

Quartic Supercyclides I: Basic Theory

Michael J. Pratt¹

*Center for Advanced Technology, Rensselaer Polytechnic Institute, Troy, NY
12180-3590, U.S.A.*

Abstract

Various classes of algebraic surfaces have been examined as to their suitability for CAGD purposes. This paper contributes further to the study of a class of quartic surfaces recently investigated by Degen, having strong potential for use in blending, and possibly also in free-form surface design. These surfaces are here put in the context of a classification of quartic surfaces originally given more than one hundred years ago. An algebraic representation is provided for them, and a simple geometric interpretation given for their rational biquadratic parametric formulation. Their theory is established from an analytic geometry viewpoint which is more straightforward than Degen's original approach and gives further useful geometric insight into their properties. A major subclass of Degen's surfaces consists of projective transforms of the Dupin cyclides; for this reason (and others, explained in the text) the name *supercyclides* is proposed for them.

Key words: Algebraic surfaces, quartics, cyclides, supercyclides, blending.

1 Introduction

The virtues of using low-degree algebraic surfaces for geometric modelling have been discussed by several authors (Bajaj, 1993; Dahmen and Thamm-Schaar, 1993; Dutta, Martin and Pratt, 1993; Sederberg, 1985). However, the question has not yet been settled as to what is the minimum degree for an algebraic surface which provides the

¹ Currently on assignment at National Institute of Standards and Technology, Mail Stop 8262, Gaithersburg, MD 20899-8262, U.S.A. (E-mail: pratt@nist.gov). The paper is based on a presentation made at the CAGD meeting held at Oberwolfach in June 1995.

most appropriate compromise between geometric flexibility for the modelling of general shapes and ease of control for the designer. It has long been recognized that the general quadrics provide insufficient freedom for free-form design. The cubic surfaces have been investigated, in particular by Sederberg (Sederberg, 1990a,b), who has suggested a method by which they may be used in the construction of piecewise-defined free-form shapes. However, this method is not intuitive for the non-mathematical designer, and, as Sederberg points out, there are certain other problems which arise with cubics, not least unexpected changes in surface topology as defining parameters are varied. While the general cubic possesses 19 degrees of freedom the general quartic has 34^2 , which poses an even greater problem in finding means for its intuitive control. For this reason investigation of the quartics for design has been restricted to special cases, notably the Steiner surfaces (Sederberg and Anderson, 1985) and the Dupin cyclides (Dutta, Martin and Pratt, 1993). The latter are easy to handle from the designer's point of view, and have been found useful for the construction of blend surfaces (Boehm, 1990; Pratt, 1990, 1995), though (Martin, de Pont and Sharrock, 1986) found that they give insufficient freedom for fully free-form design. The present paper further investigates a class of quartics identified by Degen (see Degen 1982, 1986, 1994), which proves to contain projective transformations of the Dupin cyclides and therefore permits greater freedom for modelling than those cyclides themselves. They have the advantage of naturally permitting shape modelling in terms of quadrilateral patches, as preferred in practical CAGD applications. By contrast, most other approaches to design using algebraic surfaces make use of triangular patches.

Degen's surfaces are parametrized by a conjugate net of conic curves and have the additional property that their tangent planes around any such curve envelope a quadric cone. His starting point was a paper by (Blutel, 1890), who investigated surfaces generated by a single family of conics having the tangent cone property. Degen showed that his surfaces, which he originally called *double-Blutel surfaces*, are of algebraic degree no higher than four. They include projective transformations (in general complex) of the cubic and quartic Dupin cyclides, and for this reason in his more recent papers on the topic he refers to them as *generalized cyclides*. This is an unfortunate choice of name since it may lead to confusion with the general cyclides of (Casey, 1871; Darboux, 1896), briefly discussed in Appendix A of this paper. With the exception of their special cases the Dupin cyclides, these do not possess the double-Blutel property. The name proposed here for the major subclass of Degen's surfaces is, by analogy with the superquadrics of (Barr, 1981), *supercyclides*. They appear to have significant potential for use in geometric modelling.

Degen obtained representations of the double-Blutel surfaces by means of a constructive approach, starting from the tangent cone requirement. The primary class of their

² The implicit equation of a surface of algebraic degree n contains $\frac{1}{6}(n+1)(n+2)(n+3)$ coefficients, of which one may be chosen arbitrarily.

quartic cases was subsequently characterized by (Barner, 1987) as possessing two possible generations as envelopes of families of variable quadrics, all sharing a common nondegenerate conic. This is a generalization of Dupin's original definition of his cyclide as the envelope of a variable sphere moving in constant contact with three other fixed spheres (see Dupin, 1822), the shared conic being the imaginary circle at infinity, as will be seen later. The approach taken here is analytic rather than constructive, and is based on a nineteenth-century study of a class of quartic surfaces. It is shown that this class includes the major subclass of the double-Blutel surfaces, which are thus placed in context in the established theory of quartic surfaces. The analysis is more elementary than that of either Degen or Barner, though by its nature it excludes consideration of the cubic cases covered by their treatments. A corresponding study of the other types of double-Blutel surfaces will be made in a future paper.

2 Characterization of the quartic supercyclides

The quartic algebraic surfaces were extensively studied from the point of view of analytic geometry more than a century ago. This early work contains much of interest; it has been summarized by various writers, notably (Jessop, 1916) and (Meyer, 1928). An account is given in Section 2.1 of its most important aspects for present purposes, and this will serve as the point of departure for new developments in Section 2.2.

2.1 Kummer's study of quartics with singular conics

The subclass of quartics of present interest is that of surfaces possessing a singular conic and also containing families of conics, first characterized by Kummer in 1863 as part of a wider investigation. The following description of those aspects of Kummer's work relevant in the present context is based mainly on the concise summary by (Meyer, 1928), though the material may also be found in a more dispersed form in (Jessop, 1916). Another related aspect of Kummer's work is briefly described in Appendix B.

The general implicit equation for a quartic with a double conic is

$$\phi^2 - 4p^2\chi = 0, \tag{1}$$

where $\phi = 0, \chi = 0$ are quadric surfaces and $p = 0$ is a plane. It is clear that the singular conic is the curve of intersection of $\phi = 0$ with $p = 0$. We will here take ϕ, χ and p to be homogeneous functions of the coordinates X, Y, Z, W of the complex

projective space of three dimensions S^3 . Real surfaces then result on projection from S^3 into the complex vector space V^3 with coordinates $x = X/W, y = Y/W$ and $z = Z/W$, followed by identification of cases for which x, y and z are real coordinates in its Euclidean subspace E^3 . In practice it is a simple matter to ensure the generation of real surfaces for CAGD, as will be shown later.

Quartics possessing a singular conic may also possess up to four isolated singular points. In the case of two such isolated singularities the quadratic χ has two linear factors, so that the equation of the surface becomes

$$\phi^2 - 4p^2qr = 0, \quad (2)$$

where the singular points lie on the line of intersection of the planes $q = 0, r = 0$. These planes define a pencil of planes whose axis is their line of intersection. Each plane of the pencil intersects the quartic in a pair of conics. For the planes $q = 0$ and $r = 0$ the two conics coincide, and these two planes are therefore *singular tangent planes*, sometimes referred to in older literature as *tropes*, tangent to the surface along an entire conic.

For surfaces possessing four isolated singular points, there exist two pencils of planes generating families of conic intersections. The equation of the surface³ is now

$$(st - qr - p^2)^2 - 4p^2qr = (qr - st - p^2)^2 - 4p^2st = 0. \quad (3)$$

Here the two pencils of planes are defined by the pairs of linear expressions (q, r) and (s, t) ; each such expression defines a plane tangent to the surface around a conic. For the surfaces defined by Equation (3) the singular conic is given by

$$p = 0, \quad qr - st = 0, \quad (4)$$

while the pairs of isolated singularities occur at

$$\begin{aligned} q = r = 0, \quad p^2 - st = 0, \\ s = t = 0, \quad p^2 - qr = 0. \end{aligned} \quad (5)$$

As will be shown later, the quartic Dupin cyclides belong to the class of surfaces represented by Equation (3); in their case at most one of the pairs of singular points

³ The more symmetrical form $(qr + st - p^2)^2 - 4qrst = 0$ was not apparently noted by earlier authors.

is real. If both pairs are complex the *ring* form of the surface results, while one real pair gives rise either to the *horned* or *spindle* form. Illustrations of all three forms are given by (Fladt and Baur, 1975) and (Chandru et al., 1989), showing the real points of self-intersection in the two latter cases.

2.2 Parametrization of quartics with a singular conic and four isolated singularities

A parametrization is now sought for the surfaces defined by Equation (3). It proves convenient to define a plane of the (q, r) -pencil in terms of a parameter ρ by

$$\rho^2 q - r = 0. \tag{6}$$

This plane then intersects the quartic in a curve lying on the surface obtained by substituting $r = \rho^2 q$ into Equation (3). The resulting equation factorizes in the form

$$(st - \{\rho q + p\}^2)(st - \{\rho q - p\}^2) = 0, \tag{7}$$

which represents two quadrics, confirming that the plane $\rho^2 q - r = 0$ intersects the quartic in a pair of conics, C_1 and C_2 .

A particular point on the quartic will also lie on one of the planes of the (s, t) -pencil, which will be represented in terms of a second parameter σ as

$$\sigma^2 s - t = 0. \tag{8}$$

Now the two quadrics of Equation (7) are in fact cones, having their vertices at the intersections of the planes s, t , and $\rho q \pm p$. Since both vertices lie on the axis of the (s, t) -pencil, any of its planes must intersect each cone in a pair of lines. Each such line will also intersect one of the conics C_1, C_2 ; the resulting four intersection points will lie on the quartic. To find them, we may substitute $t = \sigma^2 s$ into Equation (7). The resulting equation factorizes as

$$(p + \rho q + \sigma s)(p + \rho q - \sigma s)(p - \rho q + \sigma s)(p - \rho q - \sigma s) = 0,$$

and it therefore represents four planes. Each of them intersects $\sigma^2 s - t = 0$ along a cone generator whose intersection with $\rho^2 q - r = 0$ lies on the quartic. Three of the four resulting points on this surface may be disregarded; the others may be obtained by reversing the sign of ρ and/or σ , the effect being that of a simple reparametrization

of the quartic. The point on the surface corresponding to the parameter values ρ, σ will therefore be taken to lie on the plane

$$p + \rho q + \sigma s = 0. \quad (9)$$

Since it also lies on the planes of Equations (6) and (8) we may obtain its coordinates in terms of ρ and σ by simultaneous solution of this set of three linear equations. If we use the homogeneous coordinates of S^3 , defining $p = p_1X + p_2Y + p_3Z + p_4W$ and so on, then the system may be expressed as

$$\begin{pmatrix} \rho^2 q_1 - r_1 & \rho^2 q_2 - r_2 & \rho^2 q_3 - r_3 \\ \sigma^2 s_1 - t_1 & \sigma^2 s_2 - t_2 & \sigma^2 s_3 - t_3 \\ p_1 + \rho q_1 + \sigma s_1 & p_2 + \rho q_2 + \sigma s_2 & p_3 + \rho q_3 + \sigma s_3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -W \begin{pmatrix} \rho^2 q_4 - r_4 \\ \sigma^2 s_4 - t_4 \\ p_4 + \rho q_4 + \sigma s_4 \end{pmatrix}. \quad (10)$$

The solution for the resulting coordinate values in V^3 is given by Cramer's Rule as $x = -D_1/D_4$, $y = -D_2/D_4$ and $z = -D_3/D_4$, where D_4 is the determinant of the 3×3 matrix of Equation (10), while D_1, D_2 and D_3 are the determinants of the same matrix with its first, second and third columns respectively replaced by the column vector on the right of the equation. A cursory inspection suggests that the three coordinates are rational bicubic functions of the parameters ρ, σ , but on expansion the cubic terms in both parameters are found to cancel. These surfaces therefore have a rational biquadratic parametrization.

The most convenient way of evaluating the determinant D_4 is to express it as the sum of two determinants by splitting the elements in the first row. The new determinants have in their first rows coefficients of q and r respectively, and the first of them is multiplied by ρ^2 . Further decompositions may then be made by similarly splitting the elements of the second and third rows. Of the resulting twelve terms four are zero because their determinants exhibit repeated rows; these happen to be the terms in which ρ^3 and σ^3 occur. The result of the expansion is

$$D_4(\rho, \sigma) = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ s_1 & s_2 & s_3 \end{vmatrix} \rho^2 \sigma^2 + \begin{vmatrix} q_1 & q_2 & q_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} \rho^2 \sigma - \begin{vmatrix} q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} \rho \sigma^2 - \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ t_1 & t_2 & t_3 \end{vmatrix} \rho^2$$

$$- \begin{vmatrix} p_1 & p_2 & p_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} \sigma^2 + \begin{vmatrix} q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \\ t_1 & t_2 & t_3 \end{vmatrix} \rho - \begin{vmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} \sigma + \begin{vmatrix} p_1 & p_2 & p_3 \\ r_1 & r_2 & r_3 \\ t_1 & t_2 & t_3 \end{vmatrix}. \quad (11)$$

At this stage D_4 is defined in terms of the four singular tangent planes q, r, s and t , together with the plane p containing the singular conic of the surface. In S^3 a maximum of four of these five planes can be linearly independent. In what follows, q, r, s and t will be taken as a basis; other cases will be dealt with in a later paper. With this proviso we can express p uniquely as

$$p = \alpha q + \beta r + \gamma s + \delta t, \quad (12)$$

where $\alpha, \beta, \gamma, \delta$ are (possibly complex) scalar constants. On substitution for p in Equation (11) and decomposition of the determinants as previously it is found that many of the resulting terms vanish because their determinantal coefficients have repeated rows. The remaining terms may be collected in the form

$$D_4(\rho, \sigma) = (B_{14} + B_{24}\rho^2)(\gamma + \sigma + \delta\sigma^2) - (B_{34} + B_{44}\sigma^2)(\alpha + \rho + \beta\rho^2), \quad (13)$$

where the B_{ij} are cofactors of elements b_{ij} , $i, j \in \{1, 2, 3, 4\}$, of the matrix of coefficients of the basis planes,

$$\mathcal{B} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix}. \quad (14)$$

Turning now to the other determinants involved in the solution of Equation (10), a similar procedure shows that the equations for D_1, D_2 and D_3 have the same form as (11). The differences are that the 3×3 determinants arising lack those coefficients of the planes q, r, s and t whose suffices are 1, 2 and 3 respectively, their places being taken by coefficients with suffix 4. The column indices in these three determinants, as initially given by Cramer's Rule, thus have the orderings (4,2,3), (1,4,3) and (1,2,4). The number of column interchanges required to bring their column indices into natural or ascending order is 2, 1 and 0 respectively, each such interchange reversing the sign

of the determinant. Thus the sign associated with D_2 must be reversed on reordering the columns. Once these changes have been made the determinantal coefficients in the biquadratic expressions for D_1, D_2 and D_3 prove to be the remaining cofactors of the elements of the matrix of Equation (14). The full parametric equation of the surface, $\mathbf{Q}(\rho, \sigma) = [-D_1(\rho, \sigma), -D_2(\rho, \sigma), -D_3(\rho, \sigma), D_4(\rho, \sigma)]^T$, then takes the form

$$\mathbf{Q}(\rho, \sigma) = (\mathbf{E} + \mathbf{F}\rho^2)(\gamma + \sigma + \delta\sigma^2) - (\mathbf{G} + \mathbf{H}\sigma^2)(\alpha + \rho + \beta\rho^2), \quad (15)$$

where

$$\begin{aligned} \mathbf{E} &= (B_{11}, B_{12}, B_{13}, B_{14})^T, \\ \mathbf{F} &= (B_{21}, B_{22}, B_{23}, B_{24})^T, \\ \mathbf{G} &= (B_{31}, B_{32}, B_{33}, B_{34})^T, \\ \mathbf{H} &= (B_{41}, B_{42}, B_{43}, B_{44})^T. \end{aligned} \quad (16)$$

2.3 Significance of the vector coefficients

The determinant $|\mathcal{B}|$ of the matrix of Equation (14) can be expressed as $\sum_{j=1}^4 b_{ij}B_{ij}$, for $i = 1, 2, 3$ or 4 . On the other hand, it is known that an expansion in terms of alien cofactors, for example $\sum_{j=1}^4 b_{kj}B_{ij}$, where $k \neq i$, has the value zero since it relates to a modified determinant having two identical rows. This last fact provides a geometric interpretation for the vector-valued coefficients in the parametric representation of the surface. Consider, for example, the alien cofactor expansion

$$b_{21}B_{11} + b_{22}B_{12} + b_{23}B_{13} + b_{24}B_{14} = r_1B_{11} + r_2B_{12} + r_3B_{13} + r_4B_{14} = 0.$$

Recalling that the r_i are the coefficients defining the plane $r = 0$, we may interpret $B_{11}, B_{12}, B_{13}, B_{14}$ as the homogeneous coordinates of a point lying in that plane. The same reasoning shows that this point also lies in the planes $s = 0$ and $t = 0$. It has been assumed that q, r, s and t are linearly independent, and hence the intersection of these three planes in S^3 represents a unique point in V^3 .

Since the vector-valued coefficients $\mathbf{E}, \mathbf{F}, \mathbf{G}$ and \mathbf{H} in Equation (15) are given by $(B_{i1}, B_{i2}, B_{i3}, B_{i4})^T, i \in \{1, 2, 3, 4\}$, these coefficients therefore represent in V^3 the

vertices of the tetrahedron bounded by the four planes q, r, s and t . Two edges of this tetrahedron, the lines of intersection of the pairs of planes (q, r) and (s, t) (*i.e.* the lines through \mathbf{G}, \mathbf{H} and \mathbf{E}, \mathbf{F} respectively), play a very important role in the theory of these surfaces. They are the lines through which pass all the planes of one or other family of isoparametric conics, and they will be referred to in what follows as the *characteristic lines* of the surface. We may consider the surface to be defined either in terms of its four singular tangent planes, whose intersection points give the vectors $\mathbf{E}, \mathbf{F}, \mathbf{G}$ and \mathbf{H} , or more directly in terms of those points themselves, which, taken in sets of three, define the planes. The scalar constants α, β, γ and δ appearing in Equation (15) also affect the surface definition, since they determine the plane of its singular conic. Their influence will be discussed in more detail later in the paper.

The identification of the vector coefficients with specific points in S^3 permits a straightforward geometric interpretation of Equation (15). The first factors in each of the two terms on the right-hand side represent variable points on the lines through \mathbf{E}, \mathbf{F} and \mathbf{G}, \mathbf{H} respectively. The right-hand side as a whole then gives a further point on the line through these two variable points, since it is merely a linear combination of them. This final point lies on the surface.

2.4 The boundary cone and tangent cone properties

Several properties of the surfaces represented by Equation (15) make them particularly suitable for use in CAGD. The most important is the double tangent cone characteristic originally sought by (Degen, 1986), whose proof that it holds for these surfaces is reviewed in this section for the sake of completeness.

We already know that the isoparametric curves are conics. Consider the two particular curves given by $\sigma = \sigma_1$ and $\sigma = \sigma_2$, where σ_1, σ_2 are constants. These may be written as

$$\begin{aligned}\mathbf{Q}(\rho, \sigma_1) &= (\mathbf{E} + \mathbf{F}\rho^2)(\gamma + \sigma_1 + \delta\sigma_1^2) - (\mathbf{G} + \mathbf{H}\sigma_1^2)(\alpha + \rho + \beta\rho^2), \\ \mathbf{Q}(\rho, \sigma_2) &= (\mathbf{E} + \mathbf{F}\rho^2)(\gamma + \sigma_2 + \delta\sigma_2^2) - (\mathbf{G} + \mathbf{H}\sigma_2^2)(\alpha + \rho + \beta\rho^2).\end{aligned}$$

Then

$$\begin{aligned}(\gamma + \sigma_2 + \delta\sigma_2^2)\mathbf{Q}(\rho, \sigma_1) - (\gamma + \sigma_1 + \delta\sigma_1^2)\mathbf{Q}(\rho, \sigma_2) = \\ (\alpha + \rho + \beta\rho^2)\{(\gamma + \sigma_1 + \delta\sigma_1^2)(\mathbf{G} + \mathbf{H}\sigma_2^2) - (\gamma + \sigma_2 + \delta\sigma_2^2)(\mathbf{G} + \mathbf{H}\sigma_1^2)\}.\end{aligned}\quad (17)$$

The left-hand side of Equation (17) represents a point lying on the line through points

on the two conics having the same value of ρ . But the right-hand side represents a point in V^3 whose position depends upon the choice of σ_1, σ_2 but not upon ρ , since all components in S^3 are multiplied by the same scalar function of ρ . We conclude that all lines through corresponding points on the two curves pass through this particular point, which furthermore lies on the characteristic line defined by \mathbf{G} and \mathbf{H} , since \mathbf{E} and \mathbf{F} are absent from the equation. Thus (as pointed out by Degen) points having equal parameter values on the two isoparametric curves are in a perspective relationship with respect to a centre lying on one of the characteristic lines. The family of lines through corresponding points on the two curves are therefore generators of a quadric cone with vertex on the characteristic line. Symmetry in ρ and σ implies that the same property holds for pairs of isoparametric curves of constant ρ . A significant consequence of this result is that a supercycloids patch bounded by two pairs of isoparametric conics has coplanar corners, since the corners taken in adjacent pairs lie on two generators of the cone, which are coplanar since they meet at the vertex.

Even more importantly, it is evident that in the limit as $\sigma_2 \rightarrow \sigma_1$ the cone becomes a tangent cone, still having its vertex on the characteristic line. Analytically, on taking the limit we obtain the result

$$\mathbf{Q}(\rho, \sigma_1) - \frac{\gamma + \sigma_1 + \delta\sigma_1^2}{1 + 2\delta\sigma_1} \frac{\partial \mathbf{Q}}{\partial \sigma}(\rho, \sigma_1) = (\alpha + \rho + \beta\rho^2) \left\{ \frac{2\gamma\sigma_1 + \sigma_1^2}{1 + 2\delta\sigma_1} \mathbf{H} - \mathbf{G} \right\} \quad (18)$$

Here the left-hand side represents a line through the point $\mathbf{Q}(\rho, \sigma_1)$ in the tangent direction of the other isoparametric curve through that point. The right-hand side once again gives a point which in V^3 is independent of ρ . We conclude that all cross-tangents of the curve $\mathbf{Q}(\rho, \sigma_1)$ pass through this point. The cross-tangents are therefore generators of a cone which, since the curve is a conic, must be a quadric cone. As before, symmetry implies that isoparametric curves of constant ρ also possess tangent cones. Thus the surfaces represented by Equation (15) all possess the double tangent cone property. In the case of the Dupin cyclides the tangent cones are right circular cones, and it has been shown elsewhere (Boehm, 1990; Pratt, 1990, 1995) how this property can be used in the construction of blend surfaces. The supercycloids possess a generalization of that property which should render them still more powerful for the same purpose.

Possession of the tangent cone property necessarily requires the two families of conics to form a conjugate net on the surface. A necessary and sufficient condition for conjugacy is that at all points on any curve from either family the tangents to curves of the other family generate a developable surface (Weatherburn, 1927). Cones being developable, this property clearly holds for the supercycloids.

2.5 Isolated singular points of the supercyclide

The treatment here also follows that of (Degen, 1994). Suppose now that ρ_1 is a zero of $\alpha + \rho + \beta\rho^2$. Equation (15) gives the isoparametric conic for this value of ρ as

$$\mathbf{Q}(\rho_1, \sigma) = (\mathbf{E} + \mathbf{F}\rho_1^2)(\gamma + \sigma + \delta\sigma^2),$$

representing a fixed point in V^3 , since all four components of the homogeneous vector contain as a factor the same scalar function of σ . Thus the zeros of $\alpha + \rho + \beta\rho^2$ give two of the isolated singular points of the surface, through which all isoparametric curves of $\sigma = \text{constant}$ pass. They clearly lie on the characteristic line through \mathbf{E} and \mathbf{F} . Conversely, the zeros of $\gamma + \sigma + \delta\sigma^2$ give the other two isolated singular points, lying on the characteristic line through \mathbf{G} and \mathbf{H} . It is evident that the singular points may be either real or complex.

2.6 Inverse parametrization of the surface

A requirement frequently arising in CAD applications is that of determining the parameter values associated with a given point x, y, z lying on a surface. It is generally necessary to use an iterative method, but (de Pont, 1984) showed simple closed form solutions to exist in the particular case of the quartic Dupin cyclides. His formulae are quoted in the next section, after the necessary notation has been introduced. Corresponding results for the cubic Dupin cyclides were recently found by (Srinivas and Dutta, 1995).

Although no closed inverse parametrization formulae have so far been derived for the supercyclides, it is easy to see that an exact process for computing parameter values exists. A specified point on the surface is known to lie on the two planes given by Equations (6) and (8). The planes q, r, s and t may be assumed either known or easily determined from $\mathbf{E}, \mathbf{F}, \mathbf{G}$ and \mathbf{H} . Then it is only necessary to determine the values of ρ^2 and σ^2 giving the unique plane from either pencil containing the given point⁴; for each pencil this is a linear problem. With ρ^2 and σ^2 known there are two possible values for each parameter; the sign ambiguities may be resolved by selecting the values satisfying Equation (9).

⁴ If the plane is not unique the point lies on the axis of the pencil; it is then a singular point whose parameter values may be found as described in the last section.

2.7 Correlation with the results of Degen

As mentioned earlier, Degen adopted a constructive approach to finding classes of surfaces having double possession of the Blutel tangent cone property. Although it was derived by very different means, Equation (78) of (Degen, 1986), representing the ‘normal form’ of the major subclass of double-Blutel surfaces, is precisely equivalent to Equation (15) of this paper.

The present analysis has therefore provided an alternative means of deriving the equations of this particular subclass of double-Blutel surfaces, and has placed them in context in the established theory of quartic surfaces. Additionally, a clear interpretation has emerged of the rôle of the vector-valued coefficients in Degen’s normal form — Equation (15) of this paper — as the intersection points of a set of four linearly independent singular tangent planes of the surface. The approach has also given a simple implicit representation for the surfaces, expressed in terms of those singular tangent planes.

In (Degen, 1986, 1994) it is stated that, apart from some special cases, the double-Blutel surfaces are projective transforms of the Dupin cyclides. We will show in the ensuing sections that the surfaces defined by Equation (15) are in fact rather more general than this implies, though they certainly contain the projectively transformed Dupin cyclides as a major subclass. For this and other reasons noted in Section 1 the name *supercyclides* is proposed for them here. The relationship of the supercyclides to the general cyclides of Casey and Darboux is explained in Appendix A.

Degen’s exceptional cases will be briefly discussed in Section 3.2. They will be analyzed from the present point of view in a future paper.

To conclude this section its primary result, apparently not discovered by Kummer or any of his near contemporaries, will be restated for emphasis as follows:

Theorem *Given a quartic surface possessing a singular conic and four non-coplanar isolated singular points, previously known to bear two families of conic curves,*

- (1) *along any conic from either family the surface is tangent to a quadric cone whose vertex lies on a line through one of the pairs of singular points, and*
- (2) *the families of conics consequently form a conjugate net.*

These surfaces are identical with the major class of Degen’s double-Blutel surfaces, characterized by having skew characteristic lines.

Remark *The non-coplanarity of singular points implies that the surface has skew characteristic lines, which in turn requires the four singular tangent planes to be linearly independent. The theorem will be generalized in a future paper to cover Degen's remaining cases.*

It may be noted that all the properties of these surfaces listed in Degen's most recent paper on the subject (Degen, 1994) have emerged from the analysis of this section with one exception. This concerns the position of the central control point in the Bézier representation of a supercycloidal patch, which proves to be at the common point of intersection of the four corner tangent planes of the patch. Degen originally showed this geometrically, by using perspective relationships between sets of control points. A theorem due to the present author (Pratt, 1995) provides an alternative algebraic proof.

3 Relation to the Dupin cyclides

The properties of the Dupin cyclides pertinent for CAGD have been described by several authors, including (Chandru et al., 1989; Dutta et al., 1993) and the present writer (Pratt, 1990, 1995). (Forsyth, 1912) gives parametric equations for their quartic forms in standard position and orientation, expressible in homogeneous terms as

$$\begin{aligned}
 X &= \mu(c - a \cos \theta \cos \psi) + b^2 \cos \theta, \\
 Y &= b \sin \theta(a - \mu \cos \psi), \\
 Z &= b \sin \psi(c \cos \theta - \mu), \\
 W &= a - c \cos \theta \cos \psi,
 \end{aligned}
 \tag{19}$$

where $\theta, \psi \in [0, 2\pi)$. In these equations values of the four constants a, b, c and μ (the first three being related by $a^2 = b^2 + c^2$) distinguish a particular member of this family of surfaces. They may all be taken as non-negative with no loss of generality since, as pointed out by (de Pont, 1984), reversal of the sign of any one of them is equivalent merely to a reparametrization of the same surface.

It was mentioned earlier that formulae exist for the inverse parametrization of the quartic Dupin cyclides. (De Pont, 1984) gives the following results corresponding to

the equations above:

$$\tan \theta = \frac{by}{ax - c\mu}, \quad \sin \psi = \frac{bz}{cx - a\mu}. \quad (20)$$

Here x, y, z are the Euclidean coordinates of a point on the surface.

The trigonometric parametrization given above can be converted into a rational bi-quadratic one, and rational Bézier and B-spline formulations have been published by several authors (Boehm, 1990; Pratt, 1990; Zhou and Strasser, 1992).

The quartic Dupin cyclides may also be represented by either of the equivalent implicit equations (see Forsyth, 1912)

$$(X^2 + Y^2 + Z^2 + \{b^2 - \mu^2\}W^2)^2 - 4W^2(\{aX - c\mu W\}^2 + b^2Y^2) = 0 \quad (21)$$

or

$$(X^2 + Y^2 + Z^2 - \{b^2 + \mu^2\}W^2)^2 - 4W^2(\{cX - a\mu W\}^2 - b^2Z^2) = 0. \quad (22)$$

In the projective completion of V^3 , $W = 0$ is the plane at infinity, and it is clear that the intersection of the cyclide with this plane can be represented by

$$X^2 + Y^2 + Z^2 = W = 0,$$

the *imaginary circle at infinity*⁵. This curve is a double curve, since when $W = 0$ the quadratic factor appears raised to the power two; it is therefore a singular curve of self-intersection of the surface. The possession of the imaginary circle at infinity as a double curve characterizes the quartic cases of the *general cyclides* of Casey and Darboux (see Appendix A).

Note that if $a = b = c = 0$ in the equation for the Dupin cyclide it reduces to the equation of a (double) sphere of radius μ , centred at the origin. The significance of this will become apparent later.

It is easy to see that the Dupin cyclide is a special case of the class of surfaces under consideration by comparing Equations (3) with (21) and (22). The two sets

⁵ Any sphere in V^3 , regarded as a quadric in S^3 , intersects the plane at infinity in this imaginary circle (Pedoe, 1988).

of equations have precisely the same form, and the following identifications may be made immediately:

$$q = aX + ibY - c\mu W, \quad r = aX - ibY - c\mu W, \quad (23)$$

$$s = cX + bZ - a\mu W, \quad t = cX - bZ - a\mu W, \quad (24)$$

Note that the two singular tangent planes q and r are complex conjugates for the Dupin cyclide, though the equation of the surface is real since in its algebraic formulation they occur as the product qr . Their intersection gives a real characteristic line $aX - c\mu W = 0$, $Y = 0$.

Substitution of Equations (23) and (24) into Equation (3) reveals that the plane of the singular curve, which we know to be the plane at infinity, must be represented as

$$p = b^2 W. \quad (25)$$

The factor b^2 arises because an arbitrary choice of one coefficient in (3) has already been made in the specification of the other planes. We need a representation of p in terms of q, r, s and t , and elimination of X, Y and Z from Equations (23) and (24) gives

$$p = \frac{1}{2\mu}(c\{q + r\} - a\{s + t\}). \quad (26)$$

Then for the Dupin cyclide the coefficients in Equation (12) become

$$\alpha = \beta = c/2\mu, \quad \gamma = \delta = -a/2\mu. \quad (27)$$

The elements of the vectors **E**, **F**, **G** and **H** are given, as we have seen, by the cofactors of the matrix defined by Equation (14) after substitution of the appropriate coefficients from Equations (23) and (24). It should be noted that simply solving subsets of the latter equations to find intersection points of those planes does not generally give vectors consistently scaled for substitution into Equation (15), and use of the determinantal method is therefore advisable. After evaluation of **E**, **F**, **G** and **H** and substitution of the values of the scalar constants from Equation (27), we find that

(15) becomes

$$\mathbf{Q}(\rho, \sigma) = \left\{ \begin{pmatrix} iab^2 \\ b^3 \\ 0 \\ ib^2c/\mu \end{pmatrix} + \begin{pmatrix} iab^2 \\ -b^3 \\ 0 \\ ib^2c/\mu \end{pmatrix} \rho^2 \right\} (2\mu\sigma - a\{1 + \sigma^2\})$$

$$+ \left\{ \begin{pmatrix} ib^2c \\ 0 \\ -ib^3 \\ iab^2/\mu \end{pmatrix} + \begin{pmatrix} ib^2c \\ 0 \\ ib^3 \\ iab^2/\mu \end{pmatrix} \sigma^2 \right\} (2\mu\rho + c\{1 + \rho^2\}). \quad (28)$$

Note that the foregoing equation is not valid for $\mu = 0$; this case will be examined in a future paper. For $\mu \neq 0$, expansion gives the following results:

$$X = ib^2\{2\mu[a(1 + \rho^2)\sigma + c\rho(1 + \sigma^2)] - b^2(1 + \rho^2)(1 + \sigma^2)\},$$

$$Y = b^3(1 - \rho^2)\{2\mu\sigma - a(1 + \sigma^2)\},$$

$$Z = -ib^3(1 - \sigma^2)\{2\mu\rho + c(1 + \rho^2)\},$$

$$W = 2ib^2\{a\rho(1 + \sigma^2) + c(1 + \rho^2)\sigma\}.$$

These become identical with Equations (19), apart from a common scalar factor which may be disregarded, if the following identifications are made:

$$\frac{1 + \rho^2}{2\rho} = -\cos \theta, \quad \frac{i(1 - \rho^2)}{2\rho} = \sin \theta, \quad (29)$$

$$\frac{2\sigma}{1 + \sigma^2} = \cos \psi, \quad \frac{1 - \sigma^2}{1 + \sigma^2} = \sin \psi. \quad (30)$$

Conversely, the parameters ρ, σ are related to Forsyth's parameters θ, ψ by

$$\rho = -\cos \theta + i \sin \theta = -e^{-i\theta}, \quad (31)$$

$$\sigma = \frac{1 - \sin \psi}{\cos \psi} = \tan \frac{1}{2} \left(\frac{\pi}{2} - \psi \right). \quad (32)$$

For more general supercyclides, different substitutions may be required to obtain real forms of the equations; thus there is no virtue in making these particular replacements at an earlier stage in the analysis. On substituting for q, r, s, t, ρ and σ from Equations (23), (24), (31) and (32) into Equations (6) and (8) we obtain the planes of the isoparametric circles of the Dupin cyclide in terms of θ and ψ correctly as

$$bY \cos \theta - (aX - c\mu W) \sin \theta = 0,$$

$$bZ - (cX - a\mu W) \sin \psi = 0.$$

The application of the theory of Section 2 to the Dupin cyclide has provided a check on the correctness of the development, and we now examine the relationship between the Dupin cyclides and the supercyclides.

3.1 Projective transformations of Dupin cyclides

Throughout this section, asterisks will be used to label entities relating to Dupin cyclides. It has been shown that these surfaces are particular cases of supercyclides, being expressible in the form of Equation (3). Application of a nonsingular projective transformation maps the defining planes p^*, q^*, r^*, s^*, t^* onto the corresponding planes p, q, r, s, t of a transformed surface, which, since we are merely replacing one set of planes by another in an implicit equation of supercyclide form, will also be a supercyclide. The linear independence of q^*, r^*, s^* and t^* implies that of q, r, s and t . The planes p^* and p are given by Equation (26) with and without asterisks respectively, repeated here for convenience in the transformed case:

$$p = \frac{1}{2\mu}(c\{q + r\} - a\{s + t\}). \quad (33)$$

Comparison with Equation (12), which expresses p uniquely in terms of the four coefficients α, β, γ and δ , suggests at first sight that the only supercyclides which are projectively transformed Dupin cyclides are those for which the two *consistency conditions* $\alpha = \beta, \gamma = \delta$ are satisfied. However, given a general supercyclide as defined by Equation (15) it is in many cases possible, as shown below, to find a parameter scaling which will bring the relation between five equivalent planes into the form of Equation (33) above.

Suppose that $\alpha \neq \beta, \gamma \neq \delta$, and that for the moment none of these coefficients is zero. It is easy to verify that setting

$$\rho' = (\beta/\alpha)^{1/2} \rho, \quad \sigma' = (\delta/\gamma)^{1/2} \sigma \quad (34)$$

in Equation (15) then leads to the result

$$\mathbf{Q}(\rho, \sigma) = \mathbf{Q}(\rho', \sigma') = \left(\frac{\alpha\gamma}{\beta\delta} \right)^{1/2} \left\{ (\mathbf{E}' + \mathbf{F}'\rho'^2)(\gamma' + \sigma' + \delta'\sigma'^2) - (\mathbf{G}' + \mathbf{H}'\sigma'^2)(\alpha' + \rho' + \beta'\rho'^2) \right\}, \quad (35)$$

in which the modified vector and scalar coefficients are

$$\begin{aligned} \mathbf{E}' &= (\beta/\alpha)^{1/2} \mathbf{E}, & \mathbf{F}' &= (\alpha/\beta)^{1/2} \mathbf{F}, \\ \mathbf{G}' &= (\delta/\gamma)^{1/2} \mathbf{G}, & \mathbf{H}' &= (\gamma/\delta)^{1/2} \mathbf{H}, \end{aligned} \quad (36)$$

$$\alpha' = \beta' = (\alpha\beta)^{1/2}, \quad \gamma' = \delta' = (\gamma\delta)^{1/2}.$$

Equation (35) has the same form as Equation (15), but the coefficients $\alpha', \beta', \gamma', \delta'$ now form two consistent pairs. In V^3 the vector coefficients represent the same points as previously, since they have simply been multiplied by scalars. The surface itself is not changed in V^3 by the occurrence of a scalar factor on the right-hand side of Equation (35).

The above formulae remain valid when one of the original pairs of coefficients is consistent; in this case the related scaling multipliers reduce to unity (the ratio of two consistent zero coefficients should also be taken as unity in this context).

It therefore proves possible, when zero coefficients are absent or occur in a consistent pair, to convert an inconsistent example into a consistent one. In such a case the geometry of the original supercyclide is consistent with that of a transformed Dupin cyclide but the two parametrizations are inconsistent. The discrepancy is remedied by the parameter scaling given above.

It is instructive to examine the effect of the change of parameter from the algebraic point of view. The plane of the singular conic, expressed in terms of both the old and the new sets of coefficients, may be written as

$$p = \alpha q + \beta r + \gamma s + \delta t = \alpha' q' + \beta' r' + \gamma' s' + \delta' t' = p'.$$

Substitution of the new coefficients from Equations (36) then shows that

$$q' = (\alpha/\beta)^{1/2} q, \quad r' = (\beta/\alpha)^{1/2} r, \quad (37)$$

$$s' = (\gamma/\delta)^{1/2} s, \quad t' = (\delta/\gamma)^{1/2} t. \quad (38)$$

More generally, we could set p' equal to some scalar multiple of p , but this proves merely to multiply the implicit equation of the surface through by a constant. It is a simple matter to check that Equation (3) is unaffected when the original planes are replaced by their primed or scaled versions. Hence, by appropriately scaling the linear functions defining the basis planes with respect to each other an inconsistent case of the type under consideration may be converted into a consistent formulation of the same surface. The overall geometry of the configuration of singular tangent planes, the singular conic and hence the surface itself are unchanged in V^3 ; only the parametrization is modified.

However, if one or both of the inconsistent coefficient pairs contains a zero coefficient this transformation is no longer possible, since the relevant scaling factors become either zero or infinite. It is well known that more general rational linear parameter transformations preserve the degree of rational functions, but their only forms which also preserve the canonical form of Equation (15) are found to be parameter scalings of the type used above and inverse relationships of the form $\rho' = k/\rho$, where k is a constant. The effect of the latter is simply to reverse the roles of the related pairs of scalar or vector coefficients in each of the bracketed factors in Equation (15). It appears, then, that there is no way of converting a case of this type into a consistent one. The supercyclides may therefore be categorized into three major classes, as follows:

- 1) Regular cases:** These arise when the consistency conditions $\alpha = \beta$ and $\gamma = \delta$ are satisfied. Then, as we saw above, the supercyclide can be obtained directly by applying a nonsingular projective transformation to a Dupin cyclide with appropriately chosen (but non-unique) values of a, c and μ . Three subcases are worth mentioning:
- (a) $\alpha = \beta = 0, \quad \gamma = \delta \neq 0$. Equation (33) shows that the first condition is equivalent to $c = 0$, in which case the original Dupin cyclide is a torus having the z -axis as its axis of symmetry.
 - (b) $\alpha = \beta \neq 0, \quad \gamma = \delta = 0$. The original cyclide now has $a = 0$ but $c \neq 0$, giving a torus having the y -axis as its axis of symmetry.
 - (c) $\alpha = \beta = \gamma = \delta = 0$. For this case the singular tangent plane is a null plane. On setting $p = 0$ in Equation (3) we obtain

$$(qr - st)^2 = 0,$$

which is clearly the equation of a quadric, doubly generated. This is consistent with the fact that when $a = b = c = 0$ and $\mu \neq 0$ the Dupin cyclide becomes a double sphere, it being well known that projective transformation of a sphere gives rise to a quadric.

2) Semi-regular cases: One or both consistency conditions is violated; two of the coefficients may be zero provided they form a consistent pair.

The method for converting a semi-regular case into a regular one by means of a parameter scaling was given above.

3) Irregular cases: One or both of the consistency conditions is violated; one or both inconsistent pairs contains a zero coefficient.

These are the cases which cannot be generated by projective transformation of a Dupin cyclide. Here again, there are three subcases worthy of note (in each of them a typical example is given, the results of three further possibilities being analogous):

- (a) One nonzero coefficient; assume this to be α . The singular curve is given by Equations (4). In this case its plane is $p = \alpha q = 0$, and the curve is therefore the intersection of $q = 0$ with the quadric $qr - st = 0$. Then it also lies on the surface $st = 0$, a pair of planes whose intersection with $q = 0$ is a pair of coplanar lines. Since the planes q, r, s, t are assumed to be linearly independent these lines are non-coincident; they intersect at the common point of q, s and t , *i.e.* at the point represented by the vector \mathbf{F} .
- (b) Two nonzero coefficients. We have already dealt with the cases where $\alpha = \beta = 0$ and $\gamma = \delta = 0$. The remaining possibility is that one coefficient from each pair is zero. We will take $\beta = \delta = 0$. Then the plane of the singular curve is $p = \alpha q + \gamma s = 0$; substitution for s from this equation into $qr - st = 0$ shows that their intersection lies on the surface $q(\alpha r + \gamma t) = 0$, a pair of planes. Then once again the singular conic is a pair of coplanar non-coincident lines, the intersections of $\alpha q + \gamma s = 0$ with $q = 0$ and with $\alpha r + \gamma t = 0$. The first of these is in fact the line of intersection of $q = 0$ and $s = 0$, which passes through the points \mathbf{F} and \mathbf{H} .
- (c) Three nonzero coefficients. Assume that $\delta = 0$; then the plane of the singular curve is $\alpha q + \beta r + \gamma s = 0$. In this case it appears that the singular conic is irreducible.

An analysis of the relationship between the isolated singular points and the singular conic of the irregular supercyclides reveals why these cases are inherently distinct. There are three possible subtypes, in which respectively three, two and one of the coefficients $\alpha, \beta, \gamma, \delta$ are nonzero. Where two are zero they must be one each from (α, β) and from (γ, δ) , otherwise the surface is regular or semi-regular, as mentioned above. It is easy to show from Equations (4) and (5) that the three subcases have respectively one, two and three singular points lying on the singular conic. In the case of two, these are one from each pair⁶. By contrast, the Dupin cyclide can have zero or two singular points lying on its singular conic; if two, they must belong to the same pair. Thus the incidence characteristics of the corresponding configurations are different for the Dupin cyclide and the irregular supercyclide, showing that the latter

⁶ Recall that the isolated singular points lie two on each of the characteristic lines of a supercyclide.

may not be derived from the former by application of a projective transformation.

In all three of the cases described in this section it is possible for the singular curve to belong to one of the families of isoparametric conics. This requires that its plane p belongs to one of the pencils defined by (q, r) or (s, t) . The two cases require respectively that in Equation (12) $\gamma = \delta = 0$ or $\alpha = \beta = 0$. The planes of the isoparametric curves are given by Equations (6) and (8), and they yield the parameter value associated with the singular curve as $\rho = (-\alpha/\beta)^{1/2}$ or $\sigma = (-\gamma/\delta)^{1/2}$, depending on which pair of coefficients is nonzero.

A final point is that the distinction between regular and semi-regular cases is somewhat arbitrary. The ‘canonical’ parametrization expressed by Equation (15) results from the representations chosen earlier for the planes of the isoparametric conics. These are given in terms of ρ and σ respectively by Equations (6) and (8); a different choice would have resulted in a different canonical parametrization. Since the definition of semi-regularity depends on this choice it is clear that the only fundamental distinction is that between regular and irregular supercyclides.

3.2 Degen’s special cases

The supercyclides studied in this paper satisfy Equation (15), and belong to the major subclass of quartic double-Blutel surfaces with skew characteristic lines. (Degen, 1986) mentions three forms having different parametric representations. Firstly, he defines a class of ‘special’ surfaces, whose characteristic lines intersect. In the case of transformed Dupin cyclides, these prove to correspond to the case of $\mu = 0$, for which Equation (28) was earlier noted to be invalid. Secondly, there are what Degen refers to as ‘singly degenerate’ forms, having a character in some sense midway between the quartic and cubic cases. Finally, there are ‘doubly degenerate’ surfaces, which Degen shows to be algebraically cubic. A full analysis of these additional forms of double-Blutel surfaces is nearing completion, and will form the subject of a future paper.

4 Examples of supercyclides

Two simple examples of supercyclides are illustrated in this section. Both are regular cases, obtainable by transformation of a Dupin cyclide. In the first case, shown in Figure 1, the transformation is a real inhomogeneous scaling, resulting in a surface whose isoparametric curves are ellipses; such a surface may be termed an *ellipside*.

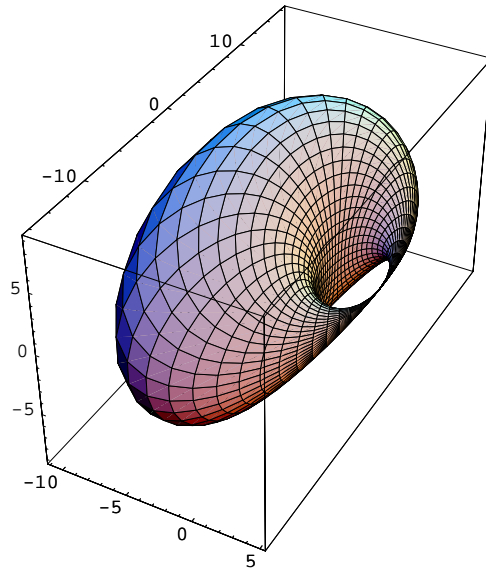


Fig. 1. An ellipsoid.

The second example, shown in Figure 2, is the surface generated by revolving the hyperbola $x^2 - 2xz - z^2 = 1$ about the z -axis. This is not one of the axes of symmetry of the hyperbola, and a double curve therefore arises, as the intersection of the sheets swept out by its two branches. Only one sheet of the supercyclide is depicted in the figure. It is easy to show that the surface is algebraically quartic, and that its double curve is the unit circle. Although it is a real surface, this supercyclide results from the application of a *complex* transformation to a Dupin cyclide — details are given in (Pratt, 1996). The complex transformation has the effect of mapping the double conic

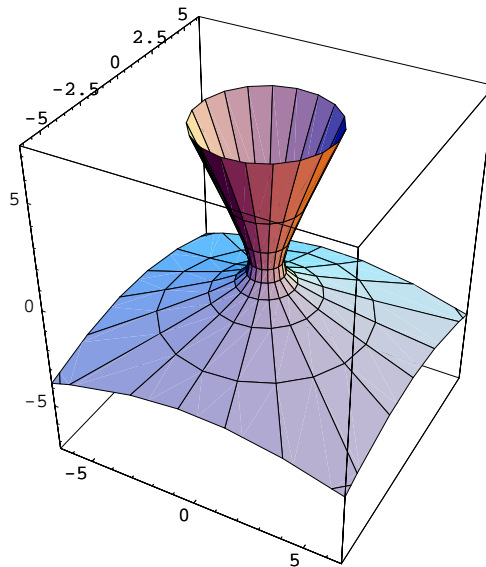


Fig. 2. A supercyclide of rotation.

of the Dupin cyclide, the circle at infinity, onto the unit circle. No real transformation can achieve this effect.

5 Discussion and conclusions

In this paper further light has been thrown on certain quartic cases of the double-Blutel surfaces investigated by Degen. In particular their place in the general classification of quartics has been identified and a comparatively straightforward method developed for the study of their properties. An implicit equation for them has been given, expressed in terms of their four singular tangent planes, and a geometric interpretation has been given of their parametric form. Additionally, their relation to the Dupin cyclides has been analyzed in some detail.

Although the analysis given has covered both real and complex forms of supercyclides it is easy to ensure the generation of real surfaces for practical applications. It is likely that implementations can most easily be made in terms of Bézier or B-spline formulations. Then, for example, specification of four real boundary curves satisfying the necessary conditions for a supercyclide patch will imply that all eight of its peripheral Bézier control points are real. The property mentioned at the end of Section 2.7 also gives a real position for the remaining central control point, and the patch as a whole will therefore be real.

It is already established that the Dupin cyclide is useful as a blending surface because of its tangent cone property. This allows the G^1 matching of two cyclide patches across a common boundary by ensuring that both are tangent to the same cone there (Pratt, 1990). For the Dupin cyclide the cone is a right circular cone, but the corresponding property for the supercyclides involves a general quadric cone. This fact, together with their much increased geometric freedom, shows that they have even greater potential as flexible blending surfaces in CAGD. It is also possible that they will provide sufficient geometric freedom for use in fully free-form surface design. In this context they have the advantage of naturally giving rise to four-sided patches, as used in existing practical CAD systems, rather than three-sided patches as generated by most other approaches to algebraic surface modelling. The existence of exact solutions to the inverse parametrization problem, as mentioned in Section 2.6, is an added advantage for the CAD use of supercyclides.

These surfaces have the further property that patches bounded by isoparametric curves on them have coplanar corners (see Section 2.4). This makes it almost trivial to generate approximate representations to any desired accuracy in terms of planar quadrilateral facets. Such a characteristic is useful for many purposes, including the efficient generation of graphical renderings. It also has potential as the basis of a

subdivision method for computing intersections between supercylide surfaces.

Apart from the Dupin cyclides, only one other class of quartic surfaces has been examined for its potential use in CAGD, that of the Steiner surfaces (Sederberg and Anderson, 1985). These are characterized by the possession of three double lines which meet in a point (Jessop, 1916), and are therefore quite distinct from the supercylides.

Much work remains to be done in the study of supercylides. On the theoretical side the remaining quartic cases of Degen's double-Blutel surfaces need further examination, as do the cubic cases. More importantly, algorithms for constructing blends and free-form surfaces from supercylides need to be established. Some preliminary work in the blending area is reported in (Degen 1994), and an alternative approach is suggested in (Allen and Dutta, 1997). The present author is preparing a companion paper to this one, in which the remaining forms of supercylides are analysed, and intends to follow this with application-related studies along the lines indicated.

Acknowledgements

The author is grateful to Professor W. L. F. Degen for a suggestion regarding the characterization of irregular supercylides.

References

- [1] S. Allen and D. Dutta. Supercylides and blending. *Computer Aided Geometric Design*, **14**, 7, 637 – 651, 1997.
- [2] C. L. Bajaj. The emergence of algebraic curves and surfaces in geometric design. In R. R. Martin, editor, *Directions in Geometric Computing*. Information Geometers Press, 1993.
- [3] M. Barner. Eine differentialgeometrische Kennzeichnung der allgemeinen Dupinschen Zykliden. *Aequationes Mathematicae*, **34**, 277 – 286, 1987.
- [4] A. H. Barr. Superquadrics and angle-preserving transformations. *IEEE Computer Graphics & Applications*, **1**, 1, 11 – 23, 1981.
- [5] E. Blutel. Recherches sur les surfaces qui sont en même temps lieux de coniques et enveloppes de cônes du second degré. *Ann. Sci. Ecole Norm. Sup.*, **3**, 7, 155 – 216, 1890.
- [6] W. Boehm. On cyclides in geometric modeling. *Computer Aided Geometric Design*, **7**, 1 – 4, 243 – 255, 1990.

- [7] J. Casey. On cyclides and sphero-quartics. *Phil. Trans. Roy. Soc. London*, **CLXI**, 585 – 721, 1871.
- [8] V. Chandru, D. Dutta, and C.M. Hoffmann. On the geometry of Dupin cyclides. *The Visual Computer*, **5**, 277 – 290, 1989.
- [9] W. Dahmen and T.-M. Thamm-Schaar. Cubicoids: Modelling and visualization. *Computer Aided Geometric Design*, **10**, 2, 93 – 108, 1993.
- [10] G. Darboux. *Sur une Classe Remarquable de Courbes et de Surfaces Algébriques*. Gauthier-Villars, Paris, 2nd edition, 1896.
- [11] J. J. de Pont. *Essays on the Cyclide Patch*. PhD thesis, Cambridge University Engineering Department, 1984.
- [12] W. L. F. Degen. Surfaces with a conjugate net of conics in projective space. *Tensor, N.S.*, **39**, 167 – 172, 1982.
- [13] W. L. F. Degen. Die zweifachen Blutelschen Kegelschnittflächen. *Manuscripta Mathematica*, **55**, 9 – 38, 1986.
- [14] W. L. F. Degen. Generalised cyclides for use in computer aided geometric design. In A. D. Bowyer, editor, *Computer Aided Surface Geometry and Design*. Oxford University Press, 1994. (Proc. 4th IMA Conf. on the Mathematics of Surfaces, Bath, England, Sept. 1990).
- [15] C. Dupin. *Applications de Géométrie et de Mécanique à la Marine, aux Ponts et Chaussées, etc.* Bachelier, Paris, 1822.
- [16] D. Dutta, R. R. Martin, and M. J. Pratt. Cyclides in surface and solid modeling. *IEEE Computer Graphics & Applications*, **13**, 1, 53 – 59, 1993.
- [17] A. Fladt and K. Baur. *Analytische Geometrie spezieller Flächen und Raumkurven*. Vieweg & Sohn, Braunschweig, 1975.
- [18] A. R. Forsyth. *Lectures on the Differential Geometry of Curves and Surfaces*. Cambridge University Press, 1912.
- [19] C. M. Jessop. *Quartic Surfaces with Singular Points*. Cambridge University Press, 1916.
- [20] R. R. Martin, J. de Pont, and T. J. Sharrock. Cyclide surfaces in computer aided design. In J. A. Gregory, editor, *The Mathematics of Surfaces*. Oxford University Press, 1986. (Proc. 1st IMA Conf. on the Mathematics of Surfaces, Manchester, England, Sept. 1984).
- [21] W. Fr. Meyer. Flächen vierter und höherer Ordnung. In W. Fr. Meyer and H. Mohrmann, editors, *Encyklopädie der Mathematischen Wissenschaften*. B. G. Teubner, Leipzig, 1921 - 1934. (Vol. 3, Part 2, Section 2B, pp. 1533 – 1779).
- [22] D. Pedoe. *Geometry: A Comprehensive Course*. Dover, 1988.

- [23] M. J. Pratt. Cyclides in computer aided geometric design. *Computer Aided Geometric Design*, **7**, 1 – 4, 221 – 242, 1990.
- [24] M. J. Pratt. Cyclides in computer aided geometric design II. *Computer Aided Geometric Design*, **12**, 1, 1 – 22, 1995.
- [25] M. J. Pratt. Dupin cyclides and supercyclides. In G. Mullineux, editor, *The Mathematics of Surfaces VI*. Oxford University Press, 1996. (Proc. 6th IMA Conf. on the Mathematics of Surfaces, Brunel University, Uxbridge, England, Sept. 1994).
- [26] T. W. Sederberg. Piecewise algebraic surface patches. *Computer Aided Geometric Design*, **2**, 1 – 3, 53 – 60, 1985.
- [27] T. W. Sederberg. Techniques for cubic algebraic surfaces: Part 1. *IEEE Computer Graphics & Applications*, **10**, 5, 14 – 25, 1990.
- [28] T. W. Sederberg. Techniques for cubic algebraic surfaces: Part 2. *IEEE Computer Graphics & Applications*, **10**, 6, 12 – 21, 1990.
- [29] T. W. Sederberg and D. C. Anderson. Steiner surface patches. *IEEE Computer Graphics & Applications*, **5**, 5, 23 – 36, 1985.
- [30] Y. L. Srinivas and D. Dutta. Rational parametric representation of the parabolic cyclide: Formulation and applications. *Computer Aided Geometric Design*, **12**, 6, 551 – 566, 1995.
- [31] C. E. Weatherburn. *Differential Geometry of Three Dimensions, Vol. 1*. Cambridge University Press, 1927.
- [32] X. Zhou and W. Strasser. A NURBS approach to cyclides. *Computers in Industry*, **19**, 2, 165 – 174, 1992.

A Appendix: The general cyclides of Casey and Darboux

The general cyclide in the sense of Casey and Darboux is the most general quartic surface having the circle at infinity as a singular curve. The simplest form of its implicit equation (Darboux, 1896; Jessop, 1916) is that of Equation (1) with

$$\phi = X^2 + Y^2 + Z^2 = 0, \quad p = W,$$

and

$$\chi = \alpha_1 X^2 + \alpha_2 Y^2 + \alpha_3 Z^2 + 2\beta_1 XW + 2\beta_2 YW + 2\beta_3 ZW + \gamma W^2 = 0. \quad (\text{A.1})$$

The Dupin cyclide is yet a further specialization, in which the coefficients occurring in Equation (A.1) are expressed in terms of the three defining parameters a, c and μ introduced in Section 3 as follows:

$$\alpha_1 = (a^2 + c^2 + \mu^2)/2, \quad \beta_1 = -ac\mu,$$

$$\alpha_2 = (a^2 - c^2 + \mu^2)/2, \quad \beta_2 = 0,$$

$$\alpha_3 = (c^2 - a^2 + \mu^2)/2, \quad \beta_3 = 0,$$

$$\gamma = (2a^2c^2 + 2c^2\mu^2 + 2\mu^2a^2 - a^4 - c^4 - \mu^4)/4.$$

This is easily verified by expansion of Equation (21) or (22).

The general cyclide may possess any number of isolated singularities from zero to four. If it has four, it is a Dupin cyclide (Jessop, 1916). Since the supercyclides always have four isolated singularities (see Section 2.2) it is therefore clear that all other forms of general cyclide are excluded from this class. For that reason they are not further discussed here, though they have many interesting properties and may well merit investigation in their own right for potential use in CAGD.

B Appendix: Kummer's cones

A brief note on this topic is added for the sake of completeness. For any quartic surface having a singular conic there may be constructed a certain quintic polynomial equation, each of whose roots in general gives rise to a quadric cone having two curves of tangency with the quartic (Jessop, 1916). Such cones are known as *Kummer's cones* after their discoverer. They are not the same as the tangent cones discussed in this paper, of which infinitely many exist for any supercyclide. However, there is some connection. For a supercyclide it is found that four of the roots of Kummer's quintic occur as repeated pairs. In these cases, instead of one cone per root we obtain a pair of planes per repeated root, so that only the fifth root actually gives rise to a cone.

The tangent planes to a Kummer cone all cut the quartic in pairs of conics. In the case of repeated roots, each pair of planes defines a pencil, any member of which cuts the quartic in a pair of conics. For the supercyclides these pencils of planes are identical with those used earlier to parametrize the surface.

The torus may be used as an illustration. The axes of the two pencils of planes (the characteristic lines of the surface) are the axis of symmetry of the torus and a line at infinity, the second pencil consisting of parallel planes perpendicular to the first axis. The single Kummer cone is a right circular cone with vertex at the centre of the torus; it is clearly tangent to the latter around two circles. Every tangent plane of this cone cuts the torus in a pair of intersecting circles, the so-called Villarceau circles. Thus, in addition to the two systems of circles generated by the pencils of planes there exist two further families. The Dupin cyclides may be obtained from a torus by inversion in a sphere, and since circles invert to circles, there are two corresponding additional families of circles lying on a Dupin cyclide. The surfaces discussed earlier in the paper as being projective transformations of Dupin cyclides therefore contain two further families of conics beyond the isoparametric curves of the given parametric representation. These do not possess the Blutel tangent cone property, however.