

Use of matrices to represent electron lenses. Matrices for the two-tube electrostatic lens

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The use of matrices to represent electron lenses is discussed. It is shown that it is more convenient and natural to represent a lens by a matrix and that the matrix elements show a more regular behavior than do the focal properties. Thus matrices are more convenient to use in computer programs. Matrix elements for the two-tube electrostatic lens are presented and their properties discussed. A simple analytical form for the matrix elements of lenses with near-unity voltage ratios is given.

INTRODUCTION

The representation of the paraxial (first-order) properties of electron lenses by focal points, focal lengths, and principal planes goes back many years and is given in all of the standard books on electron optics. Accurate values of these first-order properties for several electrostatic lenses have recently become available.¹⁻⁶ The purpose of this paper is to show, as suggested earlier,⁶ that it is more convenient to represent an electron lens by a matrix, and that the matrix elements show a more regular behavior than do the focal properties and are thus more amenable to use in computer programs. Matrix elements for the two-tube electrostatic lens are presented and their properties discussed.

REPRESENTATION OF PARAXIAL PROPERTIES

Figure 1 shows the definition of the paraxial properties of an electron lens. The two-tube electrostatic lens is shown but the same definitions are valid for any lens. Our sign convention is that all quantities to the left are negative, all quantities to the right are positive, and that focal lengths are negative if the focal point is to the left of its correspond-

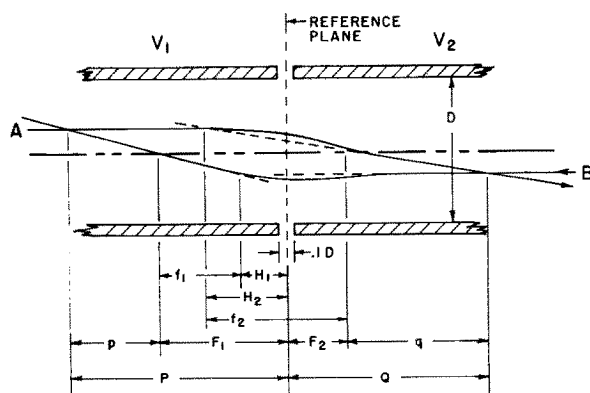


FIG. 1. Schematic diagram of the two-tube electrostatic lens showing the focal points F_1 , F_2 , focal lengths f_1 , f_2 , principal planes H_1 , H_2 and coordinates p , P , and q , Q in object and image space, respectively.

ing principal plane. This sign convention is the same as that used by Spangenberg.⁷ In many previous papers¹⁻⁴ a different sign convention was used in which all of the focal properties were positive. Recent work on very strong electrostatic lenses⁵ has necessitated changing this sign convention, since many of the quantities reverse sign.

In this sign convention, the focal properties of a decelerating lens are obtained from the focal properties of an accelerating lens by interchanging the subscripts 1 and 2 and changing the signs of all quantities. The Newtonian lens equation retains its usual form

$$pq = f_1 f_2, \quad (1)$$

and the magnification M is given by the usual formulae

$$M = -\frac{f_1}{p} = -\frac{q}{f_2}. \quad (2)$$

Matrices have been used many times⁸⁻³¹ to represent the paraxial properties of electron lenses. In our notation the matrix which applied to the position and angle coordinates of a ray (r_1, r_1') incident on the lens from the left at any point P on this ray gives the coordinate values (r_2, r_2') at any point Q on the ray leaving the lens is given by¹⁹

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \frac{1}{f_2} \begin{pmatrix} -(Q-F_2) & (P-F_1)(Q-F_2) - f_1 f_2 \\ -1 & (P-F_1) \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}. \quad (3)$$

Here each ray is treated in the asymptotic or virtual sense^{32,33}; that is, (r_1, r_1') and (r_2, r_2') are points on the entering and exiting rays as extrapolated from field-free space. Also note that P and Q do not necessarily represent conjugate points.

Equation (3) contains the distances P and Q . However, we wish to represent the intrinsic paraxial properties of a lens by a matrix with no reference to P and Q . One possibility is to use the matrix which transforms rays from the first

principal plane to the second,

$$\frac{1}{f_2} \begin{pmatrix} -(H_2 - F_2) & (H_1 - F_1)(H_2 - F_2) - f_1 f_2 \\ -1 & (H_1 - F_1) \end{pmatrix} = \frac{1}{f_2} \begin{pmatrix} f_2 & 0 \\ -1 & -f_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & -\frac{f_1}{f_2} \end{pmatrix}. \quad (4)$$

The free space transfer matrix

$$\begin{pmatrix} 1 & \Delta Z \\ 0 & 1 \end{pmatrix} \quad (5)$$

would then be used to translate rays to and from the principal planes, where ΔZ is the distance of translation. For the entering rays ΔZ would be $P - F_1$ and for the exiting rays $Q - F_2$. Although this matrix has a simple form this approach has been rejected for two reasons: (i) the focal properties are not all included in the matrix, and (ii) for lenses with voltage ratios near unity or with very large voltage ratios (near $V_2/V_1 = 6400$ for the two-tube lens⁵) H_1 and H_2 become very large, necessitating transformations to and from large distances. This procedure is somewhat artificial, since what we really want to know is how entering rays are transformed into exiting rays.

The approach actually adopted is to express ray coordinates as quantities extrapolated asymptotically to the reference plane of the lens. Then the matrix transforms entering asymptotic rays at the reference plane to exiting asymptotic rays at the reference plane and has the form

$$A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{F_2}{f_2} & \frac{F_1 F_2 - f_1 f_2}{f_2} \\ 1 & F_1 \\ -\frac{1}{f_2} & -\frac{F_1}{f_2} \end{pmatrix}. \quad (6)$$

Note that this matrix involves the quantity $F_1 F_2 - f_1 f_2$ which has been shown⁵ to be nearly constant for weak lenses. Also note that the determinant of the matrix is $-f_1/f_2$.

The process of propagating rays through systems of lenses is now straightforward. The ray is translated to the reference plane of the first lens using the free space propagator, transformed with the matrix A , propagated to the reference plane of the next lens, and the process continued until the final plane of interest is reached. All of the essential properties of the lens are now contained in the matrix. The transfer matrix is thus the most natural and convenient representation of an electron lens since it gives directly the effect of the lens on an electron beam. The focal properties, on the other hand, permit a certain lens classification, but give beam transfer properties less directly.

It is simple to calculate the imaging properties of a lens from its matrix. Consider an object at P (see Fig. 1). The matrix which propagates rays from P to any point Q in the

image space of the lens (in the asymptotic sense) is

$$\begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} + Q a_{21} & -P a_{11} + a_{12} - P Q a_{21} + Q a_{22} \\ a_{21} & -P a_{21} + a_{22} \end{pmatrix}. \quad (7)$$

For imaging we must have

$$-P a_{11} + a_{12} - P Q a_{21} + Q a_{22} = 0 \quad (8)$$

since the image point in the paraxial approximation is independent of the angle of rays leaving the object. Therefore the image is located at

$$Q = \frac{-P a_{11} + a_{12}}{P a_{21} - a_{22}}. \quad (9)$$

It is also simple to calculate the first-order focal properties from the matrix elements. The equations are

$$\left. \begin{aligned} f_1 &= -a_{12} + a_{11} a_{22} / a_{21} \\ f_2 &= -1 / a_{21} \\ F_1 &= a_{22} / a_{21} \\ F_2 &= -a_{11} / a_{21} \end{aligned} \right\} \quad (10)$$

If matrix A is given for an accelerating lens, the matrix B for a decelerating lens is found by interchanging f_1 and f_2 , and F_1 and F_2 , on the right hand side of Eq. (6). However, if we are given only the matrix elements a_{ij} for an accelerating lens, then a slightly more complicated process is required to derive the matrix B . First note that if

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = A \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}, \quad (11)$$

then

$$\begin{pmatrix} r_1 \\ r_1' \end{pmatrix} = A^{-1} \begin{pmatrix} r_2 \\ r_2' \end{pmatrix}, \quad (12)$$

where A^{-1} is the inverse of A .

On reversing the lens, both r_1' and r_2' change sign, so that the matrix B is given by

$$B \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^{-1} & -a_{12}^{-1} \\ -a_{21}^{-1} & a_{22}^{-1} \end{pmatrix}, \quad (13)$$

where a_{ij}^{-1} are the *elements* of A^{-1} . To make the use of matrices as convenient as possible, we give numerical results for both matrices A and B .

CALCULATION OF MATRICES

It is relatively straightforward to calculate matrix A using Eq. (6) and previously determined values of f_1 , f_2 , F_1 , and F_2 . However, special problems arise for weak lenses (voltage ratio near unity) and for certain very strong lenses where all of the focal properties become very large. In particular, the quantity $F_1 F_2 - f_1 f_2$ is a small number which is the difference of two very large numbers. To avoid these

TABLE I. Matrix for accelerating lens. D is diameter of lens.

V_2/V_1	a_{11}	$a_{21}D$	a_{12}/D	a_{22}	Determinant	$(V_1/V_2)^\ddagger$
1.1	0.97646	-0.0014384	0.000068238	0.97647	0.953484	0.953462
1.2	0.95539	-0.0051453	0.00024287	0.95549	0.912871	0.912871
1.3	0.93631	-0.010429	0.00049591	0.93672	0.877066	0.877058
1.4	0.91885	-0.016809	0.00080791	0.91979	0.845160	0.845154
1.5	0.90275	-0.023945	0.0011655	0.90442	0.816499	0.816497
1.7	0.87385	-0.039567	0.0019827	0.87760	0.766967	0.766965
2	0.83670	-0.064332	0.0033821	0.84486	0.707108	0.707106
5	0.62752	-0.25448	0.020184	0.70448	0.447211	0.447214
10	0.46682	-0.39562	0.045574	0.63878	0.316225	0.316228
20	0.31084	-0.49166	0.081960	0.58972	0.223603	0.223607
40	0.16977	-0.52914	0.12547	0.54026	0.158110	0.158114
100	0.020386	-0.49596	0.18294	0.45485	0.100003	0.100000
250	-0.078542	-0.39768	0.22604	0.33925	0.0632471	0.0632455
500	-0.12054	-0.30260	0.24200	0.23650	0.0447221	0.0447213
1000	-0.13830	-0.20516	0.24111	0.12901	0.0316223	0.0316228
2000	-0.13706	-0.11568	0.22384	0.025774	0.0223609	0.0223607
3000	-0.12978	-0.069805	0.20713	-0.029273	0.0182577	0.0182574
4000	-0.12246	-0.040758	0.19281	-0.064944	0.0158116	0.0158114
5000	-0.11582	-0.020358	0.18052	-0.090374	0.0141423	0.0141421
6000	-0.10990	-0.0051066	0.16984	-0.10958	0.0129101	0.0129099
6200	-0.10880	-0.0025001	0.16787	-0.11288	0.0127002	0.0127000
6400	-0.10772	0.000015914	0.16594	-0.11602	0.0125003	0.0125000
6600	-0.10666	0.0023549	0.16406	-0.11903	0.0123093	0.0123091
7000	-0.10462	0.0067859	0.16043	-0.12465	0.0119526	0.0119523
8000	-0.099878	0.016346	0.15204	-0.13682	0.0111806	0.0111803
9000	-0.095593	0.024209	0.14448	-0.14686	0.0105412	0.0105409
10000	-0.091695	0.030795	0.13761	-0.15527	0.0100002	0.0100000

problems and obtain matrix elements as accurately as possible, we have obtained the matrices A and B directly from trajectories. Two types of initial trajectories were used: parallel rays and rays which asymptotically go through the center of the midplane of the lens. The asymptotic coordinates of these rays at the midplane of the lens have the form $(r,0)$ and $0,r'$, respectively. The trajectories of these rays are then accurately propagated through the two-tube lens field using methods previously described^{4,5} and the asymptotic coordinates of the exit ray are calculated at the midplane. Using these coordinates the simultaneous equations implied by Eq. (7) were solved for the matrix elements a_{ij} and for b_{ij} .

For most of the lenses it was sufficient to use rays entering the lens parallel to the axis from the right and left. However, for lenses with voltage ratios greater than 50 the rays with midplane coordinates $(0,r')$ were found to give more accurate results.

RESULTS AND DISCUSSION

The matrices A and B for accelerating and decelerating two-tube lenses are given in Tables I and II, respectively, for voltage ratios from 1.1 to 10 000. As a test of the accuracy of the matrices, the determinant of the matrix and the square root of the voltage ratio are also given, since these quantities

TABLE II. Matrix for decelerating lens. D is diameter of lens.

V_2/V_1	b_{11}	$b_{21}D$	b_{12}/D	b_{22}	Determinant	$(V_1/V_2)^\ddagger$
1/1.1	1.0241	-0.0015085	0.000071567	1.0241	1.04879	1.04881
1/1.2	1.0467	-0.0056364	0.00026605	1.0466	1.09544	1.09545
1/1.3	1.0680	-0.011891	0.00056542	1.0676	1.14017	1.14018
1/1.4	1.0883	-0.019889	0.00095592	1.0872	1.18321	1.18322
1/1.5	1.1077	-0.029326	0.0014275	1.1056	1.22474	1.22475
1/1.7	1.1443	-0.051589	0.0025852	1.1394	1.30384	1.30384
1/2	1.1948	-0.090979	0.0047831	1.1833	1.41421	1.41421
1/5	1.5753	-0.56904	0.045135	1.4032	2.23608	2.23607
1/10	2.0200	-1.2511	0.14412	1.4762	3.16231	3.16228
1/20	2.6374	-2.1988	0.36654	1.3901	4.47221	4.47214
1/40	3.4170	-3.3467	0.79353	1.0738	6.32472	6.32456
1/100	4.5486	-4.9595	1.8293	0.20385	9.99972	10.0000
1/250	5.3638	-6.2877	3.5740	-1.2418	15.8110	15.8114
1/500	5.2881	-6.7663	5.4112	-2.6954	22.3603	22.3607
1/1000	4.0797	-6.4878	7.6243	-4.3735	31.6223	31.6228
1/2000	1.1526	-5.1731	10.010	-6.1293	44.7208	44.7214
1/3000	-1.6033	-3.8234	11.345	-7.1083	54.7715	54.7723
1/4000	-4.1074	-2.5778	12.194	-7.7449	63.2447	63.2456
1/5000	-6.3903	-1.4395	12.765	-8.1897	70.7098	70.7107
1/6000	-8.4877	-0.39555	13.156	-8.5129	77.4586	77.4597
1/6200	-8.8876	-0.19686	13.218	-8.5666	78.7388	78.7401
1/6400	-9.2815	-0.0012731	13.275	-8.6173	79.9982	80.0000
1/6600	-9.6696	0.19131	13.328	-8.6652	81.2391	81.2404
1/7000	-10.429	0.56774	13.422	-8.7529	83.6642	83.6660
1/8000	-12.238	1.4620	13.598	-8.9332	89.4406	89.4427
1/9000	-13.932	2.2966	13.706	-9.0686	94.8662	94.8683
1/10000	-15.527	3.0794	13.761	-9.1694	99.9982	100.0000

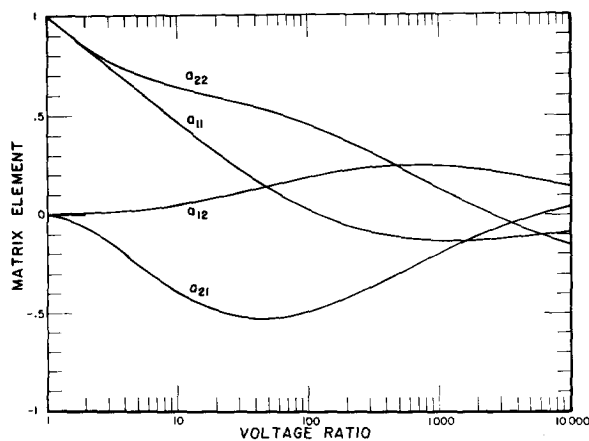


FIG. 2. The matrix elements a_{ij} for accelerating lenses.

should be equal. For all of the lenses this equality is satisfied to a few parts in 10^5 . The relationship between A and B , discussed above, is also satisfied to high precision.

The matrix elements a_{ij} and b_{ij} are shown in Figs. 2 and 3, respectively. Note that the infinities encountered in the focal properties⁵ near $V_2/V_1=1$ and 6400 are completely absent in the matrix elements. Each matrix element varies smoothly over its entire range and should be easily amenable to numerical interpolation.⁶

The matrix elements are given for a lens with unity diameter, and therefore must be scaled for lenses with different diameters by dividing a_{21} by the diameter, and multiplying a_{12} by the diameter. Alternatively, one can scale the entering trajectory coordinates in diameter units, use the matrix elements unaltered, then rescale the final trajectory coordinates.

EMPIRICAL MATRIX FOR SMALL VOLTAGE RATIOS

It was noticed that for lenses with voltage ratios near unity, the matrix elements a_{11} and a_{22} were closely equal

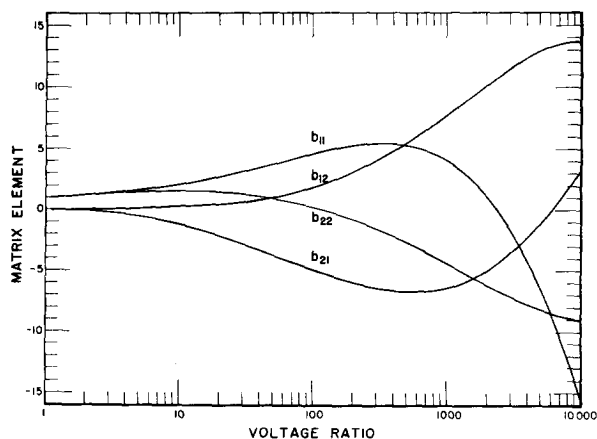


FIG. 3. The matrix elements b_{ij} for decelerating lenses.

to $(V_1/V_2)^{1/2}$. Letting $\gamma = (V_1/V_2)^{1/2}$ it was then found that, for

voltage ratios up to about 2, the matrix is well represented by

$$A = \begin{pmatrix} \gamma & 0.12308(1-\gamma)^2 \\ 2.5943(1-\gamma)^2 & \gamma \end{pmatrix}. \quad (14)$$

The matrix B is obtained simply by substituting $1/\gamma$ for γ .

Note added in proof.—The matrix B for the inverted lens (Eq. 13) is given simply by

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} V_2 \\ -V_1 \end{pmatrix}^{1/2} \begin{pmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{pmatrix}.$$

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