# Flat Surfaces:
# A Visual Invariant

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March 1992

## ABSTRACT

This paper deals with machine perception of flat surfaces. For an observer that undergoes only translational motion in parallel to a planar surface, we show that a non-linear function of optical flow produces the same value for all points on this surface, i.e., there is an optical-flow based invariant for all points that lie on a flat surface. We discuss some potential uses of this invariant.

## 1. INTRODUCTION

The problem of constant perception despite varying visual sensations, i.e., how despite different images of the same scene, the world is perceived as stationary, has puzzled many researchers in the last five decades (see for example [1]). In an attempt to answer this question another basic question arises: Is there an image-sequence transformation under which permanent properties of the environment are preserved during the motion of the eye? [1,6,7]

This paper deals with machine perception of flat surfaces. For an observer that undergoes only translational motion in parallel to a planar surface, we show that a non-linear function of optical flow produces the same value for all points on this surface, i.e., there is an optical-flow based invariant for all points that lie on a flat surface.

## 2. A FLAT SURFACE INVARIANT

The optical flow generated by a point in 3-D space as expressed in spherical coordinates is (See Appendix and [2-5]):

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{R} \begin{bmatrix} -\dfrac{\sin\theta}{\cos\phi} & \dfrac{\cos\theta}{\cos\phi} & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{bmatrix} -U-BZ+CY \\ -V-CX+AZ \\ -W-AY+BX \end{bmatrix} \tag{1}$$

Refer to Figure 1. Assume that the $Z$ axis of the camera is perpendicular to the flat surface. For the case where the camera undergoes translation only along its optical axis (the $Y$ axis) i.e., $A=B=C=U=W=0$, Equation (1) becomes:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{R} \begin{bmatrix} -\dfrac{\sin\theta}{\cos\phi} & \dfrac{\cos\theta}{\cos\phi} & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ -V \\ 0 \end{bmatrix} \tag{2}$$

For the restricted translational motion of the camera as mentioned above, the following geometrical relationship holds for *all* points on the flat surface:

$$R = \frac{Z}{\sin\phi} \tag{3}$$

By combining Equations (2) and (3) we obtain:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{V}{Z} \begin{bmatrix} -\dfrac{\cos\theta\sin\phi}{\cos\phi} \\ \sin^2\phi\ \sin\theta \end{bmatrix} \tag{4}$$

From the last equation we obtain:

$$- \frac{\dot{\theta}}{\tan\phi\cos\theta} = \frac{\dot{\phi}}{\sin^2\phi\sin\theta} = \frac{V}{Z} \tag{5}$$

In Equation (5) $V$ and $Z$ are constant for *all* points on the flat surface, and thus $-\dfrac{\dot{\theta}}{\tan\phi\cos\theta}$ and $\dfrac{\dot{\phi}}{\sin^2\phi\sin\theta}$ are constant for *all* points on the flat surface (they are also equal to each other). This also means that *any point on the flat surface produces the same values of some functions of optical flow which are, in our case,* $-\dfrac{\dot{\theta}}{\tan\phi\cos\theta}$ *and* $\dfrac{\dot{\phi}}{\sin^2\phi\sin\theta}$. The value obtained from these optical-flow-based functions is *invariant* under the given motion.

## 3. DISCUSSION

In this short paper we presented an optical-flow-based invariant of a flat surface for translational motion of a camera. The approach is a step toward the understanding of perception of 3-D rigid bodies.

This invariant can be used to build topographic maps from a camera that undergoes translation above a terrain, since for a horizontally translating camera each height above a certain reference level corresponds to a value of the optical-flow-based previously-described functions. This invariant can also be used to detect obstacles on a flat road using an on-board camera, since all points on the road will generate the same value of the previously expressed functions and any point above or below the road surface will produce a different value.

In this paper the invariant is derived for a special motion of the camera, where it undergoes translation only along its optical axis (the $Y$ axis) and the $Z$ axis of the camera is perpendicular to the flat surface. However, by reorienting the camera and applying appropriate calibration, a less restricted result can be obtained.

We are currently working on invariants for more general camera motion (in particular motions that involve fixation) and arbitrary shape objects.

## 4. ACKNOWLEDGEMENTS

The author would like to thank Jim Albus, Ernie Kent, Marty Herman and Don Orser for fruitful discussions regarding this and other invariants.

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## APPENDIX: EQUATIONS OF MOTION AND OPTICAL FLOW

This appendix describes the equations that relate a point in 3-D space to the projection of that point in the image for general six-degree-of-freedom motion of the camera.

In the following analysis, we assume a moving camera in a stationary environment. Suppose the coordinate system is fixed with respect to the camera as shown in Figure A1. Assume a pinhole camera model and that the pinhole point of the camera is at the origin of the coordinate system. We derive the optical flow components in the spherical coordinates $(R-\theta-\phi)$. In this frame, angular velocities ($\dot{\theta}$ and $\dot{\phi}$) of any point in space, say P, are identical to the optical flow values at P' in the image domain. Figure A2 illustrates this concept: $\theta$ and $\phi$ of a point in space are the same as $\theta$ and $\phi$ of the projected point P' in the image domain, and therefore there is no need to convert angular velocities of points in 3-D space to optical flow. In Figure A2 the image domain is a sphere. However, for practical purposes the surface of the image sphere can be mapped onto an image plane (or other surface).

We start with the derivation of the velocity of a 3-D point in the $XYZ$ coordinates (Figure A1). Let the instantaneous coordinates of the point $P$ be $\mathbf{R} = (X,Y,Z)^T$ (where the superscript $T$ denotes transpose). If the instantaneous translational velocity of the camera is $\mathbf{t} = (U,V,W)^T$ and the instantaneous angular velocity is $\omega = (A,B,C)^T$ then the velocity vector $\mathbf{V}$ of the point $P$ with respect to the $XYZ$ coordinate system is:

$$\mathbf{V} = -\mathbf{t} - \omega \times \mathbf{R} \qquad\qquad (A1)$$

or:

$$V_X = -U - BZ + CY \qquad\qquad (A2)$$

$$V_Y = -V - CX + AZ \qquad\qquad (A3)$$

$$V_Z = -W - AY + BX \qquad\qquad (A4)$$

where $V_X$, $V_Y$, and $V_Z$ are the components of the velocity vector $\mathbf{V}$ along the $X$, $Y$, and $Z$

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directions respectively.

To convert from $R\theta\phi$ to $XYZ$ coordinates we use the relations:

$$X = R\cos\phi\cos\theta \tag{A5}$$

$$Y = R\cos\phi\sin\theta \tag{A6}$$

$$Z = R\sin\phi. \tag{A7}$$

Similarily, to convert from $XYZ$ to $R\theta\phi$ coordinates we use:

$$R = \sqrt{X^2 + Y^2 + Z^2} \tag{A8}$$

$$\theta = \tan^{-1}\frac{Y}{X} \tag{A9}$$

$$\phi = \sin^{-1}\frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}. \tag{A10}$$

In order to find the optical flow of a 3-D point in $R\theta\phi$ coordinates, we use the following relations and transformations:

Let $V_R$, $V_\theta$, and $V_\phi$ be the components of the vector $\mathbf{V}$ in spherical coordinates, and

$$V_{R\theta\phi} = \begin{bmatrix} V_R \\ V_\theta \\ V_\phi \end{bmatrix} \tag{A11}$$

$$V_{XYZ} = \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix}. \tag{A12}$$

Then:

$$V_{R\theta\phi} = [T_\phi][T_\theta]V_{XYZ} \tag{A13}$$

where

$$[T_\theta] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{X}{\sqrt{X^2+Y^2}} & \dfrac{Y}{\sqrt{X^2+Y^2}} & 0 \\ \dfrac{-Y}{\sqrt{X^2+Y^2}} & \dfrac{X}{\sqrt{X^2+Y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A14}$$

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and

$$[T_\phi] = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{X^2+Y^2}}{\sqrt{X^2+Y^2+Z^2}} & 0 & \dfrac{Z}{\sqrt{X^2+Y^2+Z^2}} \\ 0 & 1 & 0 \\ \dfrac{-Z}{\sqrt{X^2+Y^2+Z^2}} & 0 & \dfrac{\sqrt{X^2+Y^2}}{\sqrt{X^2+Y^2+Z^2}} \end{bmatrix}. \tag{A15}$$

Also:

$$V_R = \dot{R} \tag{A16}$$

$$V_\theta = R\,\dot{\theta}\,\cos\phi \tag{A17}$$

$$V_\phi = R\,\dot{\phi} \tag{A18}$$

where dot denotes first derivative with respect to time.

Using equations (A2)-(A18) yields the following expressions:

$$\begin{bmatrix} R\,\dot{\theta}\,\cos\phi \\ R\,\dot{\phi} \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{bmatrix} -U-BZ+CY \\ -V-CX+AZ \\ -W-AY+BX \end{bmatrix} \tag{A19}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{R} \begin{bmatrix} -\dfrac{\sin\theta}{\cos\phi} & \dfrac{\cos\theta}{\cos\phi} & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{bmatrix} -U-BZ+CY \\ -V-CX+AZ \\ -W-AY+BX \end{bmatrix} \tag{A20}$$

As mentioned earlier, $\dot{\theta}$ and $\dot{\phi}$ of a point in space (i.e., the angular velocities in the camera coordinate system) are the *same* as the optical flow components $\dot{\theta}$ and $\dot{\phi}$ (Figure A2).
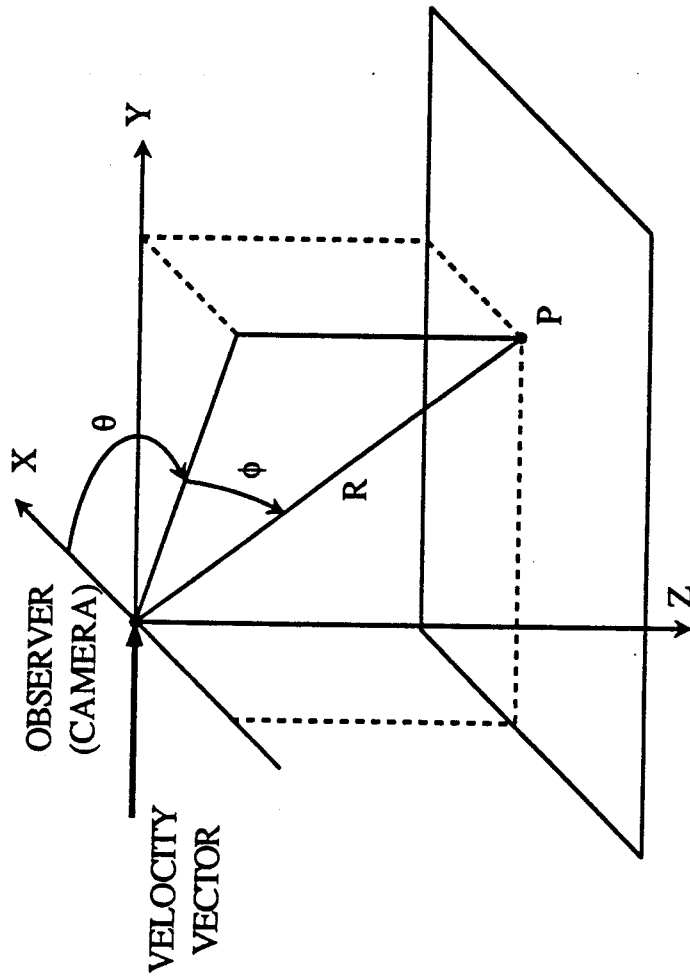
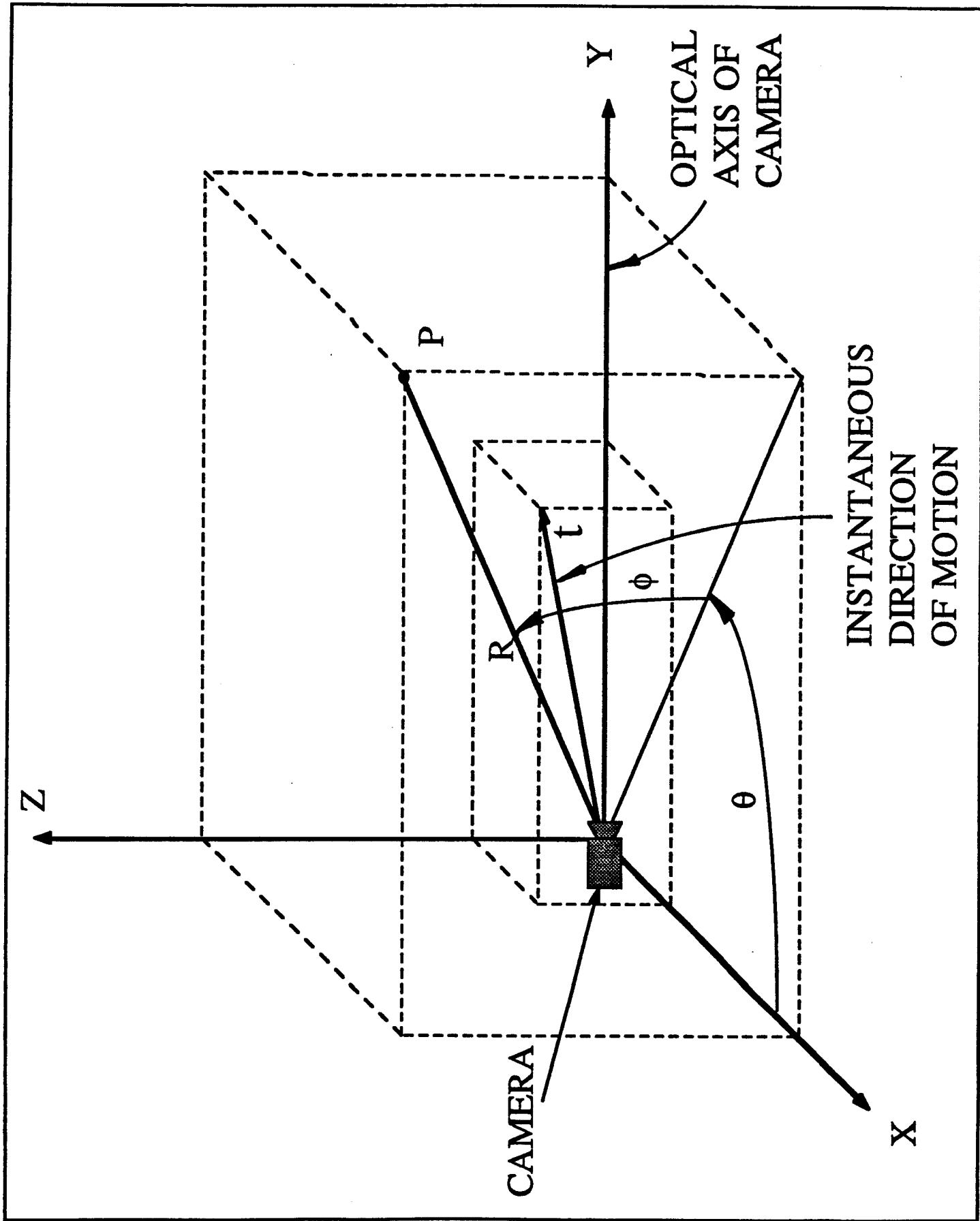Figure 1: Camera Coordinate System Relative To The Flat Surface
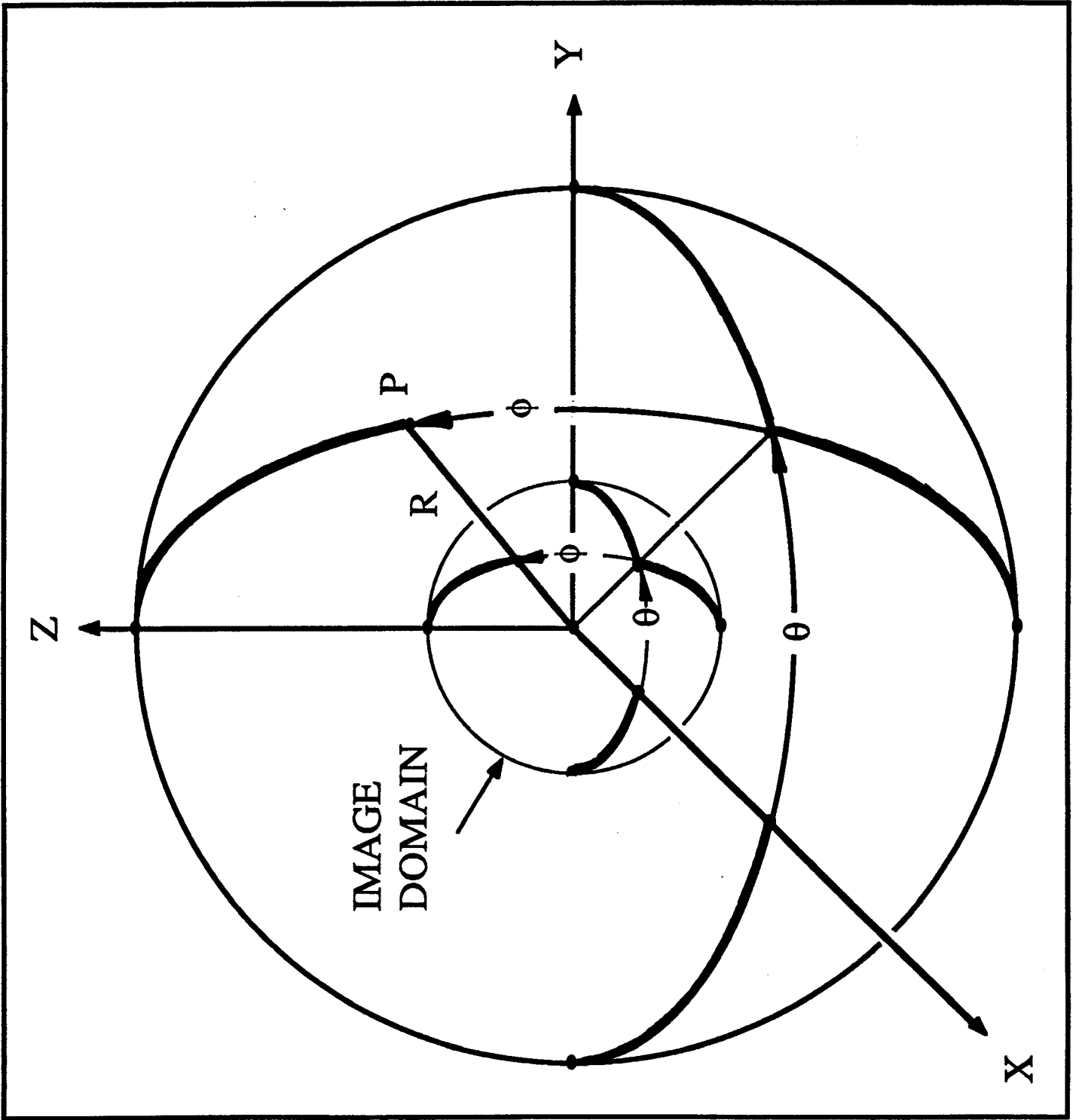
**Figure A1: Camera Coordinate Systems**

**Figure A2: Image Domain**

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| NIST-114A | U.S. DEPARTMENT OF COMMERCE | 1. PUBLICATION OR REPORT NUMBER |
|---|---|---|
| (REV. 3-90) | NATIONAL INSTITUTE OF STANDARDS AND TECHNOLOGY | NISTIR 4794 |

**BIBLIOGRAPHIC DATA SHEET**

| | |
|---|---|
| 2. PERFORMING ORGANIZATION REPORT NUMBER | |
| 3. PUBLICATION DATE | MARCH 1992 |

**4. TITLE AND SUBTITLE**

Flat Surfaces: A Visual Invariant

**5. AUTHOR(S)**

Daniel Raviv

**6. PERFORMING ORGANIZATION (IF JOINT OR OTHER THAN NIST, SEE INSTRUCTIONS)**

U.S. DEPARTMENT OF COMMERCE
NATIONAL INSTITUTE OF STANDARDS AND TECHNOLOGY
GAITHERSBURG, MD 20899

**7. CONTRACT/GRANT NUMBER**

**8. TYPE OF REPORT AND PERIOD COVERED**

**9. SPONSORING ORGANIZATION NAME AND COMPLETE ADDRESS (STREET, CITY, STATE, ZIP)**

**10. SUPPLEMENTARY NOTES**

**11. ABSTRACT (A 200-WORD OR LESS FACTUAL SUMMARY OF MOST SIGNIFICANT INFORMATION. IF DOCUMENT INCLUDES A SIGNIFICANT BIBLIOGRAPHY OR LITERATURE SURVEY, MENTION IT HERE.)**

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**12. KEY WORDS (6 TO 12 ENTRIES; ALPHABETICAL ORDER; CAPITALIZE ONLY PROPER NAMES; AND SEPARATE KEY WORDS BY SEMICOLONS)**

Visual Motion; Representations; Visual Invariants; Vision-Based Mobility; Computer Vision

| 13. AVAILABILITY | 14. NUMBER OF PRINTED PAGES |
|---|---|
| X UNLIMITED | 13 |
| FOR OFFICIAL DISTRIBUTION. DO NOT RELEASE TO NATIONAL TECHNICAL INFORMATION SERVICE (NTIS). | |
| ORDER FROM SUPERINTENDENT OF DOCUMENTS, U.S. GOVERNMENT PRINTING OFFICE, WASHINGTON, DC 20402. | 15. PRICE |
| X ORDER FROM NATIONAL TECHNICAL INFORMATION SERVICE (NTIS), SPRINGFIELD, VA 22161. | A02 |

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