provided in Table 9 and are the same as in Fig. 2. Note that $u\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$ is not used to calculate the total uncertainty for Cases 3 and 6 and for Approximation 2.

Table 1

Subrange	Temperature Range	Fixed Points						
		***i*=1**	***i*=2**	***i*=3**	***i*=4**	***i*=5**	***i*=6**	***i*=7**
1	13.8033 K to 273.16 K	e-H_2 TP	e-H_2 VP1	e-H_2 VP2	Ne TP	O_2 TP	Ar TP	Hg TP
2	24.5561 K to 273.16 K	e-H_2 TP	Ne TP	O_2 TP	Ar TP	Hg TP		
3	54.3584 K to 273.16 K	O_2 TP	Ar TP	Hg TP				
4	83.8058 K to 273.16 K	Ar TP	Hg TP					
5	−38.8344 °C to 29.7646 °C	Hg TP	Ga MP					
6	0 °C to 961.78 °C	Sn FrP	Zn FrP	Al FrP	Ag FrP			
7	0 °C to 660.323 °C	Sn FrP	Zn FrP	Al FrP				
8	0 °C to 419.527 °C	Sn FrP	Zn FrP					
9	0 °C to 231.928 °C	In FrP	Sn FrP					
10	0 °C to 156.5985 °C	In FrP						
11	0 °C to 29.7646 °C	Ga MP						

Table 2

Case	**R_{TPW} in Calculation of W**	**SPRT Use**	**One R_{TPW} value determined for:**
1	User-determined	External	Each fixed point
2	User-determined	Internal	Each fixed point
3	From calibration report	External	Each fixed point
4	User-determined	External	Entire calibration
5	User-determined	Internal	Entire calibration
6	From calibration report	External	Entire calibration

Table 3

Relevant Quantity	Mathematical Definition	Description of Elements
W^{user}	$\dfrac{\mathfrak{R}^{\text{user}}(T_{90})}{\mathfrak{R}^{\text{user}}_{\text{TPW}}} S^{\text{TPW}}_{T_{90}}$	Resistance bridge measurement by user at T_{90} Resistance bridge measurement in user TPW cell Change in physical state of SPRT between above measurements
$C^{\text{user}}_{\text{TPW}}$	$R^{\text{user}}_{\text{TPW}} / R_{\text{TPW}}$	User realization of TPW (excluding resistance measurements)
$W^{\text{cal}}_{\text{FP},i}$	$\dfrac{\mathfrak{R}^{\text{cal}}_{\text{FP},i}}{\mathfrak{R}^{\text{cal}}_{\text{TPW},i}} S^{\text{TPW}}_{\text{FP},i}$	Resistance bridge measurement in calib. lab at i^{th} fixed-point realization Resistance bridge measurement in calib. lab TPW cell for i^{th} fixed-point real. Change in physical state of SPRT between two resistance-bridge measurements Change in physical state of SPRT between calibration and meas. of W^{user}
$C^{\text{cal}}_{\text{TPW}}$	$R^{\text{cal}}_{\text{TPW}} / R_{\text{TPW}}$	Calibration lab realization of TPW (excluding resistance measurements)
$C^{\text{cal}}_{\text{FP},i}$	$R^{\text{cal}}_{\text{FP},i} / R_{\text{FP},i}$	Calibration lab realization of i^{th} fixed point (excluding resistance measurements)
N		ITS-90 Non-uniqueness

Table 4

Relevant Quantity	Mathematical Definition	Description of Elements
W^{user}	$\dfrac{\Re^{\text{user}}(T_{90})}{\Re^{\text{user}}_{\text{TPW}}} S^{\text{TPW}}_{T_{90}}$	Resistance bridge measurement by user at T_{90} Resistance bridge measurement in calib. lab TPW cell after T_{90} measurement Change in physical state of SPRT between above measurements
$W^{\text{cal}}_{\text{FP},i}$	$\dfrac{\Re^{\text{cal}}_{\text{FP},i}}{\Re^{\text{cal}}_{\text{TPW},i}} S^{\text{TPW}}_{\text{FP},i}$	Resistance bridge measurement in calib. lab at i^{th} fixed-point realization Resistance bridge measurement in calib. lab TPW cell after i^{th} fixed-point real. Change in physical state of SPRT between two resistance-bridge measurements Change in physical state of SPRT between calibration and meas. of W^{user}
$C^{\text{cal}}_{\text{TPW}}$	$R^{\text{cal}}_{\text{TPW}} / R_{\text{TPW}}$	Calibration lab realization of TPW (excluding resistance measurements)
$C^{\text{cal}}_{\text{FP},i}$	$R^{\text{cal}}_{\text{FP},i} / R_{\text{FP},i}$	Calibration lab realization of i^{th} fixed point (excluding resistance measurements)
N		ITS-90 Non-uniqueness

Table 5

Relevant Quantity	Mathematical Definition	Description of elements
R^{user}	Resistance measured by user at T_{90}	Resistance ratio measurement by user at T_{90} User resistance standard
$S_{T_{90}}^{\text{TPW}}$	$R_{\text{TPW}} / R_{\text{TPW}}$	Change in physical state of SPRT between $R_{\text{TPW}}^{\text{rep}}$ and R^{user} measurements
$C_{\text{TPW}}^{\text{cal}}$	$R_{\text{TPW},i}^{\text{cal}} / R_{\text{TPW}}$	Calibration lab realization of TPW (excluding resistance measurements)
$\Re_{\text{TPW},i}^{\text{cal}}$	$R_{\text{TPW},i}^{\text{cal}} \big/ R_{\text{stnd}}$	Resistance ratio meas. in calibration lab TPW cell after i^{th} fixed-point cell
$R_{\text{stnd}}^{\text{cal}}$	Resistance standard	Calibration lab resistance standard
$\Re_{\text{FP},i}^{\text{cal}}$	$R_{\text{FP},i}^{\text{cal}} \big/ R_{\text{stnd}}$	Resistance ratio meas. in i^{th} calibration lab fixed-point cell Change in physical state of SPRT between calibration and meas. of W^{user}
$C_{\text{FP},i}^{\text{cal}}$	$R_{\text{FP},i}^{\text{cal}} / R_{\text{FP},i}$	Calibration lab realization of i^{th} fixed point (excluding resistance measurements)
$S_{\text{FP},i}^{\text{TPW}}$	$R_{\text{TPW}} / R_{\text{TPW}}^{'}$	Change in phys. state of SPRT between measurements of $\Re_{\text{FP},i}^{\text{cal}}$ and $\Re_{\text{TPW}}^{\text{cal}}$
N		ITS-90 Non-uniqueness

Table 6

Relevant Quantity	**Mathematical Definition**	**Description of Elements**
W^{user}	$\dfrac{\Re^{\text{user}}(T_{90})}{\Re_{\text{TPW}}^{\text{user}}} S_{T_{90}}^{\text{TPW}}$	Resistance bridge measurement by user at T_{90} Resistance bridge measurement in user TPW cell Change in physical state of SPRT between above measurements
$C_{\text{TPW}}^{\text{user}}$	$R_{\text{TPW}}^{\text{user}} / R_{\text{TPW}}$	Correction factor for user TPW realization (excluding resistance measurements)
$\Re_{\text{TPW}}^{\text{cal}}$	$R_{\text{TPW}}^{\text{cal}} / R_{\text{stnd}}^{\text{cal}}$	Resistance bridge measurement for calibration lab TPW realization
$\Re_{\text{FP},i}^{\text{cal}}$	$R_{\text{FP},i}^{\text{cal}} / R_{\text{stnd}}^{\text{cal}}$	Resistance bridge measurement in calib. lab at i^{th} fixed-point realization Change in physical state of SPRT between calibration and meas. of W^{user}
$S_{\text{FP},i}^{\text{TPW}}$	$R_{\text{TPW}} / R_{\text{TPW}}^{'}$	Change in physical state of SPRT between measurements of $\Re_{\text{FP},i}^{\text{cal}}$ and $\Re_{\text{TPW}}^{\text{cal}}$
$C_{\text{TPW}}^{\text{cal}}$	$R_{\text{TPW}}^{\text{cal}} / R_{\text{TPW}}$	Correction factor for calibration lab TPW realization (excluding resistance measurements)
$C_{\text{FP},i}^{\text{cal}}$	$R_{\text{FP},i}^{\text{cal}} / R_{\text{FP},i}$	Correction for calibration lab realization of i^{th} fixed point (excluding resistance measurements)
N		ITS-90 Non-uniqueness

Table 7

Relevant Quantity	Mathematical Definition	Description of Elements
W^{user}	$\Re^{\text{user}}(T_{90}) / \Re^{\text{user}}_{\text{TPW}}$	Resistance bridge measurement by user at T_{90} Resistance bridge measurement in calib. lab TPW cell after T_{90} measurement Change in physical state of SPRT between above measurements
$\Re^{\text{cal}}_{\text{TPW}}$	$R^{\text{cal}}_{\text{TPW}} \big/ R^{\text{cal}}_{\text{stnd}}$	Resistance bridge measurement for calibration lab TPW realization
$\Re^{\text{cal}}_{\text{FP},i}$	$R^{\text{cal}}_{\text{FP},i} \big/ R^{\text{cal}}_{\text{stnd}}$	Resistance bridge measurement in calib. lab at i^{th} fixed-point realization Change in physical state of SPRT between calibration and meas. of W^{user}
$S^{\text{TPW}}_{\text{FP},i}$	$R_{\text{TPW}} / R'_{\text{TPW}}$	Change in physical state of SPRT between measurements of $\Re^{\text{cal}}_{\text{FP},i}$ and $\Re^{\text{cal}}_{\text{TPW}}$
$C^{\text{cal}}_{\text{TPW}}$	$R^{\text{cal}}_{\text{TPW}} / R_{\text{TPW}}$	Calibration lab realization of TPW (excluding resistance measurements)
$C^{\text{cal}}_{\text{FP},i}$	$R^{\text{cal}}_{\text{FP},i} / R_{\text{FP},i}$	Calibration lab realization of i^{th} fixed point (excluding resistance measurements)
N		ITS-90 Non-uniqueness

Table 8

Relevant Quantity	Definition	Description of elements
R^{user}	User-measured resistance at T_{90}	Resistance ratio measurement by user at T_{90} User resistance standard
$S_{T_{90}}^{\text{TPW}}$	$R_{\text{TPW}} / R_{\text{TPW}}$	Change in physical state of SPRT between calibration and user measurement
$C_{\text{TPW}}^{\text{cal}}$	$R_{\text{TPW},i}^{\text{cal}} / R_{\text{TPW}}$	Correction for calibration lab TPW realization (excluding resistance measurements)
$\Re_{\text{TPW}}^{\text{cal}}$	$R_{\text{TPW}}^{\text{cal}} \big/ R_{\text{stnd}}^{\text{cal}}$	Resistance ratio meas. in calibration lab TPW cell after i^{th} fixed-point cell
$R_{\text{stnd}}^{\text{cal}}$	Resistance standard	Calibration lab resistance standard
$\Re_{\text{FP},i}^{\text{cal}}$	$R_{\text{FP},i}^{\text{cal}} \big/ R_{\text{stnd}}$	Resistance ratio meas. in i^{th} calibration lab fixed-point cell Change in physical state of SPRT between calibration and meas. of W^{user}
$S_{\text{FP},i}^{\text{TPW}}$	$R_{\text{TPW}} / R'_{\text{TPW}}$	Change in physical state of SPRT between measurements of $\Re_{\text{FP},i}^{\text{cal}}$ and $\Re_{\text{TPW}}^{\text{cal}}$
$C_{\text{FP},i}^{\text{cal}}$	$R_{\text{FP},i}^{\text{cal}} / R_{\text{FP},i}$	Calibration lab realization of i^{th} fixed point (excluding resistance measurements)
N		ITS-90 Non-uniqueness

Table 9

(a)

Uncertainty	**Value**	**Caption in Fig. 1 Legend**
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right)$	0.1 mK	User TPW
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$	0.1 mK	Cal TPW
$u_T\left(W_{\mathrm{FP},1}^{\mathrm{cal}}, C_{\mathrm{FP},1}^{\mathrm{cal}}\right)$	0.4 mK	Sn FP
$u_T\left(W_{\mathrm{FP},2}^{\mathrm{cal}}, C_{\mathrm{FP},2}^{\mathrm{cal}}\right)$	0.5 mK	Zn FP
$u_T\left(W_{\mathrm{FP},3}^{\mathrm{cal}}, C_{\mathrm{FP},3}^{\mathrm{cal}}\right)$	1.1 mK	Al FP
$u_T\left(W_{\mathrm{FP},4}^{\mathrm{cal}}, C_{\mathrm{FP},4}^{\mathrm{cal}}\right)$	1.8 mK	Ag FP
$u\left(\Re_{\mathrm{TPW}}^{\mathrm{cal}}\right)$	5.0×10^{-8}	Cal TPW R Rat.
$u(W^{\mathrm{user}})$	5.0×10^{-8}	User W
$u(R^{\mathrm{user}})$	0.25 μΩ/Ω	User Resist.
$u\left(R_{\mathrm{stnd}}^{\mathrm{cal}}\right)$	0.25 μΩ/Ω	Cal Res. Stnd.
$u\left(S_{T_{90}}^{\mathrm{TPW}}\right)$	See Table 9b	SPRT Change
$u\left(S_{\mathrm{FP},i}^{\mathrm{TPW}}\right)$	See Table 9b	None
$u(N)$	0	None

(b)

Case	$u\left(S_{T_{90}}^{\mathrm{TPW}}\right)$ [μΩ/Ω]	$u\left(S_{\mathrm{FP},i}^{\mathrm{TPW}}\right)$ [μΩ/Ω]
1	0	0
2	0	0
3	1.0	0
4	0	0.3
5	0	0.3
6	1.0	0.3

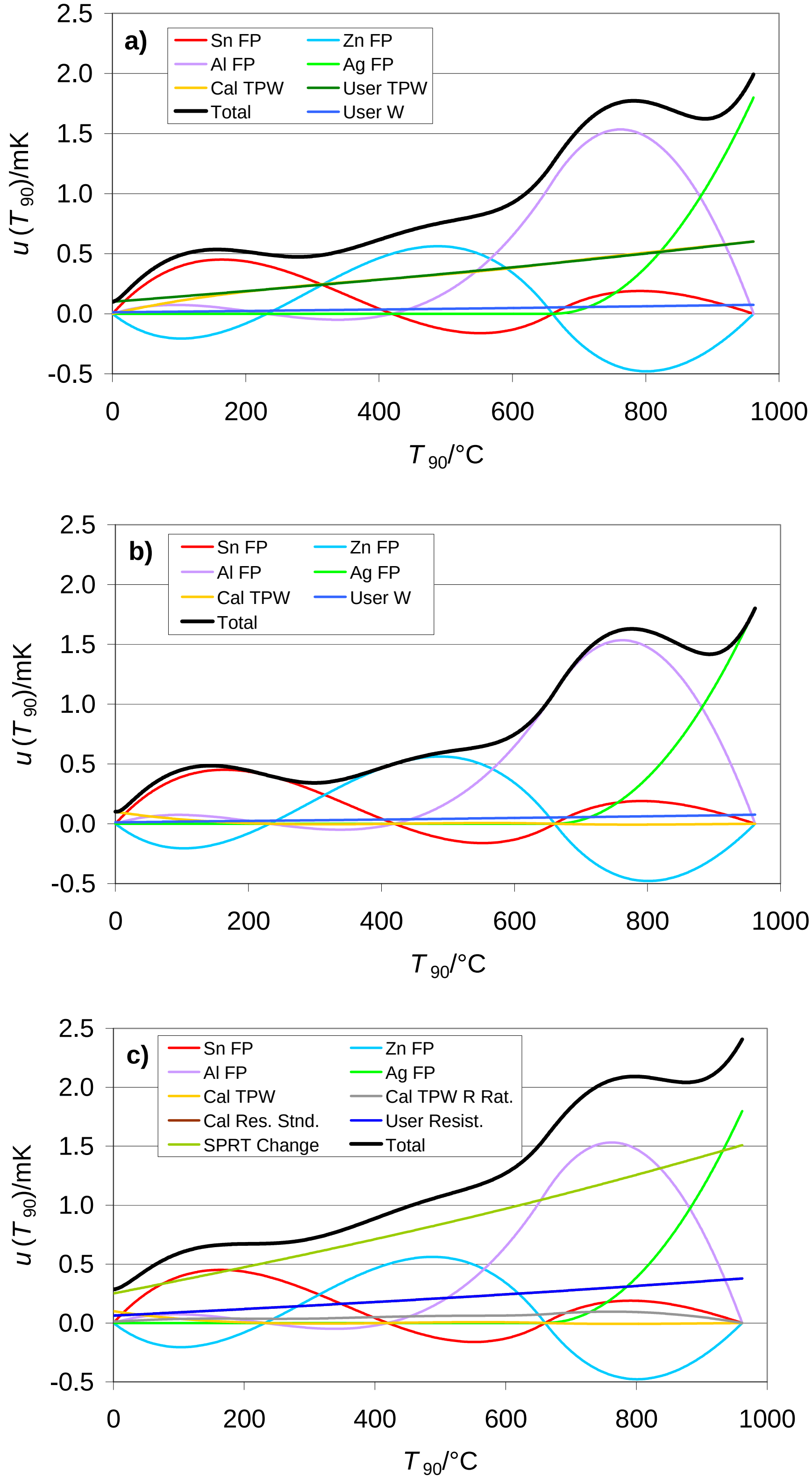

a)
Sn FP
Zn FP
Al FP
Ag FP
Cal TPW
User TPW
Total
User W
b)
Sn FP
Zn FP
Al FP
Ag FP
Cal TPW
User W
Total
c)
Sn FP
Zn FP
Al FP
Ag FP
Cal TPW
Cal TPW R Rat.
Cal Res. Stnd.
User Resist.
SPRT Change
Total
u (T 90)/mK
T 90/°C
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
0
200
400
600
800
1000

Figure 1

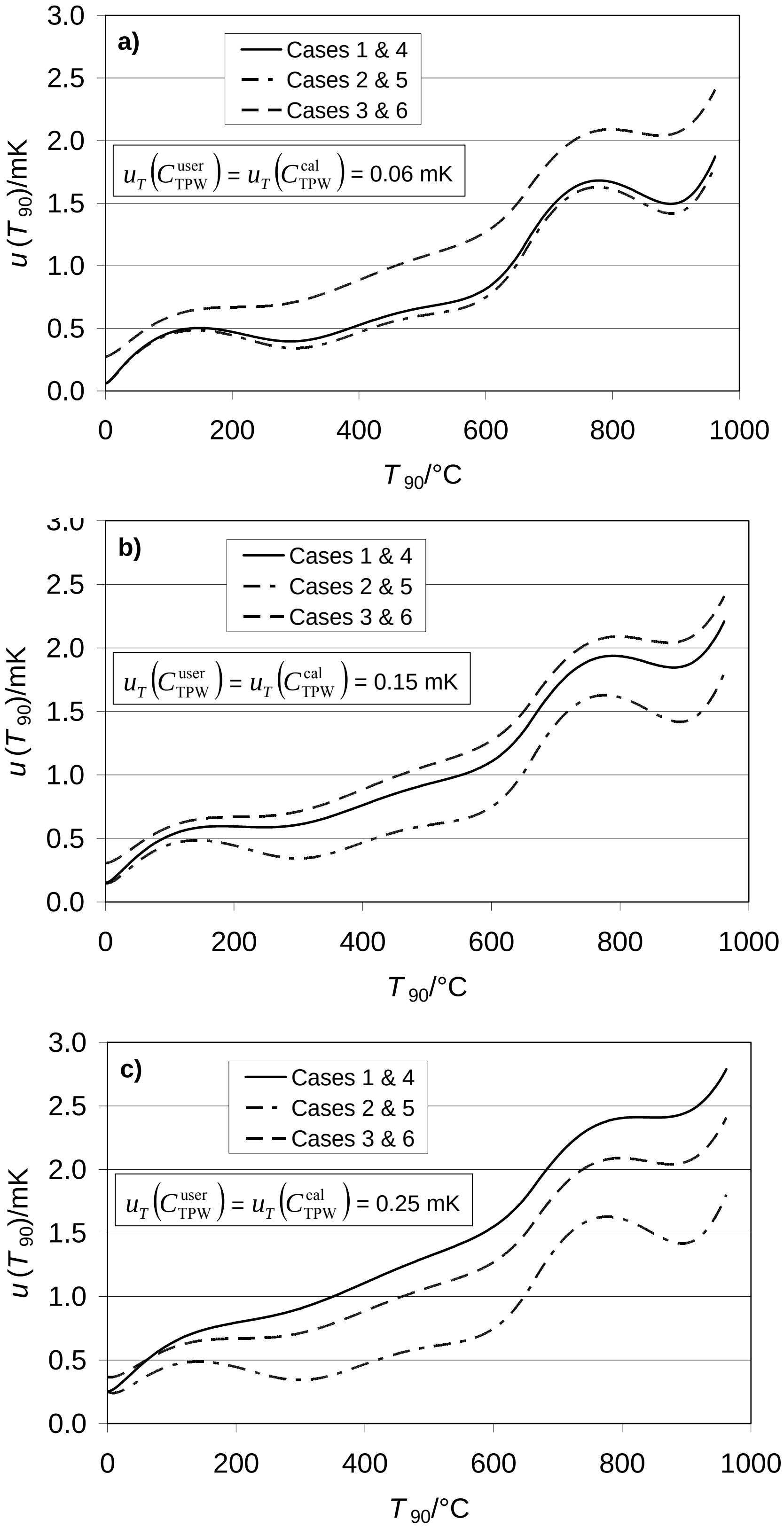
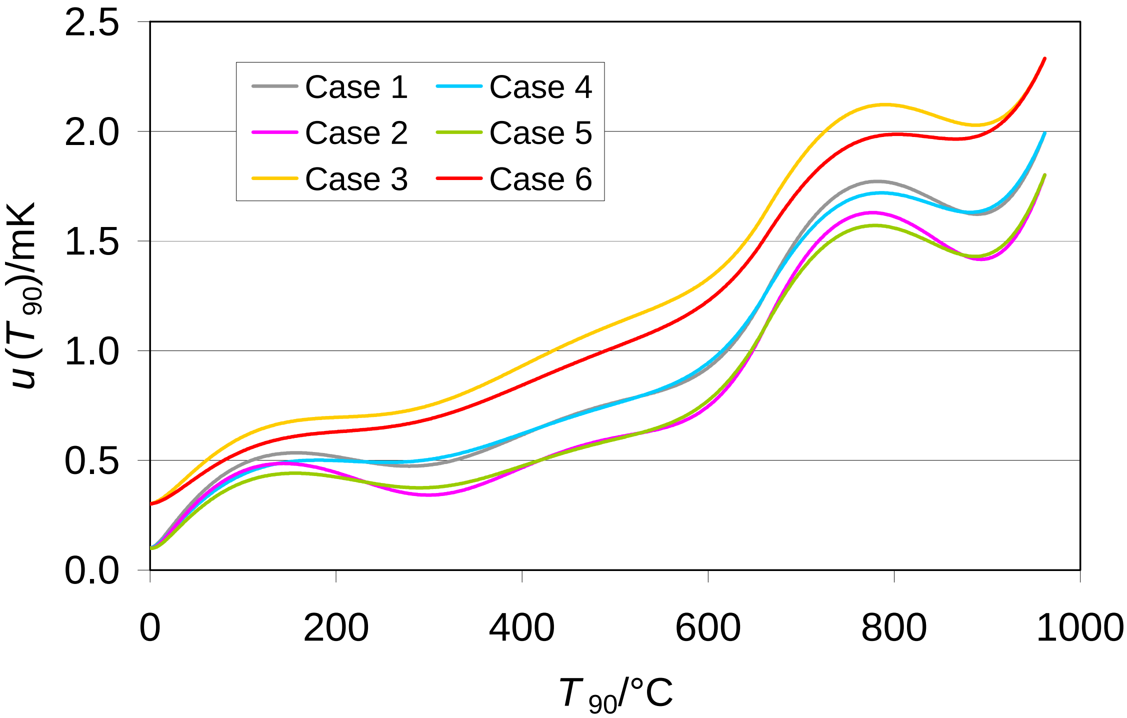

a)
Cases 1 & 4
Cases 2 & 5
Cases 3 & 6
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ = 0.06 mK
$u(T_{90})$/mK
T_{90}/°C
b)
Cases 1 & 4
Cases 2 & 5
Cases 3 & 6
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ = 0.15 mK
$u(T_{90})$/mK
T_{90}/°C
c)
Cases 1 & 4
Cases 2 & 5
Cases 3 & 6
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right)$ = 0.25 mK
$u(T_{90})$/mK
T_{90}/°C

Figure 2



2.5
2.0
1.5
1.0
0.5
0.0
0
200
400
600
800
1000
$u(T_{90})$/mK
T_{90}/°C
Case 1
Case 2
Case 3
Case 4
Case 5
Case 6

Figure 3

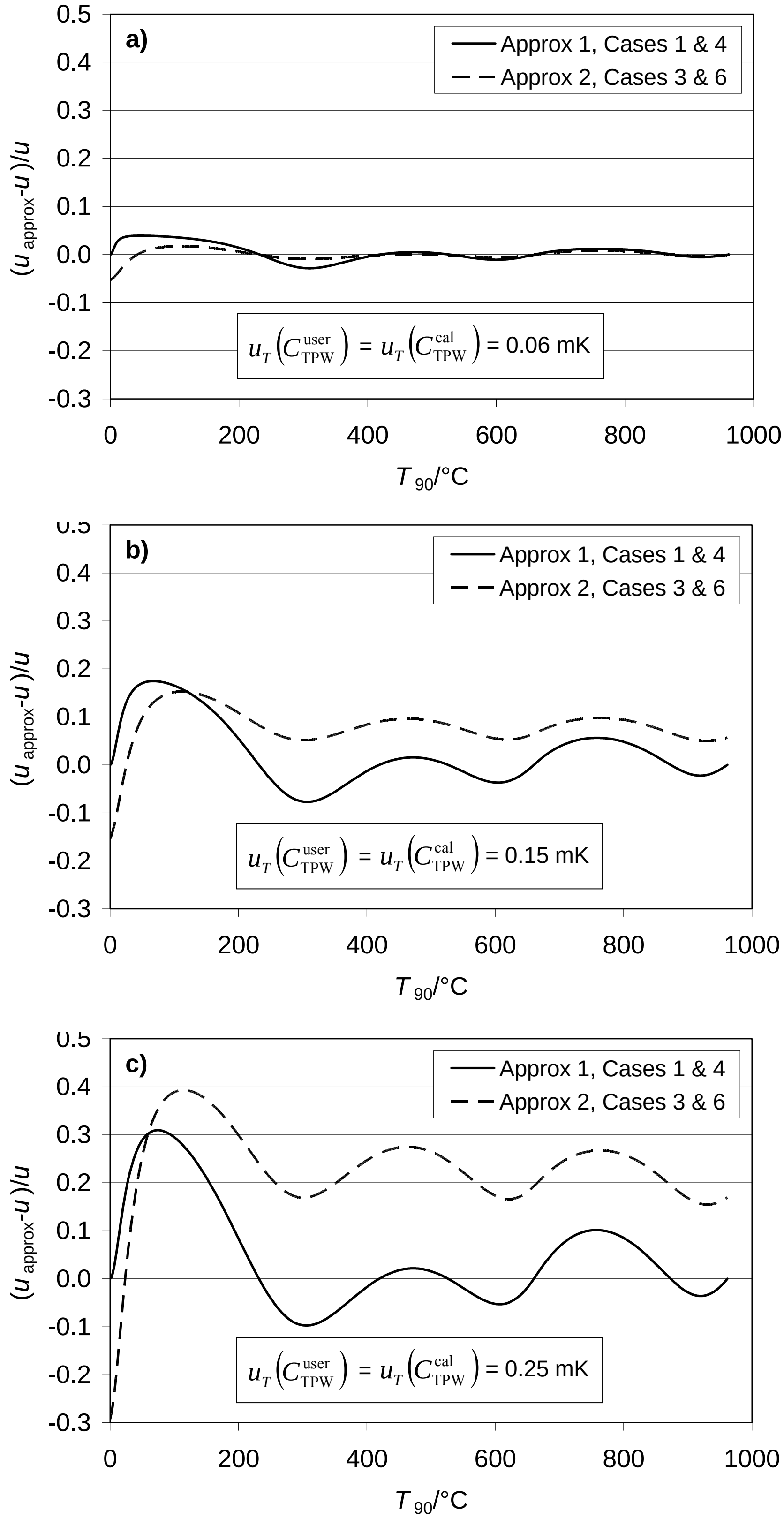

a)
b)
c)
Approx 1, Cases 1 & 4
Approx 2, Cases 3 & 6
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 0.06$ mK
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 0.15$ mK
$u_T\left(C_{\mathrm{TPW}}^{\mathrm{user}}\right) = u_T\left(C_{\mathrm{TPW}}^{\mathrm{cal}}\right) = 0.25$ mK
$(u_{\mathrm{approx}}-u)/u$
T_{90}/°C
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
0
200
400
600
800
1000

Figure 4