Analysis of the step-diagonal test

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Abstract

The step-diagonal test modifies the diagonal displacement test for machine tool performance evaluation by executing a diagonal as a sequence of single-axis motions. The proposed modification has gained interest because of the claim that the obtained additional data enables the estimation of parametric machine errors, in particular straightness and positioning errors. This paper addresses the properties of the estimated errors for machines with significant angular errors. The theoretical analysis shows that the estimated straightness and positioning errors can have significant errors if the effects of several angular errors are not corrected. A correction for some angular errors can be achieved by repeating a step-diagonal with a different sequence of axis motions. The thus estimated angular errors are expected to have larger uncertainties than those obtained through direct measurement. Use of a retro-reflector eliminates the introduced errors for some angular errors. Our analysis confirms that setup errors in the alignment of the return mirror cause significant errors in the slope of the estimated positioning errors that cannot be detected from the (step-) diagonal measurements. Correction requires information on the slope of the positioning errors of two axes. Finally, the analysis shows that it is not possible to uniquely allocate non-repeatable machine behavior to the various parametric errors.

1 Introduction

The step-diagonal or vector-measurement technique [1,2,3] is a proposed method to measure geometric errors of machine tools or coordinate measuring machines. The test differs from the diagonal displacement test described in standards for machine tool performance evaluation (e.g., ASME B5.54:2005 and ISO 230-6:2002). The diagonal displacement test involves measuring errors in the actual travel of the tool along a face or body diagonal. It provides samples for the combined effects of parametric errors, but does not allow them to be estimated.

In the step-diagonal test, each new point on the diagonal is reached through a sequence of single-axis motions. At the end of each axis step, the error in the tool

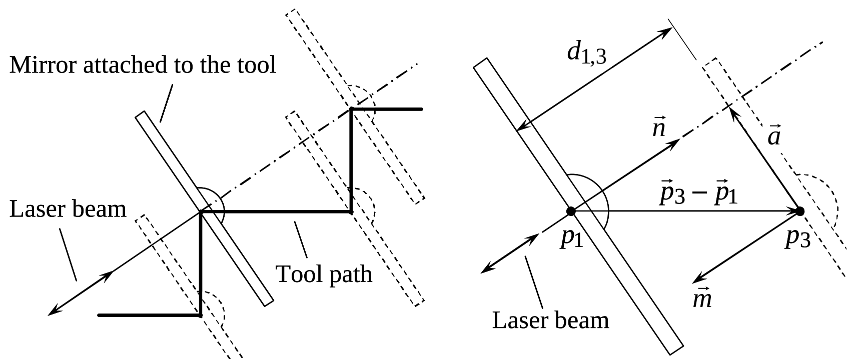

Figure 1 Principle of the step-diagonal method and symbolism used.

displacement is measured along the diagonal using a laser interferometer (Figure 1). Two configurations have been proposed [1]. One has a mirror attached to the tool that reflects the laser beam back to the interferometer. This configuration is applied in the field and is the focus of this paper. The second configuration has a large retro-reflector as a target. The retro-reflector returns the beam to a mirror on the workholding component of the machine, which in turn reflects the beam back into itself to the interferometer. In both cases, the target has to be large enough to return the laser beam when the tool is moved away from the diagonal.

For typical machines, angular errors during axis motions cause significant errors in the position and orientation of the tool throughout the workzone, and an assessment of their effects is important. Published methods to analyze data obtained with the step-diagonal test either assume that the effects of the angular errors are negligible, or alternatively, do not completely address these effects.

This paper explicitly addresses machines with significant angular errors. We describe the characteristics of the obtained error estimates when angular errors are ignored in the analysis. The combination of parametric errors that can be estimated is identified. The presented analytical solutions illustrate key properties of the obtained error estimates, but do not exploit some redundancies in step-diagonal data to reduce uncertainties. The latter can be achieved using a less-transparent linear regression approach where the identifiable error combinations are simultaneously estimated using all measurement data. The equations in this paper form a basis for such an approach.

2 Basic equations

Analysis of the step-diagonal method is achieved through two models. The kinematic model describes the errors in the position and orientation of the tool as a function of parametric machine errors, i.e., the positioning, angular, and straightness errors of each axis and the errors in the alignment of axes. The task error model describes the errors of a task, in this case the measurement of a step-diagonal, as a function of the errors in the position and orientation of the tool.

The kinematic model used assumes small angular errors, allowing for first-order approximations of their effects, and rigid-body kinematics, yielding

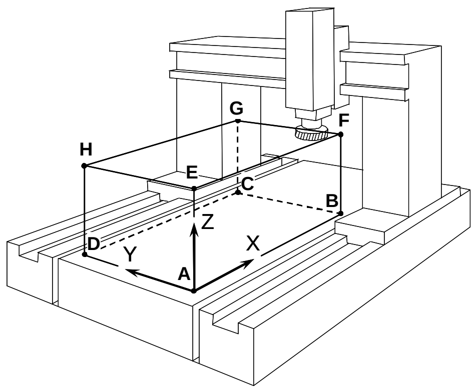

Figure 2 The studied machine configuration

parametric errors that are only affected by the position of one machine axis. For the machine depicted in Figure 2, the errors $\vec{e}=[e_x, e_y, e_z]$ and $\vec{\varepsilon}=[\varepsilon_x, \varepsilon_y, \varepsilon_z]$ in the position and orientation of the tool can be expressed as:

$$\begin{aligned} e_x &= e_x(x)+e_x(y)+e_x(z)+[\varepsilon_y(x)+\varepsilon_y(y)-S_{xz}]\cdot z-[\varepsilon_z(x)+S_{xy}]\cdot y \\ e_y &= e_y(x)+e_y(y)+e_y(z)-[\varepsilon_x(x)+\varepsilon_x(y)+S_{yz}]\cdot z \\ e_z &= e_z(x)+e_z(y)+e_z(z)+\varepsilon_x(x)\cdot y \\ \varepsilon_x &= \varepsilon_x(x)+\varepsilon_x(y)+\varepsilon_x(z) \\ \varepsilon_y &= \varepsilon_y(x)+\varepsilon_y(y)+\varepsilon_y(z) \\ \varepsilon_z &= \varepsilon_z(x)+\varepsilon_z(y)+\varepsilon_z(z), \end{aligned} \qquad (1)$$

where $e_x(y)$ and $\varepsilon_x(y)$, for example, denote the translation and angular errors in/around X when moving Y. The squareness errors S_{xy}, S_{yz} and S_{xz} have a positive value if the angle between the machine axes exceeds 90°. The position of the coordinate frame used, i.e., the reference lines where the parametric errors are defined, is arbitrary but affects the values of the positioning, straightness, and squareness errors.

The sensitive direction of both diagonal and step-diagonal measurements is determined by the direction $\vec{n}$ of the laser beam. For diagonal measurements, the beam intersects the target mirror nominally at the same point. We define this intersection as the point of the tool whose position errors are described by the kinematic model. This yields the following equation for the error $dd_{1,2}$ in a diagonal displacement $d_{1,2}$ between positions p_1 and p_2:

$$dd_{1,2} = \vec{n}\cdot[\vec{e}(\vec{p}_2)-\vec{e}(\vec{p}_1)] \qquad (2)$$

For a step-diagonal measurement, we consider a machine position p_3 that is not on the diagonal (Figure 1). At this position, the laser beam intersects the target mirror with normal $\vec{m}$ at an offset $\vec{a}$ defined by:

$$\left.\begin{array}{r}\vec{a}\cdot\vec{m}=0\\ \vec{p}_1+d_{1,3}\,\vec{n}=\vec{p}_3+\vec{a}\end{array}\right\}\Rightarrow\vec{a}=\vec{p}_1-\vec{p}_3+d_{1,3}\,\vec{n}\quad\text{and}\quad d_{1,3}=\frac{(\vec{p}_3-\vec{p}_1)\cdot\vec{m}}{\vec{m}\cdot\vec{n}}\tag{3}$$

The offset $\vec{a}$ yields an Abbe offset for the angular errors $\vec{e}$ of the tool, resulting in an additional term $\vec{n}\cdot(\vec{e}(\vec{p}_3)\times\vec{a})$ in the observed displacement error $dd_{1,3}$. This term is not present when a retro-reflector is used as target. Thus the respective step-diagonal measurements are not sensitive to angular errors that only affect tool orientation (Y-axis yaw and the angular errors of the Z-axis), unless repeated with a different tool offset.

Setup errors may cause a misalignment of the laser beam and the return mirror relative to the machine axes. A misalignment of the laser beam changes the offset $\vec{a}$ and the sensitive direction, resulting in second-order measurement errors. A misalignment $\vec{e}_m$ of the return mirror relative to the machine axes, however, results in a first-order error $\vec{n}\cdot(\vec{e}_m\times\vec{a})$ due to the offset $\vec{a}$ in the position of the beam on the mirror. The error equals the misalignment angle in the plane spanned by the laser beam and the tool path times the travel $|\vec{a}|$ of the laser spot on the target. The error reduces to zero for positions on the diagonal. The same first-order error occurs for the retro-reflector configuration. Here the offset on the return mirror is doubled, but the resulting error occurs in only half of the laser path between reference and tool. In most setups, the laser beam is first aligned to the machine motion, after which the mirror is aligned to the laser.

For the two configurations, the observed error $dd_{1,3}$ when moving from a point p_1 on the diagonal to an arbitrary point p_3 can now be summarized as:

$$\begin{array}{ll}\text{Mirror:} & dd_{1,3}=\vec{n}\cdot\left[\vec{e}(\vec{p}_3)-\vec{e}(\vec{p}_1)+\left(\vec{e}(\vec{p}_3)+\vec{e}_m\right)\times\left(\vec{p}_1-\vec{p}_3\right)\right]\\ \text{Retro-reflector:} & dd_{1,3}=\vec{n}\cdot\left[\vec{e}(\vec{p}_3)-\vec{e}(\vec{p}_1)+\vec{e}_m\times\left(\vec{p}_1-\vec{p}_3\right)\right]\end{array}\tag{4}$$

3 Diagonal displacement measurements

In this section we analyze the sensitivity of diagonal displacement measurements between the corners of the workzone to the parametric errors. Application of Equations (1) and (2) yields for the errors of body diagonals AG and DF (Fig. 2):

$$\begin{aligned}dD_{AG}=\;&n_x\,[\,\Delta e_x(x)+\Delta e_x(y)+\Delta e_x(z)+(\,\Delta e_y(x)+\Delta e_y(y)-S_{xz}\,)\cdot\Delta Z\\&\quad-(\,\Delta e_z(x)+S_{xy}\,)\cdot\Delta Y\,]+\\&n_y\,[\,\Delta e_y(x)+\Delta e_y(y)+\Delta e_y(z)-(\,\Delta e_x(y)+S_{yz}\,)\cdot\Delta Z\,]+\\&n_z\,[\,\Delta e_z(x)+\Delta e_z(y)+\Delta e_z(z)\,]\end{aligned}\tag{5}$$

$$\begin{aligned}dD_{DF}=\;&n_x\,[\,\Delta e_x(x)-\Delta e_x(y)+\Delta e_x(z)+(\,\Delta e_y(x)-S_{xz}\,)\cdot\Delta Z+S_{xy}\,\Delta Y\,]\\&n_y\,[\,-\Delta e_y(x)+\Delta e_y(y)-\Delta e_y(z)+(\,\Delta e_x(x)+S_{yz}\,)\cdot\Delta Z\,]+\\&n_z\,[\,\Delta e_z(x)-\Delta e_z(y)+\Delta e_z(z)\,],\end{aligned}\tag{6}$$

where, for example, $\Delta e_x(y)$ denotes the total change in the pitch error $e_x(y)$ of the Y-axis over the evaluated axis travel ΔY, and n_x, n_y and n_z are the direction cosines of diagonal AG. Combining the measurements of all four body diagonals, each with a nominal length D, yields:

$$\frac{dD_{AG}+dD_{BH}+dD_{CE}+dD_{DF}}{4}=n_x[\Delta e_x(x)+\frac{\Delta Z}{2}\Delta e_y(x)-\frac{\Delta Y}{2}\Delta e_z(x)] \\ +n_y[\Delta e_y(y)-\frac{\Delta Z}{2}\Delta e_x(y)]+n_z\,\Delta e_z(z) \tag{7}$$

$$\frac{-dD_{AG}-dD_{BH}+dD_{CE}+dD_{DF}}{4D\,n_y n_z}=S_{yz}+\frac{\Delta e_x(y)}{2}-[\frac{\Delta e_y(z)}{\Delta Z}+\frac{\Delta e_z(y)}{\Delta Y}] \tag{8}$$

$$\frac{-dD_{AG}+dD_{BH}+dD_{CE}-dD_{DF}}{4D\,n_x n_z}=S_{xz}-\frac{\Delta e_y(x)}{2}-\frac{\Delta e_y(y)}{2} \\ -\frac{\Delta Y\,\Delta e_x(x)}{2\Delta X}-[\frac{\Delta e_x(z)}{\Delta Z}+\frac{\Delta e_z(x)}{\Delta X}] \tag{9}$$

$$\frac{-dD_{AG}+dD_{BH}-dD_{CE}+dD_{DF}}{4D\,n_x n_y}=S_{xy}+\frac{\Delta e_z(x)}{2}+\frac{\Delta Z\,\Delta e_x(x)}{2\Delta X} \\ -\frac{\Delta Z\,\Delta e_y(y)}{2\Delta Y}-[\frac{\Delta e_x(y)}{\Delta Y}+\frac{\Delta e_y(x)}{\Delta X}]. \tag{10}$$

In Equation (7), the angular errors modify the positioning errors to represent positioning errors at lines through the center of the workzone. The squareness errors in Equations (8-10) are similarly modified to represent the average of alignment errors at the edges of the workzone. The straightness terms equal zero if straightness errors are modeled relative to the line through the axis end-points.

The equations show that errors in the total diagonals yield information on the "average" squareness errors and *one* linear combination of the modified positioning errors. Thus small diagonal errors do not imply small parametric errors, as the latter can compensate each other for the diagonal displacements [4]. The slope of positioning errors requires particular care, as exact compensation can occur for all points on a diagonal. Estimating squareness errors from diagonal measurements is challenging [4]. The sensitivity is affected by the aspect ratio of the addressed workzone. The best estimate is obtained for a machine with equal axis lengths, using face diagonals. For a machine with aspect ratio $\Delta X:\Delta Y:\Delta Z=1:b:c$, the uncertainty $u(S_{xy})$ of the estimated squareness error is related to the relative uncertainty $u(dD)/D$ in the measurement of a diagonal D by:

$$u(S_{xy})=\frac{1}{2\,|\,n_x n_y\,|}\frac{u(dD)}{D}=\frac{1+b^2+c^2}{2b}\frac{u(dD)}{D}\,. \tag{11}$$

4 Two-dimensional step-diagonal measurements

Face-diagonal measurements have an increased sensitivity to a reduced number of machine errors. In this section, we analyze the information obtained from step-diagonal measurements in a plane of two machine axes when the mirror is attached to the tool. At the end of the section we discuss the retro-reflector setup.

Figure 3 shows the evaluated measurements in the bottom XY-plane. Each new target point $i+1$ on the diagonal corresponds to one new target point for each axis, yielding $2 \times 3 = 6$ new parametric error values that affect the measurements: $e_x(x_{i+1})$, $e_y(x_{i+1})$, $e_z(x_{i+1})$, $e_x(y_{i+1})$, $e_y(y_{i+1})$ and $e_z(y_{i+1})$. The new point on the diagonal can be reached by two sequences of unidirectional single-axis motions, yielding four measurements for the errors in the change in laser reading during a single-axis step: dd_{AB}, dd_{BC}, dd_{AD}, and dd_{DC}. The four measurements are not independent, however, as the sum of the systematic errors during both sequences equals the diagonal displacement error, i.e., $dd_{AB} + dd_{BC} = dd_{AD} + dd_{DC} = dd_{AC}$. In other words, the differences between errors of parallel steps have an equal value: $dd_{DC} - dd_{AB} = dd_{BC} - dd_{AD}$.

Let $\Delta e_x(x_i)$ denote the change $e_x(x_{i+1}) - e_x(x_i)$ in the positioning error of the X-axis when moving over a distance Δx from position x_i to x_{i+1}. Application of this notation to the equations of Section 2 yields:

$$\begin{aligned} dd_{AB}(i,i) = {} & n_x\, \Delta e_x(x_i) + n_y\, \Delta e_y(x_i) - n_x y_i \Delta e_z(x_i) \\ & - n_x n_y\, d\, [e_z(x_i) + \Delta e_z(x_i) + e_z(y_i) + e_{m1}] \end{aligned} \tag{12}$$

$$dd_{BC}(i,i) = n_x\, \Delta e_x(y_i) + n_y\, \Delta e_y(y_i) + n_x n_y\, d\, [e_z(y_i) - S_{xy} + e_{m1}] \tag{13}$$

$$dd_{AB}(i,i) - dd_{DC}(i,i) = n_x n_y\, d\, \Delta e_z(y_i) \tag{14}$$

$$\begin{aligned} dd_{AB}(i,k) = {} & n_x\, \Delta e_x(x_i) - n_y\, \Delta e_y(x_i) - n_x y_k \Delta e_z(x_i) \\ & + n_x n_y\, d [e_z(x_i) + e_z(y_k) + e_{m2}] \end{aligned} \tag{15}$$

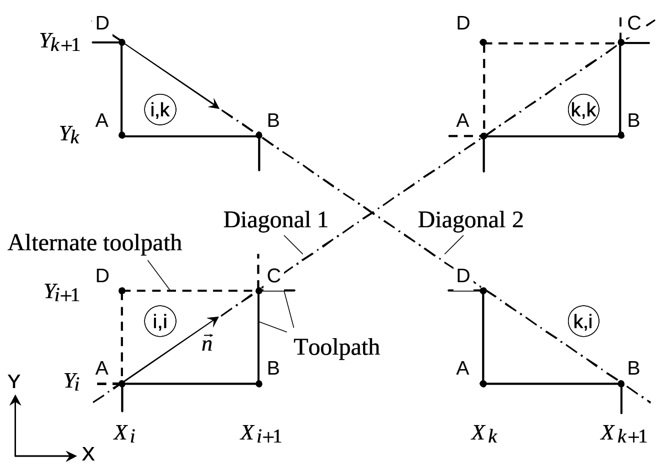

Figure 3 The evaluated step-diagonal measurements

$$
\begin{aligned}
dd_{DA}(i,k) &= -dd_{AD}(i,k) \\
&= -n_x\,\Delta e_x(y_k) + n_y\,\Delta e_y(y_k) - n_x n_y\, d\,\left[e_z(y_k) - S_{xy} + e_{m2}\right]
\end{aligned} \tag{16}
$$

$$
dd_{AB}(i,k) - dd_{DC}(i,k) = -n_x n_y\, d\,\Delta e_z(y_k), \tag{17}
$$

where n_x and n_y are the direction cosines of diagonal *AC*, d is the nominal length of a diagonal step, and e_{m1} is the error in mirror alignment for diagonal 1. Note that an error e_m in mirror alignment causes equal but opposite errors $n_x\,n_y\,d\,e_m$ in measured displacements during X- and Y-axis moves.

The difference between the errors of parallel segments of a diagonal step, e.g., dd_{DC} - dd_{AB}, is not affected by systematic errors in positioning, straightness, and squareness, as both segments have the same axis motion and sensitive direction. The difference is also not affected by any angular error that has the same constant value for both segments (Figure 4). Thus an error e_m in mirror orientation cannot be detected from the results of a step-diagonal measurement. Furthermore, the difference is not affected by a change in the angular error of the machine axis that carries the other axis ($\Delta e_z(x_i)$ in this example, see Figure 4). The center of rotation of this angular error is fixed during the motion of the other axis, and as a result does not affect the relative distance and parallelism of the mirror surface when moving this axis. Therefore, it does not cause an error in the respective segments dd_{BC} and dd_{AD}, and by extension in the difference dd_{BC} - dd_{AD} = dd_{DC} - dd_{AB}. It does cause an equal error in both dd_{DC} and dd_{AB}.

The only systematic error that causes a difference between parallel segments of a diagonal step is the change in Y-axis yaw $\Delta e_z(y_i)$. This allows this error to be estimated irrespective of errors in mirror alignment (Equation 14). The uncertainty is affected by non-repeatable measurement and machine errors:

$$
u(\Delta e_z(y_i)) = \frac{\sqrt{2}}{|n_y|}\frac{u(dd_{AB})}{\Delta x}. \tag{18}
$$

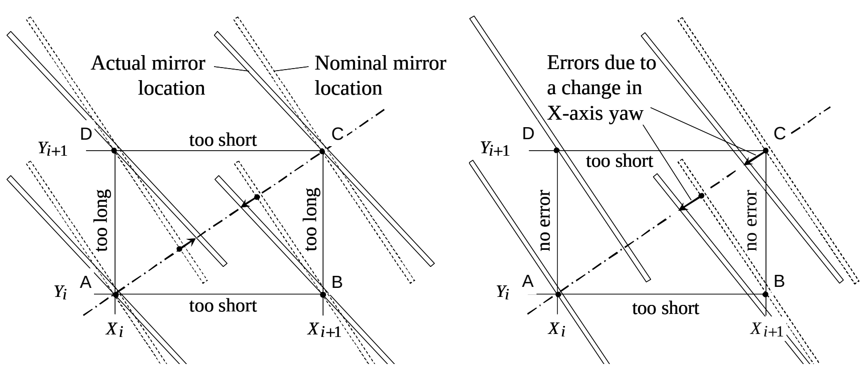

Figure 4 The difference between the errors of parallel segments, e.g., $dd_{DC} - dd_{AB}$, is not affected by a constant error in mirror orientation or a change in X-axis yaw.

For a step size of 100 mm in each axis, a standard deviation of 1 µm in dd_{AB} and dd_{DC} yields a standard deviation of 20 µrad in the estimated change of the angular error. Although the small Abbe offset used in the estimation results in a relatively high uncertainty, it may be possible to use the estimated error to compensate step-diagonal measurements, since its leverage is determined by a similar small offset.

As differences between parallel segments of a diagonal step only yield Y-axis yaw, there is only information left to estimate four independent combinations of parametric errors. Estimates for the change in positioning errors are obtained as:

$$\frac{dd_{AB}(i,i)+dd_{AB}(i,k)}{2n_x} = \Delta e_x(x_i) - \frac{y_i + y_{k+1}}{2}\Delta e_z(x_i) + \frac{\Delta y}{2}[e_z(y_k) - e_z(y_i) + e_{m2} - e_{m1}] \tag{19}$$

$$\frac{dd_{BC}(i,i)-dd_{AD}(k,i)}{2n_y} = \Delta e_y(y_i) + \frac{\Delta x}{2}[e_{m1} - e_{m2}]\,. \tag{20}$$

The first two terms of Equation (19) can be interpreted as the change in the positioning error of the X-axis at a line through the center of the workzone. The estimate is affected by the difference in mirror misalignment of both diagonals. Since the positioning error is obtained by adding the obtained estimates for the changes in the positioning error, the contamination accumulates [4]. The result is an error in the slope of the estimated positioning error, yielding an error equal to $-1/2\,(e_{m1} - e_{m2})\cdot\Delta Y$ for the total axis range. Thus, even small errors in mirror alignment cause significant errors in the estimated positioning error [4]. For the Y-axis, the mirror misalignment causes an opposite error in the estimated positioning error equal to $1/2\,(e_{m1} - e_{m2})\cdot\Delta X$. The ratio of the errors is such that they compensate each other for diagonal displacement measurements. They can only be corrected using a separate measurement for the slope of the positioning error of one of the axes. The estimated yaw error of the Y-axis can be used to further correct the obtained X-axis positioning error. The effect of a linear term in Y-axis yaw on the positioning error equals zero, as well as its contribution over the total axis range.

Estimates for the straightness errors $e_y(x_i)$ and $e_x(y_i)$ are obtained as:

$$\frac{dd_{AB}(i,i)-dd_{AB}(i,k)}{2n_y} = \Delta e_y(x_i) + \frac{1}{2}[x_k - x_{i+1}]\,\Delta e_z(x_i) - \Delta x\, e_z(x_i) - \frac{\Delta x}{2}[e_z(y_i) + e_z(y_k) + e_{m1} + e_{m2}] \tag{21}$$

$$\frac{dd_{BC}(i,i)+dd_{AD}(k,i)}{2n_x} = \Delta e_x(y_i) + \Delta y\,[e_z(y_i) - S_{xy} + \frac{e_{m1}+e_{m2}}{2}]\,. \tag{22}$$

Removing the slope through the estimated straightness errors also removes errors due to mirror misalignment. The squareness error lost in this process can be estimated from the diagonal measurements using the procedure of Section 3 [2,4], thus avoiding errors due to mirror misalignment.

The estimated straightness error is affected by yaw errors, and compensation may be required to obtain reliable estimates. The first two terms of Equation (21) can be interpreted as the change in the straightness error of the X-axis measured at a variable tool offset such that the functional point of the tool is at the center of the X-axis. X-axis yaw acts on this variable tool offset, and the large offset at the end-points of the axis may result in large errors in the estimated straightness.

The estimates for the changes in straightness errors contain a term $\Delta x\, e_z(x_i)$ or $\Delta y\, e_z(y_i)$, that after summation approximates the integrated yaw error. A yaw error of 1 µrad/m of axis travel yields a parabolic straightness profile with a peak-to-valley error of 1/8 µm/m^2. The integral can approximate the straightness profile of an axis whose straightness is determined by the guideway geometry. In that case, the obtained result does not contain information on straightness errors.

Reversal errors can be obtained by executing the step diagonals in both directions. Combining segments with the same direction of axis motion, including reversal motions, yields estimates for the uni-directional axis errors and their difference. Estimation of the repeatability of a parametric error requires care due to averaging, and may be affected by the repeatability of other parametric errors. For example, a non-repeatable positioning error, characterized by a standard deviation $s(\Delta e_x(x_i))$, affects the repeatability of the estimated straightness error $\Delta\hat{e}_y(x_i)$ of that axis:

$$s^2(\Delta\hat{e}_y(x_i)) = \frac{2}{4{n_y}^2}\, s^2(dd_{AB}) = \frac{2{n_x}^2}{4{n_y}^2}\, s^2(\Delta e_x(x_i)) + \cdots \tag{23}$$

In the step-diagonal method, a parametric error is constructed as the cumulative sum of estimates for its change during incremental axis motions. This results in an accumulation of uncertainties introduced by non-repeatable measurement and machine errors whose value changes during the required incremental motion of the other axis, e.g., due to vibrations. In general, the resulting uncertainty will be proportional to the square root of the number of incremental steps of an axis.

Step-diagonal measurements with a retro-reflector are not affected by errors in the orientation of the tool (Equation 4). This simplifies the analysis and reduces the errors in the estimated positioning and straightness errors when angular errors are not corrected. Ignoring the effects of mirror misalignment, which are similar to those of the other configuration, yields:

$$\frac{dd_{AB}(i,i) + dd_{DC}(i,k)}{2n_x} = \Delta e_x(x_i) - \frac{y_i + y_{k+1}}{2}\Delta e_z(x_i) \tag{24}$$

$$\frac{dd_{BC}(i,i) - dd_{BC}(k,i)}{2n_y} = \Delta e_y(y_i) + \Delta x \frac{e_z(x_{k+1}) - e_z(x_{i+1})}{2} \tag{25}$$

$$\frac{dd_{AB}(i,i) - dd_{DC}(i,k)}{2n_y} = \Delta e_y(x_i) + \frac{x_{k+1} - x_i}{2} \Delta e_z(x_i) \tag{26}$$

$$\frac{dd_{BC}(i,i) + dd_{BC}(k,i)}{2n_x} = \Delta e_x(y_i) - \Delta y[S_{xy} + \frac{e_z(x_{k+1}) + e_z(x_{i+1})}{2}] \tag{27}$$

$$dd_{AB}(i,i) - dd_{DC}(i,i) = n_x n_y \, d \, \Delta e_z(x_i) \, . \tag{28}$$

5 Three-dimensional step-diagonal measurements

In this section we analyze the results of three-dimensional step-diagonal measurements for the configuration where the target mirror is attached to the tool. Each new point on a diagonal can be reached through six different sequences of unidirectional single-axis movements. This provides for a maximum of seven new measurements, corresponding to the laser readings at the new corners of the cuboid spanned by the single-axis motions. The respective information is contained in the measurement of one edge of the cuboid per axis, and four independent differences between parallel edges. As in the two-dimensional case, the two differences between the parallel edges of a face have an equal value, yielding $6-1=5$ independent constraints on the 12 edges. The findings of Section 4 show that, for rigid-body kinematics, the differences between parallel edges are only affected by the change in the angular errors of the Y- and Z-axes. Application of Equation (4) shows that a difference is not affected by the change in the pitch, $\Delta e_x(y)$, of the Y-axis. The remaining angular errors can be estimated by solving the system of equations obtained from differences observed for two diagonals with a different direction in Y or Z, e.g.,:

$$dd_{DC}(i,i,i) - dd_{AB}(i,i,i) = \Delta x \, n_z \Delta e_y(y_i) - \Delta x \, n_y \Delta e_z(y_i) \tag{29}$$

$$dd_{EF}(i,i,i) - dd_{AB}(i,i,i) = \Delta x \, n_z \Delta e_y(z_i) - \Delta x \, n_y \Delta e_z(z_i) \tag{30}$$

$$dd_{EH}(i,i,i) - dd_{AD}(i,i,i) = -\Delta y \, n_z \Delta e_x(z_i) + \Delta y \, n_x \Delta e_z(z_i) \tag{31}$$

$$dd_{DC}(i,i,k) - dd_{AB}(i,i,k) = \Delta x \, n_z \Delta e_y(y_i) + \Delta x \, n_y \Delta e_z(y_i) \tag{32}$$

$$dd_{EF}(k,k,i) - dd_{AB}(k,k,i) = \Delta x \, n_z \Delta e_y(z_i) + \Delta x \, n_y \Delta e_z(z_i) \tag{33}$$

$$dd_{EH}(k,k,i) - dd_{AD}(k,k,i) = -\Delta y \, n_z \Delta e_x(z_i) - \Delta y \, n_x \Delta e_z(z_i) \, , \tag{34}$$

where each diagonal has a positive direction cosine in Z. With the estimation of the above angular errors we have exhausted the information in the differences between parallel edges. This leaves three edges per new point on a diagonal, one for each axis. Combining edges parallel to Z for the four step-diagonals yields:

$$\frac{dd_{CG}iii + dd_{DH}kii + dd_{AE}kki + dd_{BF}iki}{4n_z} = \Delta e_z(z_i) \tag{35}$$

$$\frac{dd_{CG}iii - dd_{DH}kii - dd_{AE}kki + dd_{BF}iki}{4n_x} = \Delta e_x(z_i) - \Delta z[S_{xz} + e_y(z_i)] \tag{36}$$

$$\frac{dd_{CG}iii + dd_{DH}kii - \Delta d_{AE}kki - \Delta d_{BF}iki}{4n_y} = \Delta e_y(z_i) + \Delta z[-S_{yz} + e_x(z_i)] \quad (37)$$

$$dd_{CG}iii - dd_{DH}kii + dd_{AE}kki - dd_{BF}iki = 0, \quad (38)$$

where errors due to mirror misalignment have been ignored. For Z-axis motion, the mirror misalignment causes an error $n_z\, d_{xy}\, e_m$ in each measurement, where e_m is the misalignment angle of the mirror around a horizontal axis parallel to the mirror surface, and d_{xy} the length of the diagonal step in the XY plane. This yields an error in the estimated positioning error of the Z-axis equal to the average mirror misalignment angle times the total length of the diagonal in the XY plane. The angular machine errors affect the estimated straightness errors in a manner similar to the two-dimensional case. A correction can be obtained using separate measurements for the angular errors, or the estimates for the angular errors obtained from measurements of parallel edges (Equations 29-34).

Combining the errors observed for edges parallel to the Y-axis yields:

$$\frac{dd_{BC}iii + dd_{AD}kii - dd_{BC}iik - dd_{AD}kik}{4n_y} = \Delta e_y(y_i) - \frac{z_i + z_{k+1}}{2}\Delta e_x(y_i) - \frac{\Delta z}{2}[e_x(z_i) - e_x(z_k)] \quad (39)$$

$$\frac{dd_{BC}iii - dd_{AD}kii - dd_{BC}iik + dd_{DA}kik}{4n_x} = \Delta e_x(y_i) + \frac{z_i + z_{k+1}}{2}\Delta e_y(y_i) + \Delta y[-S_{xy} + \frac{2e_z(y_i) + e_z(z_i) + e_z(z_k)}{2}] \quad (40)$$

$$\frac{dd_{BC}iii + dd_{AD}kii + dd_{BC}iik + dd_{AD}kik}{4n_z} = \Delta e_z(y_i) + \frac{y_k - y_{i+1}}{2}\Delta e_x(y_i) - \Delta y[\frac{2e_x(y_i) + e_x(z_i) + e_x(z_k)}{2}] \quad (41)$$

$$\frac{dd_{BC}iii - dd_{AD}kii + dd_{BC}iik - dd_{AD}kik}{4n_x} = -\frac{z_k - z_{i+1}}{2}\Delta e_y(y_i) + \frac{\Delta y}{2}[e_z(z_i) - e_z(z_k)]. \quad (42)$$

The first two terms of Equation (39) yield the positioning error of the Y-axis at a line through the center of the workspace. The result is affected by the pitch error of the Z-axis motion. The first two terms of Equation (40) yield the straightness error of the Y-axis in X at a line through the center of the workspace. The first two terms of Equation (41) yield the “straightness” error of the Y-axis in Z, measured at a variable tool offset such that the tool is at the center of the Y-axis. Equation (42) can, in principle, yield an estimate for the change in the roll error of the Y-axis. The respective uncertainty increases near the center of the axis.

Combining the errors observed for edges parallel to the X-axis yields:

$$\frac{dd_{AB}iii - dd_{AB}ikk - dd_{AB}iik + dd_{AB}iki}{4n_x} = \Delta e_x(x_i) + \frac{z_i + z_{k+1}}{2}\Delta e_y(x_i)$$
$$- \frac{y_i + y_{k+1}}{2}\Delta e_z(x_i) - \frac{\Delta y}{2}[e_z(y_i) - e_z(y_k)] + \frac{\Delta z}{2}[e_y(z_i) - e_y(z_k)] \quad (43)$$

$$\frac{dd_{AB}iii + dd_{AB}ikk - dd_{AB}iik - dd_{AB}iki}{4n_y} = \Delta e_y(x_i) - \frac{z_i + z_{k+1}}{2}\Delta e_x(x_i)$$
$$+ \frac{x_k - x_{i+1}}{2}\Delta e_z(x_i) - \Delta x\frac{2e_z(x_i) + e_z(y_i) + e_z(y_k) + e_z(z_i) + e_z(z_k)}{2} \quad (44)$$

$$\frac{dd_{AB}iii + dd_{AB}ikk + dd_{AB}iik + dd_{AB}iki}{4n_z} = \Delta e_z(x_i) + \frac{y_i + y_{k+1}}{2}\Delta e_x(x_i)$$
$$- \frac{x_k - x_{i+1}}{2}\Delta e_y(x_i) + \Delta x\frac{2e_y(x_i) + e_y(y_i) + e_y(y_k) + e_y(z_i) + e_y(z_k)}{2} \quad (45)$$

$$\frac{dd_{AB}iii - dd_{BA}ikk + dd_{BA}iik - dd_{AB}iki}{2n_x} = \Delta z[e_y(y_i) - e_y(y_k)]$$
$$- \Delta y[e_z(z_i) - e_z(z_k)]. \quad (46)$$

The first three terms of Equation (43) yield the positioning error of the X-axis at a line through the center of the workspace. The first three terms of Equations (44) and (45) yield the "straightness" error of the X-axis at a line through the center of the workzone, measured at a variable tool offset such that the tool is at the center of the X-axis. The last terms in Equations (44) and (45) result in an approximation of the integrated average pitch and yaw errors for each of the points used in the estimation.

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