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Equivalency of LMSF and Alpha-Beta-Gamma filters

Henry Schneiderman and Albert J. Wavering
Building 220, Room B127
Robot Systems Division
National Institute of Standards and Technology
Gaithersburg, MD 20899

Abstract

In this paper, it is shown that a second order least-mean-square-fit (LMSF) smoothing filter of infinite length may be expressed in the form of an $\alpha-\beta-\gamma$ filter. The relationships for α, β, γ, are derived such that equivalency to a LMSF is achieved.

1. Introduction

One method for filtering a time sequence is to perform a least squares fit of the previous values of the sequence to a polynomial function of time. Such a procedure is sometimes referred to as a least-mean-square-fit (LMSF) filter [1]. Usually such a least squares fit is computed over a finite length of previous samples of the sequence with equal weighting of the samples. However, if the polynomial is quadratic and the fit is computed over the entire infinite sequence of previous samples with an exponential decay weighting, the steady-state response of the resulting IIR filter can be shown to be equivalent to that of an $\alpha-\beta-\gamma$ filter. In this paper, this equivalency is established and the relationships for α, β, γ are derived in such a filter. In the development that follows, it is assumed that the sample period (T) is 1.

2. LMSF filter specification

To fit a 2nd order polynomial to previous values of a time sequence, a least squares residual, J, is minimized through choice of a_0, a_1, a_2:

$$J = \sum_{t=-\infty}^{0} \lambda^{-t}\left(x_m(t) - \left(a_0 + a_1 t + a_2 t^2\right)\right)^2 \tag{1}$$

where

t is the time (current time: $t = 0$), in sample periods

$x_m(t)$ are the measured values of the time sequence

a_0, a_1, a_2 are the polynomial coefficients

An exponential decay profile is achieved in (1) by multiplying previous terms by increasing powers of λ where $0 < \lambda \leq 1$.

The computed values, $\hat{a}_0, \hat{a}_1, \hat{a}_2$, that minimize J can then be used to generate the filtered values of the time sequence:

$$x_{f-LMSF}(t|0) = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2 \tag{2}$$

This formulation can be used for both prediction, $t > 0$, and smoothing, $t \leq 0$.

3. α–β–γ filter specification

For the case where the sample period is 1, the state equations for the α–β–γ filter [2][3], are:

Prediction

$$x_p(k) = x_s(k-1) + v_s(k-1) + \frac{1}{2}a_s(k-1),\ v_p(k) = v_s(k-1) + a_s(k-1),\ a_p(k) = a_s(k-1). \tag{3}$$

Smoothing

$$x_s(k) = x_p(k) + \alpha[x_m(k) - x_p(k)],\ v_s(k) = v_p(k) + \beta[x_m(k) - x_p(k)], \tag{4}$$

$$a_s(k) = a_p(k) + \frac{\gamma}{2}[x_m(k) - x_p(k)]. \tag{5}$$

To use the α–β–γ filter to predict the position $x_{f-\alpha\beta\gamma}$ at some future time t, the following equation is used:

$$x_{f-\alpha\beta\gamma}(t|k) = x_s(k) + (t-k)v_s(k) + \frac{1}{2}(t-k)^2 a_s(k). \tag{6}$$

4. LMSF equivalency to α–β–γ filter

To show that the LMSF and α–β–γ filter formulations are equivalent, a frequency domain representation is used. Converting (1) to the z-domain and expressing it in matrix form gives:

$$J = (b' - Ha)^T W (b' - Ha) = b'^T W b' - 2b'^T W H^T a + a^T H^T W H a \tag{7}$$

$$a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \\ \ldots & \ldots & \ldots \end{bmatrix} \quad W = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda^2 & \\ & & & \ldots \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ \ldots \end{bmatrix} \quad b' = bX_m(z) \tag{8}$$

To minimize J with respect to a, the derivative of J with respect to a is set to zero and solved for a:

$$\frac{dJ}{da} = -H^T W b' + H^T W H a = 0 \tag{9}$$

$$\hat{a} = \left(H^T W H\right)^{-1} H^T W b X_m(z) \tag{10}$$

Substituting $\hat{a}$ into (2) gives the filtered output $X_{f\text{-}LMSF}(z)$:

$$X_{f-LMSF}(z) = d^T \hat{a} = d^T\left(H^T W H\right)^{-1} H^T W b X_m(z) \qquad d = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} \tag{11}$$

The transfer function relating the measured values to the filtered values is then given by:

$$T_{LMSF}(z) = \frac{X_{f-LMSF}(z)}{X_m(z)} = d^T\left(H^T W H\right)^{-1} H^T W b \tag{12}$$

$$H^T W H = \begin{bmatrix} \sum_{i=0}^{\infty} \lambda^i & -\sum_{i=0}^{\infty} i\lambda^i & \sum_{i=0}^{\infty} i^2\lambda^i \\ -\sum_{i=0}^{\infty} i\lambda^i & \sum_{i=0}^{\infty} i^2\lambda^i & -\sum_{i=0}^{\infty} i^3\lambda^i \\ \sum_{i=0}^{\infty} i^2\lambda^i & -\sum_{i=0}^{\infty} i^3\lambda^i & \sum_{i=0}^{\infty} i^4\lambda^i \end{bmatrix} \qquad H^T W b = \begin{bmatrix} \sum_{i=0}^{\infty} \left(\frac{\lambda}{z}\right)^i \\ -\sum_{i=0}^{\infty} i\left(\frac{\lambda}{z}\right)^i \\ \sum_{i=0}^{\infty} i^2\left(\frac{\lambda}{z}\right)^i \end{bmatrix} \tag{13}$$

Each of the summations in (13) can be expressed in terms of the product of a function of λ and an arithmogeometric series of the form:

$$s = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots \tag{14}$$

which has the following closed form solution:

$$s = \frac{a}{1-r} + \frac{rd}{(1-r)^2} \tag{15}$$

Making substitutions based on (15) into (13) and then substituting the result into (12) gives the following form for the LMSF transfer function:

$$\begin{aligned} T_{LMSF}(z) = \frac{X_{f-LMSF}(z)}{X_m(z)} &= \Big(1-\lambda^3+1.5t\left(1-\lambda-\lambda^2+\lambda^3\right)+0.5t^2\left(1-3\lambda+3\lambda^2-\lambda^3\right) \\ &+z^{-1}\left(-3\lambda+3\lambda^3-2t\left(1-3\lambda^2+2\lambda^3\right)+t^2\left(-1+3\lambda-3\lambda^2+\lambda^3\right)\right) \\ &+z^{-2}\left(3\lambda^2-3\lambda^3+0.5t\left(1+3\lambda-9\lambda^2+5\lambda^3\right)+0.5t^2\left(1-3\lambda+3\lambda^2-\lambda^3\right)\right)\Big)\Big/\left(1-3\lambda z^{-1}+3\lambda^2 z^{-2}-\lambda^3 z^{-3}\right) \end{aligned} \tag{16}$$

Similarly the frequency response for an α–β–γ filter, (3), (4), (5), (6), is given by:

$$T_{\alpha\beta\gamma}(z) = \frac{X_{f-\alpha\beta\gamma}(z)}{X_m(z)} = \frac{\left(\alpha+\beta t+\frac{\gamma t^2}{4}\right)+\left(\beta-2\alpha+\frac{\gamma}{4}-2\beta t-\frac{\gamma t^2}{2}+\frac{\gamma t}{2}\right)z^{-1}+\left(\alpha-\beta+\frac{\gamma}{4}+\beta t+\frac{\gamma t^2}{4}-\frac{\gamma t}{2}\right)z^{-2}}{1+\left(\alpha+\beta+\frac{\gamma}{4}-3\right)z^{-1}+\left(\frac{\gamma}{4}-2\alpha-\beta+3\right)z^{-2}+(\alpha-1)\,z^{-3}} \tag{17}$$

This is computed by arranging the state equations (3), (4), (5), (6) of the filter into a signal flow graph and using Mason's general gain rule [4] to compute the transfer function.

Equivalency between (16) and (17) is achieved with the following choices:

$$\alpha = 1-\lambda^3 \qquad \beta = 1.5-1.5\lambda-1.5\lambda^2+1.5\lambda^3 \qquad \gamma = 2-6\lambda+6\lambda^2-2\lambda^3 \tag{18}$$

This can be verified by substitution of (18) into (17).

These relationships for α, β, and γ can be used to minimize the sum of the exponentially-weighted squared errors of position estimates for all prior samples. Alternative relationships for α, β, and γ can be chosen using other techniques [5][6].

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