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EVALUATING ENGINEERING FUNCTIONS WITH IMPRECISE QUANTITIES

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Abstract

There is an increased awareness of the benefits of modeling imprecision in engineering problems, but success is limited by two problems associated with the calculus of imprecision: (1) Solving equations for a parameter is difficult because the common operators of addition and multiplication lack true inverses. (2) The multiple occurrence of imprecise quantities in engineering functions can lead to incorrectly very imprecise results because each occurrence of the parameter is treated as a separate parameter with the same range rather than multiple occurrences of the same parameter. Results obtained in the area of interval analysis are extended to the area of the calculus of imprecision. New operators are defined for functions of imprecise quantities that alleviate these two obstacles, and thus provide a general framework for including imprecision directly in engineering calculations.

Keywords: Fuzzy arithmetic fuzzy equations, fuzzy numbers, intervals, manufacturability.

1. INTRODUCTION

There has been an increased interest in modeling imprecision other than what can be described by stochastic uncertainty in engineering applications. The foundation of this approach is that many concepts cannot be accurately measured and modeled because imprecision is intrinsic to the parameters and relationships in these problems. In these situations, parameters can be modeled as imprecise quantities that restrict the value of a parameter to a partially ordered set. This approach has been used to solve problems in concurrent engineering and manufacturing [8, 10, 11]. When modeling these physical systems, two problems arise due to the operator's mathematical properties. First there is, in general, no inverse for the extended algebraic operators addition and multiplication. Consequently fuzzy equations cannot be solved by inverting the operators. Second, when multiple occurrences of a parameter occur in a function the standard mathematics overstates the imprecision of the result. Consequently, the result contains the actual set as a subset. These limitations hinder the development of systems for modeling the imprecision intrinsic to many engineering applications.

This paper is organized as follows: Section 2 presents the two problems encountered in engineering models and related work to overcome these problems. Section 3 discusses notation and functions with imprecise quantities. Section 4 extends three operators from work conducted in interval analysis to the domain of imprecise quantities. In section 5 an example is provided to demonstrate the benefits of this approach in modeling engineering systems.

2. ISSUES AND LIMITATIONS OF FUZZY MATHEMATICS APPLIED TO ENGINEERING SYSTEMS

This section discusses the two anomalies associated with applying the extension principle to physical systems.

2.1 LACK OF INVERSE PREVENTS SOLVING EQUATIONS

Solving equations is complicated due to the lack of inverse operators, that is $\oplus$ and $\otimes$ are not group operators, but form a semi-group with identity 0 and 1 respectively [6]. Consequently, algebraic equations of the form, $A \oplus X = B$ cannot be solved for unknown X as $X = B \ominus A$. The solution to the inverted operation will enclose the actual solution, that is if we denote X as the actual solution and X' as the solution obtained from the inverse then $X \subseteq X'$.

2.2 MULTIPLE OCCURRENCE OF PARAMETERS IN AN EXPRESSION RESULT IN INCREASED IMPRECISION

Multiple occurrence of parameters in an expression causes the solution to be more imprecise, that is it brackets the actual solution [7]. Let $g(x) = \frac{x}{x-2}$ and an equivalent representation of the function is $f(x) = 1 + \frac{2}{x-2}$. If $x \to \langle 3, 4, 5 \rangle$ then $g(x) \to \langle 1, 2, 5 \rangle$ and $f(x) \to \langle 1.67, 2, 3 \rangle$. The function *g(x)* is called an *improper representation* of the function because it treats each occurrence *x* as a separate parameter with the same range, when the intent is that it is the same parameter. Therefore, *g(x)* obtains a more imprecise results (larger set of values) than *f(x)*, the *proper representation* [6].

2.3 RELATED WORK

Buckley and Qu [3] examine the problem of solving linear and quadratic equations and present the conditions governing the existence of a solution when using the α-cut method. The issue of multiple occurrence of parameters in an expression was addressed in Dong and Wong [5] as part of an algorithm for computing fuzzy weighted averages (FWA). A combinatorial interval analysis scheme was used to account for multiple occurrences of a parameter. Inverses of functions can be determined by the FWA discretization algorithm since all the discretized points are stored internally by the system. Thus, it is possible to map the output into the input, although it is not possible to solve equations using this algorithm. Wood, *et al.*, [18] and Otto, *et al.* [16] have extended this approach to encompass more functions and combination metrics.

Giachetti, *et al.*, [11] exploited the imprecision increasing property of fuzzy equations by ordering the constraints in a hierarchy. The solutions to constraint networks higher in the hierarchy were used to place an additional constraint on the solution to lower level more precise constraint networks. This procedure, called *precision convergence*, was used in a concurrent engineering design problem to reduce the problem's imprecision.

The absence of an inverse and the multiple occurrence of parameters are well known problems in the domain of interval analysis [1, 14]. Ward, *et al.*, [17] have extensively examined the use of interval analysis in the mechanical engineering design domain. They developed four operators, the range operator and three inverses to range that are used to solve interval equations of three parameters. Finch and Ward [9] extended these results to arbitrary relationships over *n* parameters and show how to obtain useful information pertinent to the analysis of physical systems. They accomplish this by making an important distinction between physically dependent and independent parameters.

3. FUZZY QUANTITIES AND MAPPINGS

An *imprecise quantity Q* is a partially ordered set of real numbers. Each element $x \in Q$ has an associated *membership value* $\mu_Q(x)$ representing the degree *x* belongs to *Q*. It is a mapping $\mu_Q : x \to [0,1]$. Common practice is to impose restrictions on the shape of $\mu_Q(x)$ to either a triangular or trapezoidal distribution [4, 12]. We will use trapezoidal distributions, so then the imprecise quantities can be represented by a quadruple that defines the membership function's endpoints as,

$$x \to \langle \underline{\underline{x}}, \underline{x}, \overline{x}, \overline{\overline{x}} \rangle \tag{1}$$

An imprecise quantity defines a set of closed intervals called α-cut sets that are described by, $Q_\alpha = \{ x | \mu_Q(x) \geq \alpha \}, \forall \alpha \in (0, 1]$. The α-cut set at α is represented by the interval,

$$Q_\alpha = [\underline{x}_\alpha, \overline{x}_\alpha] \tag{2}$$

The interval $[\underline{\underline{x}}, \overline{\overline{x}}]$ is the α-cut set at α=0 and is called the support set. The interval $[\underline{x}, \overline{x}]$ is the α-cut set at α=1 and is called the core set of the imprecise quantity. Throughout this paper the double bar notation will be used to represent the endpoints of the support set and the single bar to represent the endpoints of the core set.

For an isotonic function *f* , the α-cut endpoints of the evaluated function are equal to that function evaluated on the α-cut endpoints of the individual parameters [6]. Formally, we state,

$$[f(Q_1, Q_2)]_\alpha = f(Q_{1\alpha}, Q_{2\alpha}), \quad \forall \alpha \in (0,1] \tag{3}$$

Equation (3) provides the justification used in Section 4 to evaluate functions based on the imprecise quantity's endpoints given by expression (1).

3.1 FUZZY MAPPINGS

Figure 1 shows a monotonically increasing mapping (function) $f : R \to R$ and the inverse mapping

$f^{-1}: R \to R$. The membership function $\mu(x)$ is mapped by function f and induces $\mu(y)$ defined via the extension principle as,

$$\mu(y) = \sup\left\{\min\{\mu(x), \mu(m)\} \middle| f(x) = y\right\} \qquad (4)$$

$$\mu(y) = 0 \text{ if } f^{-1}(y) = \varnothing .$$

Let $m \to \langle 1.5, 2, 2, 2.5 \rangle$, $x \to \langle 5, 5.5, 5.5, 6 \rangle$ and if the function f is $y = m \otimes x$, then it induces $y \to \langle 7.5, 10, 10, 15 \rangle$. When $\mu(y)$ is mapped by the inverse, f^{-1}, then the extension principle induces $\mu(x')$ on R and not $\mu(x)$. In Figure 1 the support sets $\left[\underline{\underline{x}}, \overline{\overline{x}}\right]$ and $\left[\underline{\underline{x'}}, \overline{\overline{x'}}\right]$ are shown on the horizontal axis and $\left[\underline{\underline{y}}, \overline{\overline{y}}\right]$ on the vertical axis. Note that $x_\alpha \subseteq x'_\alpha$, *i.e.* x_α is more *precise* than x'_α. In engineering applications it is commonly desirable to retrieve x_α but this is not possible when strictly using the extension principle. The extension principle is pessimistic and determines the largest interval possible. The reason imprecise mappings lack an inverse is that there is more than one forward mapping from the input to a single output value. (*e.g.* there are two combinations of x and m values, {10, 1.5} and {6, 2.5} that map through $y=mx$ to y = 15). The lack of an inverse for imprecise quantities occurs with all functions. The extension principle does not differentiate between which values are desired but in physical systems there is an important distinction. Oftentimes, it is desirable to obtain the subset x_α.

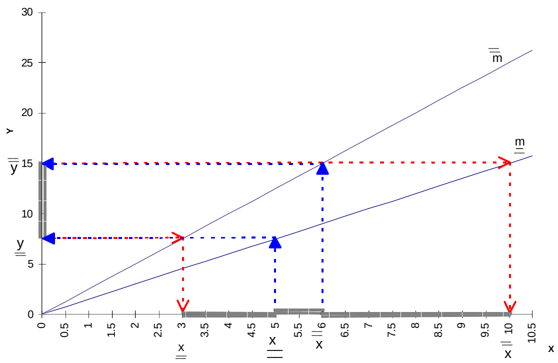

Figure 1. Mapping Imprecise Quantities

3.2 Models of Physical Systems

The parameters in engineering application models have a domain specific connotation. The causality between the imprecise engineering parameters can be exploited to achieve better results in domain specific models. Dubois, *et al.*, [8] discuss the significance of controllable versus uncontrollable parameters in the context of job-shop scheduling. If the parameter is controllable then the fuzzy set represents preference for a value. Fuzzy sets of uncontrollable parameters represent a possibility distribution that constrain the values the parameter can assume. Likewise, the partitioning of parameters into design, tuning, and noise parameters has been advantageously applied by Otto and Antonsson [15] for mechanical engineering design. The distinction of the causality of parameters is significant to the interpretation of these engineering models. *Physically independent parameters* are those that temporally occur first and determine the physically dependent parameter values. The *physically dependent parameters* cannot be directly specified by the designer. This notion of dependency does not correspond to the typical mathematical definition.

4. Operators to Perform Mappings with Imprecise Quantities

This section presents three operators adapted from [9] in the terminology relevant to imprecise quantities that will be used to evaluate engineering functions.

4.1 Decreasing and Increasing Parameters Subset

The *decreasing parameters subset*, D_f is the subset of parameters for the function denoted by the subscript such that the function $f(q_1, \ldots, q_n)$ is monotonically decreasing. $f(x, q_1, \ldots, q_n)$ is *monotonically decreasing* w.r.t. x if and only if for $x > x'$ and when $q_i, \ldots, q_n$ is constant, then $f(x, q_1, \ldots, q_n) < f(x', q_1, \ldots, q_n)$ [2].

The *increasing parameters subset*, I_f is the subset of parameters for when the function $f(q_1, \ldots, q_n)$ is monotonically increasing. A function $f(x, q_1, \ldots, q_n)$ is called *monotonically increasing* w.r.t. x if and only if for $x > x'$ and when $q_i, \ldots, q_n$ is constant, then $f(x, q_1, \ldots, q_n) \geq f(x', q_1, \ldots, q_n)$.

4.2 IMAGE DEFINITION

The image determines the possibility distribution of the physically dependent output from the input domain. This is equivalent to the extension principle and is considered "pessimistic" since it finds the largest possible set resulting from the physically independent parameters. It is included here to maintain a consistent notation with the inverses to image.

Image: $f(q_1,\ldots,q_n)=p$ then

$$p \to \left\langle f\left(\overline{\overline{D}}_f,\underline{\underline{I}}_f\right), f\left(\overline{D}_f,\underline{I}_f\right), f\left(\underline{D}_f,\overline{I}_f\right), f\left(\underline{\underline{D}}_f,\overline{\overline{I}}_f\right)\right\rangle$$

If $x \in D_f$ then the notation $\overline{\overline{D}}_f$ denotes the parameters in D_f at their $\overline{\overline{x}}$ values according to expression (1).

4.3 DOMAIN DEFINITION

An inverse of the image is the domain operator. Domain determines the physically independent parameter such that the forward mapping will always be restricted by the physically dependent parameter *p*.

Domain: $f^{-1}(q_1,\ldots,q_n,p)=q_k$

for $p \in I_{f^{-1}}$ then $q_k \to$

$$\left\langle \begin{array}{l} f^{-1}\left(\left\{\underline{\underline{D}}_{f^{-1}} \cup \underline{\underline{p}}\right\},\left\{\overline{\overline{I}}_{f^{-1}} - \overline{\overline{p}}\right\}\right), \\ f^{-1}\left(\left\{\underline{D}_{f^{-1}} \cup \underline{p}\right\},\left\{\overline{I}_{f^{-1}} - \overline{p}\right\}\right), \\ f^{-1}\left(\left\{\overline{D}_{f^{-1}} \cup \overline{p}\right\},\left\{\underline{I}_{f^{-1}} - \underline{p}\right\}\right), \\ f^{-1}\left(\left\{\overline{\overline{D}}_{f^{-1}} \cup \overline{\overline{p}}\right\},\left\{\underline{\underline{I}}_{f^{-1}} - \underline{\underline{p}}\right\}\right) \end{array} \right\rangle$$

for $p \in D_{f^{-1}}$ then $q_k \to$

$$\left\langle \begin{array}{l} f^{-1}\left(\left\{\underline{\underline{D}}_{f^{-1}} - \underline{\underline{p}}\right\},\left\{\overline{\overline{I}}_{f^{-1}} \cup \overline{\overline{p}}\right\}\right), \\ f^{-1}\left(\left\{\underline{D}_{f^{-1}} - \underline{p}\right\},\left\{\overline{I}_{f^{-1}} \cup \overline{p}\right\}\right), \\ f^{-1}\left(\left\{\overline{D}_{f^{-1}} - \overline{p}\right\},\left\{\underline{I}_{f^{-1}} \cup \underline{p}\right\}\right), \\ f^{-1}\left(\left\{\overline{\overline{D}}_{f^{-1}} - \overline{\overline{p}}\right\},\left\{\underline{\underline{I}}_{f^{-1}} \cup \underline{\underline{p}}\right\}\right) \end{array} \right\rangle$$

Where $\left\{\underline{D}_{f^{-1}} - \underline{p}\right\}$ denotes the set of parameters when the expression $f(q_1,\ldots,q_n,p)$ is decreasing less the parameter *p*. $\left\{\underline{I}_{f^{-1}} \cup \underline{p}\right\}$ denotes the set of monotonically increasing parameters and the parameter *p*.

4.4 SUFFICIENT ELEMENTS

The independent parameters are partitioned into uncontrolled q' and controlled q'' subsets. Sufficient elements is an inverse of the image that determines the physically independent parameter sets in one partition such that parameters in the second partition will always map into the physically dependent parameter set.

SufElements: $f^{-1}(q',q'',p)=q_k$

for $p \in D_{f^{-1}}$ *then* $q_k \to$

$$\left\langle \begin{array}{l} f^{-1}\left(\overline{\overline{I'}}_{f^{-1}} \cup \overline{\overline{D''}}_{f^{-1}}, \underline{\underline{p}} \cup \underline{\underline{D'}}_{f^{-1}} \cup \underline{\underline{I''}}_{f^{-1}}\right), \\ f^{-1}\left(\overline{I'}_{f^{-1}} \cup \overline{D''}_{f^{-1}}, \underline{p} \cup \underline{D'}_{f^{-1}} \cup \underline{I''}_{f^{-1}}\right), \\ f^{-1}\left(\underline{I'}_{f^{-1}} \cup \underline{D''}_{f^{-1}}, \overline{p} \cup \overline{D'}_{f^{-1}} \cup \overline{I''}_{f^{-1}}\right), \\ f^{-1}\left(\underline{\underline{I'}}_{f^{-1}} \cup \underline{\underline{D''}}_{f^{-1}}, \overline{\overline{p}} \cup \overline{\overline{D'}}_{f^{-1}} \cup \overline{\overline{I''}}_{f^{-1}}\right) \end{array} \right\rangle$$

for $p \in I_{f^{-1}}$ *then* $q_k \to$

$$\left\langle \begin{array}{l} f^{-1}\left(\overline{\overline{D'}}_{f^{-1}} \cup \overline{\overline{I''}}_{f^{-1}}, \underline{\underline{p}} \cup \underline{\underline{I'}}_{f^{-1}} \cup \underline{\underline{D''}}_{f^{-1}}\right), \\ f^{-1}\left(\overline{D'}_{f^{-1}} \cup \overline{I''}_{f^{-1}}, \underline{p} \cup \underline{I'}_{f^{-1}} \cup \underline{D''}_{f^{-1}}\right), \\ f^{-1}\left(\underline{D'}_{f^{-1}} \cup \underline{I''}_{f^{-1}}, \overline{p} \cup \overline{I'}_{f^{-1}} \cup \overline{D''}_{f^{-1}}\right), \\ f^{-1}\left(\underline{\underline{D'}}_{f^{-1}} \cup \underline{\underline{I''}}_{f^{-1}}, \overline{\overline{p}} \cup \overline{\overline{I'}}_{f^{-1}} \cup \overline{\overline{D''}}_{f^{-1}}\right) \end{array} \right\rangle$$

These three operators show how to obtain the parameters of expression (1). The entire membership function can be obtained via two methods, discretization [5, 16, 18] or the parametered fuzzy numbers approach [12]. Both methods are approximations but they reduce the computational complexity and obtain useful results. Giachetti and Young [12] analyzed fuzzy algebraic operators and set forth guidelines for determining the accuracy of the parametered fuzzy number approach. They defined a spread ratio, as $\lambda = \left(\underline{\underline{x}} / \underline{x}\right)$ for the left spread and $\rho = \left(\overline{x} / \overline{\overline{x}}\right)$ for the right spread. When $\lambda < 1.3$ and $\rho > 0.8$ then a linear approximation for α-cut endpoints between 0 and 1 yields results within 10% of the actual value.

5. Machining Example Problem

A *manufacturing process capability* is the physical ability of a manufacturing process to perform one or more feature-generating operations to some level of accuracy and precision. A commonly encountered engineering problem is to provide the design engineer an indication of the machine capability so that he can specify part features such that they are easy to manufacturing (*i.e.* well within the machine capabilities) [10]. In slab milling the theoretical arithmetic average surface roughness when upmilling is R_i (mm) and is determined by an equation derived by Martellotti [13],

$$R_i = \frac{0.125 f^2}{(d/2) + (f n_t / \pi)} \qquad (5)$$

where f = chip load (mm/tooth); d = diameter of milling cutter (mm); and n_t = number of teeth on cutter. Information pertinent to the machine capabilities can be obtained by evaluating expression (5) with the image, domain and sufficient elements operators. Using the terminology of [9] the parameters in the machining example are classified based on their physical causality. The surface roughness is a *physically dependent parameter* since it is determined by the feed rate and tool geometry. Consequently, it is determined using the image operator. Both chip load and cutting tool diameter and number of teeth are *physically independent* in expression (5) since they are determined first by the manufacturing expert. If a desired surface roughness is specified first then the physically independent chip load to achieve it can be found with the domain operator. If the tool diameter is selected first then it is classified as *uncontrollable*. The chip load f is *controllable* since it can be adjusted on the machine. These terms are used in the sufficient elements operator to solve for the tool diameter such that for any chip load in the set f the desired surface roughness will be achieved. Note that this notion of physical dependency is different than mathematical dependency since expression (5) could be rewritten to solve for d as a function of surface roughness but d would still be physically independent even though it is mathematically dependent on R_i.

5.1 Evaluation of Machine Capabilities

5.1.1 Image

The physically dependent parameter surface roughness is determined using the image operator. Let the parameters be defined as; $d \to \langle 58, 60, 62, 64 \rangle$ mm, $n_t \to \langle 4,5,5,6 \rangle$ teeth, and $f \to \langle 0.30, 0.35, 0.40, 0.45 \rangle$ mm/tooth.

According to definition 4.1 the set of increasing parameters I_f is {f} and the set of decreasing parameters D_f is {n_t, d}. Although f occurs multiple times in expression (5) it is treated here as a single parameter. The image of expression (5) is,

$$R_i \to \langle 0.35, 0.49, 0.66, 0.85 \rangle \ \mu\text{m}$$

This is the induced preference range of surface roughness that can be expected if the machinist stays within the preference functions for chip load and cutter tool diameter.

5.1.2 Domain

The physically independent chip load is determined such that it is restricted to map forward into the desired surface roughness output. Expression (8) is rewritten to obtain the quadratic equation in f as,

$$0.125 f^2 - \frac{R_i n_t}{\pi} f - \frac{R_i d}{2} = 0 \qquad (9)$$

Equation (9) is solved for f and the increasing subset of parameters $I_{f^{-1}}$ is {R_i, n_t, d} and the decreasing subset of parameters $D_{f^{-1}}$ is {$\varnothing$}. The physically dependent parameter p is R_i. The domain expression is used and the chip load is obtained as,

$$f \to \langle 0.30, 0.35, 0.40, 0.45 \rangle \ \text{mm/tooth.}$$

If the extension principle was used then the resulting chip load would be,

$$f' \to \langle 0.29, 0.34, 0.41, 0.47 \rangle \ \text{mm/tooth.}$$

Note that $f \subseteq f'$. If the machinist relied on f' then a possible result is that the roughness would not be within the desired range. The extended result f' indicates the machinist has a greater range of possible chip loads for attaining the surface roughness than what actually exists. Consequently, we see the need for an inverse to the image operator.

5.1.3 Sufficient Elements

The sufficient elements is used to determine the tool diameter such that the chip load can be adjusted and still yield the desired surface roughness. Equation (5) is rewritten as a function of d,

$$d = \frac{0.25 f^2}{R_i} - \frac{2 n_t f}{\pi}$$

The sufficient elements are,

$d \rightarrow \langle 58.4, 59.3, 61.4, 63.1 \rangle$ mm.

The extension principle would result in,

$d' \rightarrow \langle 57.8, 59.3, 61.4, 63.5 \rangle$ mm.

$d \subseteq d'$ with the possible result of falling outside of the desired range of surface roughness if d' was used.

6. CONCLUSION

This paper extended the operators developed by Finch and Ward [9] to fuzzy quantities. It demonstrated how these operators can be used to overcome two common problems encountered when evaluating fuzzy equations; the lack of an inverse and the multiple occurrence of parameters in a relationship. The operators developed have applicability to a wide range of functions. The operators can be applied to trigonometric functions limited to domains where they are monotonically increasing or decreasing. A significant range of problems can be tackled using this methodology to find more precise resulting sets. Further work is required to classify engineering parameters to better evaluate models that contain imprecision.

7. ACKNOWLEDGMENT

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