Computers & Structures Vol. 28, No. 3, pp. 395–406, 1988
Printed in Great Britain.

0045-7949/88 $3.00 + 0.00

FINITE ELEMENT PROCEDURES FOR LARGE STRAIN ELASTIC–PLASTIC THEORIES

JAMES D. LEE

Robot Systems Division, National Bureau of Standards, Gaithersburg, MD 20899, U.S.A.

(*Received* 21 *March* 1987)

Abstract—In a unified approach, the virtual work equations in rate form and in incremental forms are derived rigorously for elastic–plastic continuum subjected to large strains. The finite element procedures for the analyses of elastic–plastic solid based on Lee's theory and the Green–Naghdi theory are presented. Also, it is shown that transformations can be performed among the Eulerian, the Total-Lagrangian, and the Updated-Lagrangian formulations, and among different forms of constitutive relations, without any approximation.

1. INTRODUCTION

The variational principle, or the principle of virtual work, for the analyses of solid mecahnics with large strains involved has been treated extensively by Washizu [1], and by, for example, Hill [2], Eringen [3], Bathe [4], Malvern [5], Oden [6], Hibbitt *et al.* [7], McMeeking and Rice [8], Scharnhorst and Pian [9], Lubarda and Lee [10], etc. The virtual work equations can be expressed in rate form and in three incremental forms. In Sec. 2, it is shown that, by starting from any one of the three forms of Cauchy's law of motion—the equilibrium equation, a universal virtual work equation in rate form can be derived rigorously. In Sec. 3, in a straightforward and rigorous way, the Eulerian, Total-Lagrangian, and Updated-Lagrangian forms of incremental virtual work equations are derived, and it is shown that the three forms of incremental virtual work equations can be transformed from one to the other without any approximation. It is shown also that the incremental virtual work equations are exactly the same as the virtual work equation in rate form in the limiting case—the size of the incremental step approaching zero. In Sec. 4, the finite element formulations based on the Total-Lagrangian and the Updated-Lagrangian incremental virtual work equations are made in detail for elastic–plastic continuum divided into general, three-dimensional, solid elements. The emphasis is put on the calculation of nodal forces, which is an exact treatment. The calculated nodal forces are then taken as the basis to check whether the equilibrium is reached pointwise. These formulations can then be applied to any theory of elastic–plastic solid with large strains.

Among numerous theories of elastic–plastic solid, Lee's theory [10–15] is the most unique one in the sense that the decomposition of total deformation into the elastic and the plastic parts is made at the deformation gradient level, while Green–Naghdi's theory [16–21] is the most general one—it allows material anisotropy and various kinds of hardening rules to be incorporated into the formulation. In Secs 5 and 6, the detailed iterative procedures are outlined for Lee's theory and Green–Naghdi's theory, respectively.

Throughout this paper, the standard tensor summation convention is adopted: the rectangular Eulerian coordinates, $x_k (k = 1, 2, 3)$, and Lagrangian coordinates, $X_K (K = 1, 2, 3)$, are employed; an index, k or K, after a comma indicates a partial differentiation with respect to the coordinate, x_k or X_K; a superposed dot indicates the material time derivative; and some standard notations appearing in Eringen's book [3] are utilized.

2. VIRTUAL WORK EQUATION IN RATE FORM

The equilibrium equation may be written in one of the following forms [3]:

$$\sigma_{ij,j} + \rho f_i = 0 \tag{1}$$

$$T_{Ki,K} + \rho_0 f_i = 0 \tag{2}$$

$$(T_{KL} x_{i,L})_{,K} + \rho_0 f_i = 0, \tag{3}$$

where: σ_{ij} is the Cauchy stress tensor; ρ and ρ_0 are the mass density in the deformed and the undeformed state respectively; f_i is the body force per unit mass; $x_{i,K}$ is the deformation gradient; and the first order and the second order Piola–Kirchhoff stress tensors are defined respectively as

$$T_{Ki} \equiv J X_{K,j} \sigma_{ji} \tag{4}$$

$$T_{KL} \equiv J X_{K,i} X_{L,j} \sigma_{ij}. \tag{5}$$

In eqns (4) and (5), J is the Jacobian of the transformation between the deformed configuration and the undeformed configuration, i.e.

$$J \equiv \det(x_{k,K}), \quad (6)$$

and $X_{K,i}$ is related to $x_{i,K}$ as

$$x_{i,K} X_{K,j} = \delta_{ij}, \quad X_{K,i} x_{i,L} = \delta_{KL}. \quad (7)$$

It is noticed that, in this work, attention is focused on the static problems, although the formulations could be extended to the dynamic cases without major difficulty.

If eqn (1) is differentiated with respect to time and then multiplied by the virtual velocity δv_i and then integrated over the deformed volume, the following is obtained:

$$\int \frac{\mathrm{d}}{\mathrm{d}t} [\sigma_{ij,j} + \rho f_i] \delta v_i \, \mathrm{d}v = 0. \quad (8)$$

Similarly, from eqns (2) and (3), one may have the following:

$$\int [(\dot{T}_{Ki}),_K + \rho_0 \dot{f}_i] \delta v_i \, \mathrm{d}V = 0 \quad (9)$$

$$\int \left\{ \frac{\mathrm{d}}{\mathrm{d}t} [(T_{KL} x_{i,L})],_K + \rho_0 \dot{f}_i \right\} \delta v_i \, \mathrm{d}V = 0. \quad (10)$$

After some mathematical manipulations, eqns (8)–(10) are all reduced to the same virtual work equation in rate form as follows:

$$\int \rho \dot{f}_i \delta v_i \, \mathrm{d}v + \int_{s^*} \frac{\mathrm{d}}{\mathrm{d}t} (\sigma_{ij} \, \mathrm{d}a_j) \delta v_i = \int [\mathring{\sigma}_{ij} \delta \, \mathrm{d}_{ij} + \sigma_{ij} v_{k,j} \delta v_{k,i}] \, \mathrm{d}v, \quad (11)$$

where: s^* is part of surface surrounding v over which the surface traction, defined as $T_i \equiv \sigma_{ij} n_j$($n_j$ is the unit outward normal of the surface), is specified; $\mathrm{d}a_j$ is the differential area vector; d_{ij} is the deformation rate tensor; and the Truesdell stress rate tensor $\mathring{\sigma}_{ij}$ is defined as [22, 23]

$$\mathring{\sigma}_{ij} \equiv \dot{\sigma}_{ij} - \sigma_{ik} v_{j,k} - \sigma_{kj} v_{i,k} + \sigma_{ij} v_{k,k}. \quad (12)$$

Also, one may readily show that eqn (11) can be rewritten as

$$\int \rho \dot{f}_i \delta v_i \, \mathrm{d}v + \int_{s^*} \frac{\mathrm{d}}{\mathrm{d}t} (\sigma_{ij} \, \mathrm{d}a_j) \delta v_i = \int J^{-1} [\mathring{\tau}_{ij} \delta \, \mathrm{d}_{ij} - 2\tau_{ij} \, \mathrm{d}_{kj} \delta \, \mathrm{d}_{ik} + \tau_{ij} v_{k,j} \delta v_{k,i}] \, \mathrm{d}v, \quad (13)$$

where the Kirchhoff stress tensor is defined as

$$\tau_{ij} \equiv J \sigma_{ij}, \quad (14)$$

the Jaumann stress rate tensor is defined as [10]

$$\mathring{\tau}_{ij} \equiv \dot{\tau}_{ij} - \omega_{ik} \tau_{kj} + \tau_{ik} \omega_{kj}, \quad (15)$$

and ω_{ij} is the spin tensor.

It should be noted that eqn (13) is the same result as that obtainable from the Hill's variational principle [2, 8, 10]; also, the second term on the left-hand side of eqn (13) indicates that the follower type loading is automatically incorporated in the formulation. It should be emphasized that the virtual work equation in rate form, eqn (11) or eqn (13), is as general and exact as the equilibrium equation, eqns (1), (2), or (3). However, the rate form may not be practical at the problem-solving level, but it can and will be used as a basis to check the validity of the virtual work equation in incremental forms, which will be formulated in the next section.

3. EULERIAN AND LAGRANGIAN FORMULATIONS

By multiplying eqn (1) by the virtual displacement δu_i and integrating over the deformed volume, one may obtain the virtual work equation in the following form:

$$\int_{s^*} \sigma_{ij} \delta u_i \, \mathrm{d}a_j + \int \rho f_i \delta u_i \, \mathrm{d}v = \int \sigma_{ij} \delta e_{ij} \, \mathrm{d}v, \quad (16)$$

where the infinitesimal strain tensor, e_{ij}, is defined as

$$e_{ij} \equiv (u_{i,j} + u_{j,i})/2. \quad (17)$$

If either eqn (2) or eqn (3) is multiplied by the virtual displacement δu_i and integrated over the undeformed volume, the virtual work equation is obtained as

$$\int_{s^*} T_{KL}(\delta_{NL} + u_{N,L}) \delta u_N \, \mathrm{d}A_K + \int \rho_0 f_K \delta u_K \, \mathrm{d}V = \int T_{KL} \delta E_{KL} \, \mathrm{d}V, \quad (18)$$

where $\mathrm{d}A_K$ is the differential area vector in the undeformed state; the Green–Lagrangian strain tensor, E_{KL}, is defined as

$$E_{KL} \equiv (x_{k,K} x_{k,L} - \delta_{KL})/2 \equiv (C_{KL} - \delta_{KL})/2 = (u_{K,L} + u_{L,K} + u_{M,K} u_{M,L})/2. \quad (19)$$

u_K and f_K are respectively the displacement vector and the body force vector expressed in the Lagrangian coordinates, i.e.

$$u_k = \delta_{kK} u_K \quad f_k = \delta_{kK} f_K, \tag{20}$$

where δ_{kK} is the direction cosine between the Eulerian coordinates, x_k, and the Lagrangian coordinates, X_K.

Suppose the solutions at state 1 are known; the solutions at state 2 can be expressed as the sums of the solutions at state 1 and the incremental solutions, e.g.

$$T_{KL}(2) = T_{KL}(1) + \Delta T_{KL}. \tag{21}$$

Then, eqns (16) and (18) can be rewritten as

$$\int_{s^*} [\sigma_{ij}(1) + \Delta\sigma_{ij}]\delta\Delta u_i \, da_j(2) + \int \rho(2)[f_i(1) + \Delta f_i]\delta\Delta u_i \, dv(2) = \int [\sigma_{ij}(1) + \Delta\sigma_{ij}]\delta\Delta e_{ij} \, dv(2) \tag{22}$$

$$\int_{s^*} [T_{KL}(1) + \Delta T_{KL}][\delta_{NL} + u_{N,L}(1) + \Delta u_{N,L}]\delta\Delta u_N \, dA_K + \int \rho_0[f_K(1) + \Delta f_K]\delta\Delta u_K \, dV = \int [T_{KL}(1) + \Delta T_{KL}]\delta\Delta E_{KL} \, dV. \tag{23}$$

It is worthwhile to mention that, in deriving eqns (22) and (23), in other words, in the process of seeking the incremental solutions, the variations of the solutions at state 1 are vanishing.

The incremental Truesdell stresses and the incremental Washizu strains are defined as [1]:

$$\Delta\sigma^*_{ij} \equiv \frac{1}{J(1)} x_{i,K}(1)\, x_{j,L}(1)\, \Delta T_{KL} \tag{24}$$

$$\Delta\varepsilon^*_{ij} \equiv X_{K,i}(1)\, X_{L,j}(1)\, \Delta E_{KL}. \tag{25}$$

Combining eqn (24) with eqn (5), it can be readily shown that

$$\sigma_{ij}(1) + \Delta\sigma^*_{ij} = \frac{1}{J(1)} x_{i,K}(1)\, x_{j,L}(1)\, [T_{KL}(1) + \Delta T_{KL}] \tag{26}$$

$$\sigma_{ij}(2) \equiv \sigma_{ij}(1) + \Delta\sigma_{ij} = \frac{J(1)}{J(2)} [\delta_{im} + \Delta u_{i,m}(1)][\delta_{jn} + \Delta u_{j,n}(1)] \times [\sigma_{mn}(1) + \Delta\sigma^*_{mn}], \tag{27}$$

where

$$\Delta u_{i,j}(1) \equiv \frac{\partial \Delta u_i}{\partial x_j(1)}. \tag{28}$$

Eringen [3] has shown that

$$\dot{T}_{KL} \equiv \lim_{\Delta t \to 0} \frac{\Delta T_{KL}}{\Delta t} = J X_{K,i} X_{L,j} \dot{\sigma}_{ij}. \tag{29}$$

Combining eqn (24) with eqn (29), it is seen that

$$\dot{\sigma}_{ij} = \lim_{\Delta t \to 0} \frac{\Delta\sigma^*_{ij}}{\Delta t}, \tag{30}$$

which indicates the meaning of the incremental Truesdell stresses. The incremental Washizu strains are the incremental Green–Lagrangian strains taking state 1 as the reference state, and it has been shown that [1]

$$\Delta\varepsilon^*_{ij} = [\Delta u_{i,j}(1) + \Delta u_{j,i}(1) + \Delta u_{k,i}(1)\, \Delta u_{k,j}(1)]/2. \tag{31}$$

Also, it should be noted that

$$\lim_{\Delta t \to 0} \frac{\Delta\varepsilon^*_{ij}}{\Delta t} = \dot{E}_{KL} X_{K,i} X_{L,j} = d_{ij}, \tag{32}$$

which indicates the meaning of the incremental Washizu strains.

Starting from either eqn (22) or eqn (23), one may prove that the virtual work equation can be expressed as

$$\int_{s^*} [\sigma_{ij}(1) + \Delta\sigma^*_{ij}][\delta_{jk} + \Delta u_{k,j}(1)]\delta\Delta u_k \, da_i(1) + \int \rho(1)[f_i(1) + \Delta f_i]\delta\Delta u_i \, dv(1) = \int [\sigma_{ij}(1) + \Delta\sigma^*_{ij}]\delta\Delta\varepsilon^*_{ij} \, dv(1). \tag{33}$$

It is noticed that the integrations of eqns (22), (23) and (33) are to be performed, respectively, on the configurations of state 2, initial state, and state 1; therefore, Eulerian, Total-Lagrangian, and Updated-Lagrangian formulations are named for eqns (22), (23) and (33), respectively. It should be emphasized that these three equations are nothing but the same virtual work equation expressed in different incremental forms and, during the process of derivation, no approximation has ever been made.

It is straightforward to show that eqn (33) can be written as

$$\int \Delta\sigma^*_{ij}\,\delta\Delta\varepsilon^*_{ij}\, dv(1) + \int \sigma_{ij}(1)\Delta u_{k,j}(1)\delta\Delta u_{k,i}(1)\, dv(1) = \int \rho(1)\Delta f_i \delta\Delta u_i \, dv(1) + \int_{s^*} [\Delta\sigma^*_{ij} + \sigma_{kj}(1)\Delta u_{i,k}(1)]\delta\Delta u_i \, da_j(1) + \int_{s^*} \Delta\sigma^*_{ij}\Delta u_{k,j}(1)\delta\Delta u_k \, da_i(1) + R, \tag{34}$$

where

$$R \equiv \int \rho(1) f_i(1) \delta \Delta u_i \, dv(1) + \int_{s^*} \sigma_{ij}(1) \delta \Delta u_j \, da_i(1) - \int \sigma_{ij}(1) \delta \Delta e_{ij} \, dv(1). \quad (35)$$

In view of eqn (16) and by recalling that $\sigma_{ij}(1)$ and $\rho(1)$ are the solutions at state 1 subjected to the body force $f_i(1)$ and the surface traction $\sigma_{ij}(1) n_i(1)$ on s^*, it is concluded that R is vanishing. Then, with the following equalities

$$\frac{d}{dt}(\sigma_{ij} \, da_j) = (\dot{\sigma}_{ij} + \sigma_{kj} v_{i,k}) \, da_j \quad (36)$$

$$\lim_{\Delta t \to 0} \left\{ \frac{\Delta \sigma^*_{ij}}{\Delta t}; \ \frac{\Delta \varepsilon^*_{ij}}{\Delta t}; \ \frac{\Delta u_{i,j}}{\Delta t}; \ \frac{\Delta u_i}{\Delta t}; \ \frac{\Delta f_i}{\Delta t} \right\} = \{ \mathring{\sigma}_{ij}; \ d_{ij}; \ v_{i,j}; \ v_i; \ \dot{f}_i \}, \quad (37)$$

it can be shown that

$$\lim_{\Delta t \to 0} \frac{\text{eqn (34)}}{\Delta t^2} = \text{eqn (11)}, \quad (38)$$

which means, in the limiting case, the virtual work equation in incremental form is exactly the same as the virtual work equation in rate form. Therefore, the validity of the incremental virtual work equations, eqns (22), (23) and (33) is established.

4. FINITE ELEMENT FORMULATIONS

Let a solid body be devided into many elements; each element consists of N nodal points. Correspondingly, there are N shape functions, N_γ, $\gamma = 1, 2, 3, \ldots, N$, so that the Eulerian coordinates of a generic point (x, y, z) within the element can be linked to the nodal point coordinates $(\bar{x}_\gamma, \bar{y}_\gamma, \bar{z}_\gamma)$ in the following matrix form:

$$x_i = N_{i\alpha} \bar{x}_\alpha, \quad (39)$$

where:

$$\mathbf{x} \equiv (x, y, z)^T;$$

$$\bar{x}_\alpha \equiv (\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{x}_2, \bar{y}_2, z_2, \ldots, \bar{x}_N, \bar{y}_N, \bar{z}_N)^T;$$

and $N_{i\alpha}$ is a $(3 * 3N)$ matrix which can be expressed as

$$N_{i\alpha} = \begin{vmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_N & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_N & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_N \end{vmatrix}. \quad (40)$$

Similarly, in Lagrangian coordinates, the counterpart of eqn (39) can be written as

$$X_K = N_{K\alpha} \bar{X}_\alpha, \quad (41)$$

where $\bar{X}_\alpha$ is the nodal point coordinate vector in the undeformed state expressed in Lagrangian coordinates, and $N_{K\alpha}$ has the same form as $N_{i\alpha}$. Also, the displacements and incremental displacements of a generic point within the element can be linked to the counterparts of the nodal points as

$$\begin{vmatrix} u_i \\ \Delta u_i \end{vmatrix} = N_{i\alpha} \begin{vmatrix} u_\alpha \\ \Delta u_\alpha \end{vmatrix}$$

$$\begin{vmatrix} u_K \\ \Delta u_K \end{vmatrix} = N_{K\alpha} \begin{vmatrix} U_\alpha \\ \Delta U_\alpha \end{vmatrix}. \quad (42)$$

Through a very standard procedure [4, 24, 25], the displacement and the incremental displacement gradients can be expressed as

$$(u_{i,j}; \ \Delta u_{i,j})^T = B_{ij\alpha}(u_\alpha; \ \Delta u_\alpha)^T \quad (43)$$

$$(u_{K,L}; \ \Delta u_{K,L})^T = B_{KL\alpha}(U_\alpha; \ \Delta U_\alpha)^T, \quad (44)$$

where $\mathbf{B}$ is a $(3 * 3 * 3N)$ matrix than can be obtained through the shape functions and the nodal point coordinates.

In the Updated-Lagrangian approach, suppose the solutions at state 1 are obtained; then eqn (16) written in the following form has to be satisfied:

$$\int \sigma_{ij}(1) \, \delta e_{ij}(1) \, dv(1) - \int \rho(1) f_i(1) \, \delta u_i \, dv(1) - \int_{s^*} \sigma_{ij}(1) \, \delta u_i \, da_j(1) = 0, \quad (45)$$

which can be rewritten as

$$\delta u_\alpha \left[\int \sigma_{ij}(1) \, B_{ij\alpha}(1) \, dv(1) - \int \rho(1) f_i(1) \, N_{i\alpha} \, dv(1) - \int_{s^*} \sigma_{ij}(1) \, N_{i\alpha} \, da_j(1) \right] \equiv \delta u_\alpha F_\alpha = 0. \quad (46)$$

It is noticed that F_α is the nodal force vector of a generic element. If eqn (46) is summed over all the elements of the solid body, one may obtain the following:

$$\delta u_\zeta F_\zeta = 0 \quad \zeta = 1, 2, 3, \ldots N^d, \tag{47}$$

where N^d stands for the number of degrees of freedom of the entire solid body. Because eqn (47) should be valid for arbitrary δu_ζ except for those components which are specified through displacement boundary conditions, therefore, one may conclude that: (1) for all the components where the displacements are specified, the reactive forces are obtained through the calculations indicated in eqn (46); and (2) for all the other components, $F_\zeta = 0$, which means the equilibrium is reached pointwise. The calculation of nodal forces may now be expressed by

$$\mathbf{F} = \sum_{\text{elements}} \mathbf{F}^e \tag{48}$$

$$F^e_\alpha = \int \sigma_{ij} B_{ij\alpha}\, \mathrm{d}v - \int \rho f_i N_{i\alpha}\, \mathrm{d}v - \int_{s^\sigma} \sigma_{ij} N_{i\alpha}\, \mathrm{d}a_j. \tag{49}$$

Similarly, in the Total-Lagrangian approach, recall eqn (19) and notice that the variation of the Green–Lagrangian strains can be expressed as

$$\delta E_{KL} = 1/2(B_{KL\alpha} + B_{LK\alpha} + B_{MK\beta}\, U_\beta\, B_{ML\alpha} + B_{ML\beta}\, U_\beta\, B_{MK\alpha})\delta U_\alpha. \tag{50}$$

For the calculation of nodal forces, the counterpart of eqn (49) can be written as

$$F^e_\alpha = \int T_{KL}\, \hat{B}_{KL\alpha}\, \mathrm{d}V - \int \rho_0 f_K\, N_{K\alpha}\, \mathrm{d}V - \int_{s^\sigma} T_{KL}(\delta_{ML} + B_{ML\beta}\, U_\beta) N_{M\alpha}\, \mathrm{d}A_K, \tag{51}$$

where

$$\hat{B}_{KL\alpha} \equiv B_{KL\alpha} + B_{MK\beta}\, \mathrm{U}_\beta\, \mathrm{B}_{\mathrm{ML}\alpha}. \tag{52}$$

In order to solve for the incremental solutions using the Updated-Lagrangian approach, the terms in eqn (33) are treated as follows:

$$\begin{aligned}\int \Delta\sigma^*_{ij}\, \delta\Delta\varepsilon^*_{ij}\, \mathrm{d}v(1) &\simeq \int a^*_{ijmn}(1)\, \Delta\varepsilon^*_{mn}\, \delta\Delta\varepsilon^*_{ij}\, \mathrm{d}v(1) \\ &\simeq \int a^*_{ijmn}(1)\, \Delta e_{mn}\, \delta\Delta e_{ij}\, \mathrm{d}v(1) \\ &= \delta\Delta u_\beta\, \Delta u_\alpha \int a^*_{ijmn}(1)\, B_{mn\alpha}\, B_{ij\beta}\, \mathrm{d}v(1) \\ &\equiv \delta\Delta u_\beta\, \Delta u_\alpha\, K^{(1)}_{\alpha\beta}\end{aligned} \tag{53}$$

$$\begin{aligned}\int \sigma_{ij}(1)\, \delta\Delta\varepsilon^*_{ij}\, \mathrm{d}v(1) &= \int \sigma_{ij}(1)\, \delta\Delta e_{ij}\, \mathrm{d}v(1) + \int \sigma_{ij}(1)\, \Delta u_{k,j}\, \delta\Delta u_{k,i}\, \mathrm{d}v(1) \\ &= \left[\int \sigma_{ij}(1)\, B_{ij\beta}\, \mathrm{d}v(1)\right]\delta\Delta u_\beta + \delta\Delta u_\beta\, \Delta u_\alpha \int \sigma_{ij}(1)\, B_{kj\alpha}\, B_{ki\beta}\, \mathrm{d}v(1) \\ &\equiv -F^{(1)}_\beta \delta\Delta u_\beta + \delta\Delta u_\beta\, \Delta u_\alpha\, K^{(2)}_{\alpha\beta}\end{aligned} \tag{54}$$

$$\begin{aligned}\int \rho(1)[f_i(1) + \Delta f_i]\delta\Delta u_i\, \mathrm{d}v(1) &= \left[\int \rho(1)[f_i(1) + \Delta f_i] N_{i\beta}\, \mathrm{d}v(1)\right]\delta\Delta u_\beta \\ &\equiv F^{(2)}_\beta \delta\Delta u_\beta\end{aligned} \tag{55}$$

$$\begin{aligned}\int_{s^\sigma} [\sigma_{ij}(1) + \Delta\sigma^*_{ij}][\delta_{jk} + \Delta u_{k,j}]\delta\Delta u_k\, \mathrm{d}a_i(1) &\simeq \int_{s^\sigma} [\sigma_{ij}(1) + \Delta\sigma_{ij}]\delta\Delta u_j\, \mathrm{d}a_i(1) \\ &= \delta\Delta u_\beta \int_{s^\sigma} [\sigma_{ij}(1) + \Delta\sigma_{ij}] N_{j\beta}\, \mathrm{d}a_i(1) \\ &\equiv F^{(3)}_\beta\, \delta\Delta u_\beta,\end{aligned} \tag{56}$$

where a^*_{ijmn} specifies the constitutive relation for the elastic–plastic solid, i.e.

$$\mathring{\sigma}_{ij} = a^*_{ijmn}\, \mathrm{d}_{mn}. \tag{57}$$

The detailed expression of a^*_{ijmn} will be indicated in the next section. Now, the governing matrix equations for a generic element become

$$[K^{(1)}_{\alpha\beta} + K^{(2)}_{\alpha\beta}]\Delta u_\beta = F^{(1)}_\alpha + F^{(2)}_\alpha + F^{(3)}_\alpha. \tag{58}$$

In order to solve for the incremental solution using the Total-Lagrangian approach, first, the incremental Green–Lagrangian strains can be found, by using eqn (19), as

$$\Delta E_{KL} = 1/2[\Delta u_{K,L} + \Delta u_{L,K} + u_{M,K}\, \Delta u_{M,L} + u_{M,L}\, \Delta u_{M,K} + \Delta u_{M,K}\, \Delta u_{M,L}]. \tag{59}$$

Define ΔG_{KL} and ΔH_{KL} as

$$\Delta G_{KL} \equiv \Delta u_{K,L} + u_{M,K}\, \Delta u_{M,L} = \hat{B}_{KL\alpha}\, \Delta U_\alpha, \tag{60}$$

$$\Delta H_{KL} \equiv 1/2 \Delta u_{M,K}\, \Delta u_{M,L}. \tag{61}$$

Now, the terms in eqns (23) are treated as

$$\int \Delta T_{KL}\,\delta\Delta E_{KL}\,\mathrm{d}V$$

$$\simeq \int A_{KLMN}(1)\,\Delta E_{MN}\,\delta\Delta E_{KL}\,\mathrm{d}V$$

$$\simeq \int A_{KLMN}(1)\,\Delta G_{MN}\,\delta\Delta G_{KL}\,\mathrm{d}V$$

$$= \delta\Delta U_\beta\,\Delta U_\alpha \int A_{KLMN}(1)\,\hat{B}_{MN\alpha}\,\hat{B}_{KL\beta}\,\mathrm{d}V$$

$$\equiv \delta\Delta U_\beta\,\Delta U_\alpha\,K^{(1)}_{\alpha\beta} \tag{62}$$

$$\int T_{KL}(1)\,\delta\Delta E_{KL}\,\mathrm{d}V$$

$$= \left[\int T_{KL}(1)\,\hat{B}_{KL\beta}\,\mathrm{d}V\right]\delta\Delta U_\beta$$

$$+\,\delta\Delta U_\beta\,\Delta U_\alpha \int T_{KL}(1)\,B_{MK\alpha}\,B_{ML\beta}\,\mathrm{d}V$$

$$\equiv -F^{(1)}_\beta\,\delta\Delta U_\beta + \delta\Delta U_\beta\,\Delta U_\alpha\,K^{(2)}_{\alpha\beta} \tag{63}$$

$$\int \rho_0[f_K(1)+\Delta f_K]\delta\Delta u_K\,\mathrm{d}V$$

$$= \delta\Delta U_\beta \int \rho_0[f_K(1)+\Delta f_K]N_{K\beta}\,\mathrm{d}V$$

$$\equiv \delta\Delta U_\beta\,F^{(2)}_\beta \tag{64}$$

$$\int_{s^*} [T_{KL}(1)+\Delta T_{KL}][\delta_{ML}+u_{M,L}(1)$$

$$+\,\Delta u_{M,L}]\delta\Delta u_M\,\mathrm{d}A_k$$

$$\simeq \int_{s^*} [T_{KL}(1)+\Delta T_{KL}]$$

$$\times\,[\delta_{ML}+u_{M,L}(1)]\delta\Delta u_M\,\mathrm{d}A_k$$

$$= \delta\Delta U_\beta \int_{s^*} [T_{KL}(1)+\Delta T_{KL}]$$

$$\times\,[\delta_{ML}+B_{ML\gamma}\,U_\gamma(1)]N_{M\beta}\,\mathrm{d}A_k$$

$$\equiv \delta\Delta U_\beta\,F^{(3)}_\beta, \tag{65}$$

where A_{KLMN} specifies the following constitutive relation of the elastic–plastic solid.

$$\dot{T}_{KL} = A_{KLMN}\,\dot{E}_{MN}. \tag{66}$$

The detailed expression of A_{KLMN} will be indicated in Sec. 6. Now, the governing matrix equations for a generic element assume the same form as eqn (58). When the matrix equations for all the elements in the solid body are summed up, the grand matrix equations of the entire mechanical system are obtained. After the displacement-specified boundary conditions are imposed, one may solve for the incremental nodal point displacements. It should be emphasized that whether a set of solutions is acceptable or not sepends on whether the equilibrium of nodal forces, the calculation of which is an exact treatment, is reached or not; therefore, the approximations made in deriving eqns (53), (56), (62) and (65) in order to obtain the incremental solutions in an iterative process do not imply that the final accepted solutions are approximated ones. The point will be seen clearly in the following sections.

5. ITERATIVE PROCEDURES FOR LEE'S THEORY

Lee and his co-workers [10–15] have formulated a theory of plasticity based on the exact nonlinear kinematics of elastic and plastic deformations. Among all the existing theories of plasticity, Lee's theory is a very unique one. Instead of assuming that the total strain is the summation of the elastic strain and the plastic strain, Lee's theory begins with the fact that the deformation gradient is the product of the elastic and the plastic deformation gradients. Therefore, it is worthwhile to demonstrate how the finite element formulations derived in the previous sections can be applied for Lee's theory.

To begin with, let Lee's theory be briefly described in the following [10]. First, let the deformation gradient $x_{i,K}$ be expressed as

$$x_{i,K} = F^e_{ij}\,F^p_{jK}, \tag{67}$$

where $\mathbf{F}^e$ and $\mathbf{F}^p$ may be named, respectively, as the elastic part and plastic part of the deformation gradient and $\mathbf{F}^e$ specifies the mapping from the unstressed plastically deformed configuration to the elastically–plastically deformed configuration. However, the decomposition as indicated in eqn (67) is not unique because any arbitrary local rotation in the unstressed state gives another unstressed configuration. Therefore, further restrictions have to be imposed on $\mathbf{F}^e$, namely, $\mathbf{F}^e$ is symmetric and has the same principal directions as the stress tensor and the plastic part of the deformation rate tensor. These restrictions imply that the material under consideration is isotropic and obeys the isotropic hardening rule. Define $\bar{\mathbf{F}}^e$ and $\bar{\mathbf{F}}^p$ to be the inverse of $\mathbf{F}^e$ and $\mathbf{F}^p$, respectively, i.e.

$$F^e_{ij}\,\bar{F}^e_{jk} = \bar{F}^e_{ij}\,F^e_{jk} = \delta_{ik} \tag{68}$$

$$F^p_{kK}\,\bar{F}^p_{Km} = \delta_{km}$$

$$\bar{F}^p_{Kk}\,F^p_{kL} = \delta_{KL}. \tag{69}$$

The elastic and the plastic parts of the deformation rate tensor and the spin tensor are defined as

$$d^e_{ij} \equiv 1/2(\dot{F}^e_{ik}\bar{F}^e_{kj} + \dot{F}^e_{jk}\bar{F}^e_{ki}) \tag{70}$$

$$\omega^e_{ij} \equiv 1/2(\dot{F}^e_{ik}\bar{F}^e_{kj} - \dot{F}^e_{jk}\bar{F}^e_{ki}) \tag{71}$$

$$d^p_{IJ} \equiv 1/2(\dot{F}^p_{IK}\bar{F}^p_{KJ} + \dot{F}^p_{JK}\bar{F}^p_{KI}) \tag{72}$$

$$\omega^p_{IJ} \equiv 1/2(\dot{F}^p_{IK}\bar{F}^p_{KJ} - \dot{F}^p_{JK}\bar{F}^p_{KI}). \tag{73}$$

It can be shown that

$$\begin{aligned} d_{ij} &= d^e_{ij} + 1/2(F^e_{im}\omega^p_{mn}\bar{F}^e_{nj} \\ &\quad + F^e_{jm}\omega^p_{mn}\bar{F}^e_{ni}) + d^p_{ij} \\ &\equiv \hat{d}_{ij} + d^p_{ij} \end{aligned} \tag{74}$$

$$\omega_{ij} = \omega^e_{ij} + 1/2(F^e_{im}\omega^p_{mn}\bar{F}^e_{nj} - F^e_{jm}\omega^p_{mn}\bar{F}^e_{ni}), \tag{75}$$

where eqn (74) indicates that the deformation rate tensor is not just the sum of the elastic and the plastic parts—a difference between Lee's theory and other theories of plasticity. Then, Lubarda and Lee [10] obtained the constitutive relation between the Kirchhoff stress tensor and the elastic Cauchy–Green tensor as

$$\tau_{ij} = 2c_{ik}\frac{\partial\Sigma}{\partial c_{kj}}, \tag{76}$$

where Σ is the Helmholtz free energy density and

$$c_{ij} \equiv F^e_{ik}F^e_{jk}. \tag{77}$$

Since the material is isotropic, Σ must be the function of the three invariants of c_{ij}, which are defined as

$$I_1 \equiv c_{ii}, \tag{78}$$

$$I_2 \equiv (I_1^2 - c_{ij}c_{ij})/2, \tag{79}$$

$$I_3 \equiv 1/6 e_{ijk}e_{rst}c_{ir}c_{js}c_{kt} = \det(\mathbf{c}). \tag{80}$$

Then, eqn (76) can be written as

$$\tau_{ij} = 2[J_1 c_{ij} + J_2(I_1 c_{ij} - c_{ik}c_{kj}) + J_3 I_3 \delta_{ij}], \tag{81}$$

where

$$J_i \equiv \frac{\delta\Sigma}{\partial I_i}. \tag{82}$$

It is noticed that eqn (81) does imply that the Kirchhoff stress tensor and the elastic Cauchy–Green tensor, defined in eqn (77), have the same principal directions. With the equality

$$\hat{c}_{ij} = \hat{d}_{ij}c_{kj} + c_{ik}\hat{d}_{kj} \tag{83}$$

one can show that

$$\hat{\tau}_{ij} = b_{ijmn}\hat{d}_{mn}, \tag{84}$$

where $\hat{c}_{ij}$ and $\hat{\tau}_{ij}$ are the Jaumann rates of c_{ij} and τ_{ij} respectively, and the detailed expression of b_{ijmn} is

$$\begin{aligned} b_{ijmn} &= J_1\alpha_{ijmn} + J_2[I_1\alpha_{ijmn} - \beta_{ijmn} + 4c_{ij}c_{mn} \\ &\quad - 2c_{im}c_{jn} - 2c_{in}c_{jm}] \\ &\quad + 4J_3 I_3 \delta_{ij}\delta_{mn} + 4J_{11}c_{ij}c_{mn} \\ &\quad + 4J_{22}g_{ij}g_{mn} + 4J_{33}I_3^2\delta_{ij}\delta_{mn} \\ &\quad + 4J_{12}[c_{ij}g_{mn} + c_{mn}g_{ij}] \\ &\quad + 4J_{13}I_3[c_{ij}\delta_{mn} + c_{mn}\delta_{ij}] \\ &\quad + 4J_{23}I_3[\delta_{ij}g_{mn} + \delta_{mn}g_{ij}], \end{aligned} \tag{85}$$

in which

$$J_{ij} \equiv \frac{\partial^2\Sigma}{\partial I_i \partial I_j} \tag{86}$$

$$g_{ij} \equiv I_1 c_{ij} - c_{ik}c_{kj} \tag{87}$$

$$\alpha_{ijmn} \equiv \delta_{im}c_{jn} + \delta_{in}c_{jm} + \delta_{jm}c_{in} + \delta_{jn}c_{im} \tag{88}$$

$$\begin{aligned} \beta_{ijmn} &\equiv \delta_{im}c_{jk}c_{kn} + \delta_{in}c_{jk}c_{km} \\ &\quad + \delta_{jm}c_{ik}c_{kn} + \delta_{jn}c_{ik}c_{km}. \end{aligned} \tag{89}$$

It is noticed that

$$b_{ijmn} = b_{mnij} = b_{jimn} = b_{ijnm}, \tag{90}$$

and, also, $\mathbf{b}$ is a function of $\mathbf{c}$. In view of eqn (81), $\mathbf{b}$ can be transformed to be a function of the stresses. For the plastic part of the constitutive relations, let the yield function be

$$f = 3/2\tau'_{ij}\tau'_{ij} - S^2, \tag{91}$$

where S is the current yield strength and τ'_{ij} is the Kirchhoff stress deviator, defined as

$$\tau'_{ij} \equiv \tau_{ij} - 1/3\tau_{kk}\delta_{ij}. \tag{92}$$

The loading rate, L, defined as

$$L \equiv \frac{\partial f}{\partial \tau_{ij}}\hat{\tau}_{ij}, \tag{93}$$

can be shown to be

$$L = 3\tau'_{ij}\hat{\tau}_{ij}. \tag{94}$$

Upon loading, defined as

$$f = 0, \quad L > 0, \tag{95}$$

it is assumed that

$$d^p_{ij} = q \frac{\partial f}{\partial \tau_{ij}} L, \tag{96}$$

which can be rewritten as

$$d^p_{ij} = 9q\tau'_{ij}\tau'_{mn}\mathring{t}_{mn}, \tag{97}$$

where q can be a function of the invariants of τ_{ij}. Due to the requirement of continuity in plastic loading [16],

$$S^2 = 3/2\tau'_{ij}\tau'_{ij} \tag{98}$$

holds for the entire loading process. In unloading, defined as $f < 0$ or $L < 0$, and in neutral loading, defined as $f = 0$ and $L = 0$, it is assumed that

$$d^p_{ij} = \dot{S} = 0. \tag{99}$$

From eqn (97) it is noticed that

$$d^p_{ii} = 0, \tag{100}$$

which means that the plastic strains do not contribute to volume change, Drucker's normality condition is imposed [26, 27], and the stress tensor has the same principal directions as d^p_{ij}. Combining eqns (84) and (97) with eqn (74), one obtains

$$\mathring{t}_{ij} = a_{ijmn} d_{mn}, \tag{101}$$

where a_{ijmn} is the inverse of

$$\bar{a}_{ijmn} \equiv \bar{b}_{ijmn} + 9q\tau'_{ij}\tau'_{mn} \tag{102}$$

and $\bar{b}_{ijmn}$ is the inverse of b_{ijmn}, i.e.

$$\begin{vmatrix} a \\ b \end{vmatrix}_{ijmn} \begin{vmatrix} \bar{a} \\ \bar{b} \end{vmatrix}_{mnpq} = 1/2(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}). \tag{103}$$

Now, defining a^*_{ijmn} as

$$a^*_{ijmn} \equiv a_{ijmn}/J - (\delta_{im}\sigma_{jn} + \delta_{in}\sigma_{jm} + \delta_{jm}\sigma_{in} + \delta_{jn}\sigma_{im})/2, \tag{104}$$

then the Truesdell stress rate tensor $\mathring{\sigma}_{ij}$ can be related to the deformation rate tensor d_{mn} as

$$\mathring{\sigma}_{ij} = a^*_{ijmn} d_{mn}. \tag{105}$$

Also, it is noticed that, in the neutral loading or unloading case, $a_{ijmn} = b_{ijmn}$. At this moment, for those elements which are still in the elastic region, i.e. no plastic strain has occurred, it is worthwhile to prove the following equality:

$$\begin{aligned} 2x_{i,K}x_{j,L}\frac{\partial \Sigma}{\partial C_{KL}} &= 2F^e_{im}F^p_{mK}F^e_{jn}F^p_{nL}\frac{\partial \Sigma}{\partial c_{pq}}\frac{\partial c_{pq}}{\partial C_{KL}} \\ &= 2F^e_{im}F^e_{jn}F^p_{mK}F^p_{nL} \\ &\quad \times F^p_{Kp}F^p_{Lq}\frac{\partial \Sigma}{\partial c_{pq}} \\ &= 2F^e_{im}F^e_{jn}\frac{\partial \Sigma}{\partial c_{mn}} \\ &= 2F^e_{im}F^e_{nm}\frac{\partial \Sigma}{\partial c_{nj}} \\ &= 2c_{in}\frac{\partial \Sigma}{\partial c_{nj}} = \tau_{ij}, \end{aligned} \tag{106}$$

and, also, $\mathbf{F}^p$ can be only an orthogonal transformation matrix, which means

$$\begin{aligned} C_{KK} &= I_1, \\ (C_{KK}C_{LL} - C_{KL}C_{KL})/2 &= I_2, \\ \det(\mathbf{C}) &= I_3. \end{aligned} \tag{107}$$

This implies that, for elements still in the elastic region, τ_{ij} can be calculated as if it is an elastic solid without even changing the form of the Helmholtz free energy density.

The finite element procedures for large strain elastic–plastic solid based on Lee's theory can now be outlined as follows:

Step 1

Based on the current stresses, the elastic Cauchy–Green tensor c_{ij} can be calculated according to eqn (81) and then b_{ijmn} and a^*_{ijmn} are calculated according to eqn (85) and eqns (102)–(104) respectively. $B_{ij\alpha}$ is calculated based on the current geometry. Then, the element stiffness matrices, $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$, and the forcing terms, $\mathbf{F}^{(1)}$, $\mathbf{F}^{(2)}$, and $\mathbf{F}^{(3)}$, are obtained by using eqns (53) and (54) and eqns (54)–(56) respectively.

Step 2

The grand matrix equations can be formed as

$$\mathbf{K}\,\Delta\mathbf{u} = \mathbf{F}. \tag{108}$$

After the displacement-specified boundary conditions are imposed, one may solve for $\Delta\mathbf{u}$, the nodal point incremental displacement vector.

Step 3

The incremental Washizu strains are calculated as

$$\begin{aligned} \Delta\varepsilon^*_{ij} &= 1/2[\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}] \\ &= 1/2[B_{ij\alpha} + B_{ji\alpha} + {}_{ki\beta}\Delta u_\beta B_{kj\alpha}]\Delta u_\alpha. \end{aligned} \tag{109}$$

Step 4

Approximate the constitutive relation in rate form, eqn (105), by

$$\Delta\sigma^*_{ij} = c_{ijmn}\,\Delta\varepsilon^*_{mn}, \tag{110}$$

where

$$c_{ijmn} = \text{average}\,\{a^*_{ijmn}(\sigma(1)),\quad a^*_{ijmn}(\sigma(2))\}. \tag{111}$$

This is actually the first approximation made in this paper. Also, in view of eqns (30) and (32), it is noticed that, in the limiting case, eqn (110) approaches eqn (105). Then, recall eqn (27) and rewrite it as

$$\sigma_{ij}(2) = \frac{J(1)(\delta_{im} + B_{im\alpha}\Delta u_\alpha)(\delta_{jn} + B_{jn\beta}\Delta u_\beta)}{\det\,[\delta_{kK} + (u_k + \Delta u_k)_{,K}]} \times [\sigma_{mn}(1) + \Delta\sigma^*_{mn}]. \tag{112}$$

It is seen that eqns (110)–(112) involve iterations. Let S_1 and S_2 be defined as

$$S_1^2 \equiv 3/2\,\tau'_{ij}(1)\,\tau'_{ij}(1)$$

$$S_2^2 \equiv 3/2\,\tau'_{ij}(2)\,\tau'_{ij}(2). \tag{113}$$

If $S_2 \le S$ or $S_1 = S$ and $S_2 \ge S$, let

$$c_{ijmn} = 1/2\,\{a^*_{ijmn}(\sigma(1)) + a^*_{ijmn}(\sigma(2))\}. \tag{114}$$

If $S_1 < S$ and $S_2 > S$, let

$$\bar{c}_{ijmn} = \{\bar{a}^*_{ijmn}(\sigma(1))(S - S_1) + \bar{a}^*_{ijmn}(\sigma(2))(S_2 - S)\}/(S_2 - S_1), \tag{115}$$

where $\bar{\mathbf{c}}$ and $\bar{\mathbf{a}}^*$ are the inverse of $\mathbf{c}$ and $\mathbf{a}^*$, respectively.

Step 5

For those elements which never experience plastic deformation, Step 3 and 4 should be bypassed. Instead, the following will be calculated.

$$x_{k,K} = \delta_{kK} + (u_k + \Delta u_k)_{,K} = \delta_{kK} + [N_{k\alpha}(u_\alpha + \Delta u_\alpha)]_{,K} \tag{116}$$

$$\sigma_{ij}(2) = \frac{2}{J}\frac{\partial\Sigma}{\partial C_{KL}}\,x_{i,K}\,x_{j,L}. \tag{117}$$

It is noticed that this step does not involve the approximation made in eqn (110).

Step 6

Update the geometry, the displacement field, and the stress distributions as

$$\bar{x}_\alpha \leftarrow \bar{x}_\alpha + \Delta u_\alpha \tag{118}$$

$$u_\alpha \leftarrow u_\alpha + \Delta u_\alpha \tag{119}$$

$$\sigma_{ij} \leftarrow \sigma_{ij}(2). \tag{120}$$

Then, the $B_{ij\alpha}$ matrix should be updated also and all the other quantities, if needed, can be updated.

Step 7

The nodal force vector is now calculated according to eqns (48) and (49). If the calculated nodal force vector indicates that the equilibrium is not reached pointwise, then, in view of eqns (49), (54), (55) and (56), the nodal force vector, **F**, indicated in eqn (48), actually serves as the forcing term for further iteration, namely, one should go back to Step 1.

If the equilibrium is reached pointwise, or, at least, the error is within certain given error tolerance, one may change the applied loading, or even the displacement-specified boundary conditions, and go back to Step 1 to seek the solutions at the next incremental level.

6. ITERATIVE PROCEDURES FOR THE GREEN–NAGHDI THEORY

Green and Naghdi [16, 17] formulated a very general theory for elastic–plastic continuum. Later, Casey and Naghdi [18–21] addressed further issues related to that general theory. Since the Green–Naghdi theory has been referenced by many researchers in the field of plasticity, it has been decided to demonstrate how the finite element formulations derived previously can be applied for that theory.

The basic constitutive relations, with the thermomechanical coupling being neglected, of the Green–Naghdi theory are listed in the following [16]:

$$E_{KL} = E^e_{KL} + E^p_{KL} \tag{121}$$

$$\Sigma = \Sigma(\mathbf{E}^e) \tag{122}$$

$$T_{KL} = \frac{\partial\Sigma}{\partial E^e_{KL}} = T_{KL}(\mathbf{E}^e) \tag{123}$$

$$F = f(\mathbf{T}, \mathbf{E}^p) - K \tag{124}$$

$$L = \frac{\partial f}{\partial T_{KL}}\,\dot{T}_{KL} \tag{125}$$

$$\dot{E}^p_{KL} = \lambda\frac{\partial f}{\partial T_{KL}}\,L \text{ (loading)} \tag{126}$$

$$\dot{K} = \left(1 + \lambda\frac{\partial f}{\partial T_{KL}}\frac{\partial f}{\partial E^p_{KL}}\right)L \text{ (loading)} \tag{127}$$

$$\dot{E}^p_{KL} = \dot{K} = 0 \text{ (neutral loading or unloading)}, \tag{128}$$

where $\mathbf{E}^e$ is the elastic Green–Lagrangian strain; $\mathbf{E}^p$ is the plastic Green–Lagrangian strain; $F = 0$ specifies the yield surface; K may be called the hardening parameter; L is the loading rate; the case of loading is defined as $F = 0$ and $L > 0$; the case of neutral loading is defined as $F = 0$ and $L = 0$; and the case of unloading is defined as $F < 0$ or $L < 0$. It is seen that, first, the decomposition of elastic and plastic parts is made at the strain level, instead of at the deformation gradient level as in the Lee's theory. Second, the yield function, f, and the hardening rule, eqn (127), are more general than the counterparts in the Lee theory—this means the size and the location of the yield surface in the stress space can be changed during the loading process. Third, the Green–Naghdi theory is not restricted to isotropic material with isotropic hardening rule. Also, it is noticed that the continuity condition is imposed by eqn (127), i.e. in loading,

$$\dot{F} = \frac{\partial f}{\partial T_{KL}} \dot{T}_{KL} + \frac{\partial f}{\partial E^p_{KL}} \dot{E}^p_{KL} - \dot{K}$$

$$= L\left\{1 + \lambda \frac{\partial f}{\partial E^p_{KL}} \frac{\partial f}{\partial T_{KL}}\right\} - \dot{K} = 0 \quad (129)$$

and the Drucker's normality condition is imposed by eqn (126), in which λ can be a function of T_{KL} and E^p_{KL}.

Equations (123) and (126) can also be written as

$$\dot{T}_{KL} = \frac{\partial^2 \Sigma}{\partial E^e_{KL}\, \partial E^e_{MN}} \dot{E}^e_{MN} \equiv B_{KLMN} \dot{E}^e_{MN} \quad (130)$$

$$\dot{E}^p_{KL} = \lambda \frac{\partial f}{\partial T_{KL}} \frac{\partial f}{\partial T_{MN}} \dot{T}_{MN}. \quad (131)$$

Combining eqns (130) and (131) obtains

$$\dot{T}_{KL} = A_{KLMN} \dot{E}_{MN}, \quad (132)$$

where $\mathbf{A}$ is the inverse of

$$\bar{A}_{KLMN} \equiv \bar{B}_{KLMN} + \lambda \frac{\partial f}{\partial T_{KL}} \frac{\partial f}{\partial T_{MN}} \quad (133)$$

and $\bar{\mathbf{B}}$ is the inverse of B_{KLMN}. In case of neutral loading or unloading, $A_{KLMN} = B_{KLMN}$. In finite element analysis, the following incremental constitutive relation is needed:

$$\Delta T_{KL} = \hat{A}_{KLMN} \Delta E_{MN}, \quad (134)$$

and it is proposed to approximate $\hat{A}$ by

$$\hat{A}_{KLMN} = \text{average}\,\{A_{KLMN}[\mathbf{T}(1), \mathbf{E}^p(1)],$$

$$A_{KLMN}[\mathbf{T}(2), \mathbf{E}^p(2)]\}. \quad (135)$$

Now, the finite element procedures for large strain elastic–plastic solid based on the Green–Naghdi theory can be outlined as follows.

Step 1

Based on the current stresses, plastic strains, and displacements, the element stiffness matrices $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ are calculated according to eqns (62) and (63) and the forcing terms, $\mathbf{F}^{(1)}$, $\mathbf{F}^{(2)}$ and $\mathbf{F}^{(3)}$ are calculated according to eqns (63)–(65).

Step 2

The grand matrix equations can be formed as

$$\mathbf{K}\Delta\mathbf{U} = \mathbf{F}. \quad (136)$$

After the displacement-specified boundary conditions are imposed, one may solve for $\Delta\mathbf{U}$, the nodal point increment displacement vector.

Step 3

The incremental Green–Lagrangian strains are calculated, according to eqn (59), as

$$\Delta E_{KL} = 1/2(\hat{B}_{KL\alpha} + \hat{B}_{LK\alpha})\Delta \mathbf{U}_\alpha$$

$$+ 1/2\, B_{MK\alpha} B_{ML\beta} \Delta\mathbf{U}_\alpha \Delta\mathbf{U}_\beta. \quad (137)$$

Step 4

Calculate $\mathbf{E}^e(1)$ according to eqn (123) as

$$T_{KL}(1) = T_{KL}(\mathbf{E}^e(1)). \quad (138)$$

Let $\mathbf{E}^e(2)$ be $\mathbf{E}^e(1) + \Delta\mathbf{E}$ and then calculate the following:

$$T_{KL}(2) = T_{KL}(\mathbf{E}^e(2)) \quad (139)$$

$$F = f[\mathbf{T}(2),\ \mathbf{E}^p(1)] - \mathrm{K}. \quad (140)$$

If $F \le 0$, then T_{KL} is updated and go to Step 6; otherwise, go to Step 5.

Step 5

(A) Make an educated guess for $T_{KL}(2)$, and calculate E^e_{KL} according to eqn (139).

(B) Calculate $\hat{E}^p_{KL}(2)$ according to

$$\hat{E}^p_{KL}(2) = E^e_{KL}(1) + E^p_{KL}(1) + \Delta E_{kL} - E^e_{KL}(2). \quad (141)$$

(C) Calculate the following:

$$f_1 = f[\mathbf{T}(1),\ \mathbf{E}^p(1)],$$

$$f_2 = f[\mathbf{T}(2),\ \hat{\mathbf{E}}^p(2)] \quad (142)$$

$$A_{KLMN}(1) = A_{KLMN}[\mathbf{T}(1),\ \mathbf{E}^p(1)]$$

$$A_{KLMN}(2) = A_{KLMN}[\mathbf{T}(2),\ \hat{\mathbf{E}}^p(2)]. \quad (143)$$

(D) If $f_1 = K$ and $f_2 > K$, let

$$\hat{\mathbf{A}} = 1/2[\mathbf{A}(1) + \mathbf{A}(2)] \quad (144)$$

and if

$$f_1 < K \quad \text{and} \quad f_2 > K,$$

let

$$\mathbf{C} \equiv \{\bar{A}(1)(K - f_1) + \bar{A}(2)(f_2 - K)\}/(f_2 - f_1) \quad (145)$$

and

$$\hat{A}_{IJKL} C_{KLMN} = 1/2(\delta_{IM}\delta_{JN} + \delta_{IN}\delta_{JM}), \quad (146)$$

where $\bar{\mathbf{A}}$ stands for the inverse of $\mathbf{A}$.

(E) Calculate $T'_{KL}(2)$ as

$$T'_{KL}(2) = T_{KL}(1) + \hat{A}_{KLMN}\Delta E_{MN}. \quad (147)$$

If T'_{KL} is approximately equal to $T_{KL}(2)$, within a given error tolerance, then update the following:

$$T_{KL}(2) \leftarrow T'_{KL}(2) \quad (148)$$

$$K \leftarrow f_2 \quad (149)$$

$$E^p_{KL}(2) \leftarrow \hat{E}^p_{KL}(2), \quad (150)$$

and go to Step 6; otherwise, go back to (A).

Step 6

Now the nodal point displacements are updated as

$$U_\alpha \leftarrow U_\alpha + \Delta U_\alpha, \quad (151)$$

and, also, $\hat{B}_{KL\alpha}$ should be updated according to eqn (52). Calculate the nodal force vector according to eqns (48) and (51). If the equilibrium is not reached pointwise, or the error is not within a given error tolerance, then in view of eqns (51), (63), (64) and (65), the calculated nodal force vector actually serves as the forcing term for further iteration, namely, one should go back to Step 1; otherwise, one may change the force and/or displacement specified boundary conditions and go back to Step 1 to seek the solutions at the next incremental level.

7. DISCUSSION

The virtual work equation in rate form, eqn (11) or eqn (13), is obtained exactly. The virtual work equations in incremental forms, namely, the Eulerian, the Total-Lagrangian, and the Updated-Lagrangian formulations, are rigorously derived to be eqns (22), (23) and (33), respectively. Actually, the three incremental forms can be transformed from one to the other without any approximation. Besides, in the limiting case, the incremental forms and the rate forms are identical. Also, it is noticed that if the constitutive relation, in rate form, of any theory of plasticity can be expressed by any one of the following:

$$\mathring{t}_{ij} = a_{ijmn} d_{mn} \quad (152)$$

$$\mathring{\sigma}_{ij} = a^*_{ijmn} d_{mn} \quad (153)$$

$$\mathring{T}_{KL} = A_{KLMN}\dot{E}_{MN}, \quad (154)$$

then they can be transformed from one to the other by the following rules.

$$A_{IJMN} = J a^*_{ijmn} X_{I,i} X_{J,j} X_{M,m} X_{N,n} \quad (155)$$

$$a^*_{ijmn} = 1/J a_{ijmn} - 1/2(\delta_{im}\sigma_{jn} + \delta_{in}\sigma_{jm} + \delta_{jm}\sigma_{in} + \delta_{jn}\sigma_{im}). \quad (156)$$

The linearizations made in eqns (53) and (56) or in eqns (62) and (65), in order to evaluate the stiffness matrix and the forcing term, do not present approximations for the overall formulation because the validity of the solutions is based on the equilibriums of the nodal forces which are calculated exactly. Also, it is noticed that the follower-type forces are incorporated automatically into this formulation. On the other hand, it should be mentioned again that the following approximations,

$$c_{ijmn} = \text{average}\,\{a^*_{ijmn}[\sigma(1)], a^*_{ijmn}[\sigma(2)]\} \quad (157)$$

$$\hat{A}_{KLMN} = \text{average}\,\{A_{KLMN}[\mathbf{T}(1), \mathbf{E}^p(1)], A_{KLMN}[\mathbf{T}(2), \mathbf{E}^p(2)]\}, \quad (158)$$

have been made for Lee's theory and the Green–Naghdi theory, respectively. However, this kind of approximation perhaps cannot be avoided if the incremental finite element procedures are employed. It is believed that, in view of eqns (114) and (115) or eqns (144) and (145), this approximation is reasonable and accurate. Besides, this approximation is not employed for all those elements which have never experienced plastic deformation, in Lee's theory, and for all those elements in the process of neutral loading or unloading (including elastic theory), in the Green–Naghdi theory. Actually, for realistic problems, the majority of the elements are in that category. On the other hand, it is recognized that the solutions for elastic–plastic solid are path-dependent, if the incremental step is too large, and even if the stable solutions are obtained, the path-dependency introduced numerically may not be acceptable. On the other hand, if the size of the incremental step is reduced, say, to half, and no appreciable difference is observed, then the doubt can be eliminated.

REFERENCES

1. K. Washizu, *Variational Methods in Elasticity and Plasticity*, 2nd Edn. Pergamon Press, Oxford (1975).

2. R. Hill, *J. Mech. Phys. Solids* **7,** 209 (1959).
3. A. C. Eringen, *Mechanics of Continua*. Krieger, Huntington, NY (1980).
4. K. J. Bathe, *Finite Element Procedures in Engineering Analysis*. Prentice-Hall, Englewood Cliffs, NJ (1982).
5. L. E. Malvern, *Introduction to the Mechanics of a Continuous Medium*. Prentice-Hall, Englewood Cliffs, NJ (1969).
6. J. T. Oden, *Finite Elements of Nonlinear Continua*. McGraw-Hill, New York (1972).
7. H. D. Hibbitt, P. V. Marcal and J. R. Rice, *Int. J. Solids Struct.* **6,** 1069 (1970).
8. R. M. McMeeking and J. R. Rice, *Int. J. Solids Struct.* **11,** 601 (1975).
9. T. Scharnhorst and T. H. H. Pian, *Int. J. Numer. Meth. Engng* **12,** 665 (1978).
10. V. A. Lubarda and E. H. Lee, *J. appl. Mech.* **48,** 35 (1981).
11. E. H. Lee, *J. appl. Mech.* **36,** 1 (1969).
12. E. H. Lee, *Int. J. Solids Struct.* **17,** 859 (1981).
13. E. H. Lee, In *Numerical Methods in Industrial Forming Processes* (Edited by Pittmann, Wood, Alexander and Zienkiewicz). Pineridge Press, Swansea, U.K. (1982).
14. E. H. Lee and R. M. McMeeking, *Int. J. Solids Struct.* **16,** 715 (1980).
15. E. H. Lee, *Constitutive Equations: Macro and Computational Aspects*, p. 103. ASME, New York (1984).
16. A. E. Green and P. M. Naghdi, *Archs ration. Mech. Analysis* **18,** 251 (1965).
17. A. E. Green and P. M. Naghdi, in *Proc. IUTAM Symposium on Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluids* (Edited by Parkus and Sedov). Springer, Berlin (1966).
18. J. Casey and P. M. Naghdi, *J. appl. Mech.* **48,** 285 (1981).
19. J. Casey and P. M. Naghdi, *Quart. J. Mech. appl. Math.* **37,** 231 (1984).
20. J. Casey and P. M. Naghdi, *Constitutive Equations: Macro and Computational Aspects*, p. 53. ASME, New York (1984).
21. J. Casey and P. M. Naghdi, *J. appl. Mech.* **50,** 350 (1983).
22. C. Truesdell, *Comm. Pure appl. Math.* **8,** 123 (1955).
23. C. Truesdell, *J. Rat. Mech. Anal.* **4,** 83 (1955).
24. L. J. Segerlind, *Applied Finite Element Analysis*. John Wiley, New York (1976).
25. O. C. Zienkiewicz, *The Finite Element Method*, 3rd Edn. McGraw-Hill, London (1983).
26. D. C. Drucker, *Proc. 1st U.S. National Congr. Appl. Mech.*, p. 487. ASME, New York (1951).
27. A. C. Palmer, G. Maier and D. C. Drucker, *J. appl. Mech.* **34,** 464 (1967).