Noise Characterization of Multiport Amplifiers

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***Abstract*—This paper addresses the issue of the definition and measurement of the noise figure and parameters to characterize multiport devices, particularly differential amplifiers. A parameterization in terms of the noise matrix appears to be the most practical. The noise figure for a given output port is defined and related to the noise matrix and scattering parameters of the device, as well as the correlations between different input noise waves. The degradation of the signal-to-noise ratio is obtained from a special choice of the input correlation function. Two examples are considered in detail: a three-port differential amplifier and a four-port mixed-mode amplifier, both with reflectionless terminations. The noise figures, effective input temperatures, and gains are related to the results of a series of hot–cold measurements, as in the familiar two-port case.**

***Index Terms*—Amplifier noise, differential amplifier, multiport amplifier, noise, noise figure, noise matrix, thermal noise.**

I. Introduction

THERE ARE several equivalent parameterizations for the noise characteristics of two-port amplifiers and transistors, including the well known IEEE noise parameters [1], [2] and their variants, the noise matrix in either its voltage–current [3] or its wave amplitude [4], [5] incarnation, and the terminal-invariant set of Engen [6]. The noise figure or effective input noise temperature of a two-port amplifier as a function of source impedance or reflection coefficient can be expressed in terms of any of these sets. For more than two ports, or for more than one mode in a port, the situation is not so well developed. The basic multiport noise-matrix formalism was introduced long ago [3], but the expression of multiport noise figures in terms of a common set of parameters has not been developed. Even the definition of multiport noise figures is not well established. The IEEE definitions [7] allow for multiple input ports, as well as different input and output frequencies (since they were developed with receivers in mind), but they are restricted to one output port and, even for that case, they stop short of defining a noise figure. Differential amplifiers present two complications not included in the two-port noise-figure definition and parameterization: multiple input ports and a signal input channel that is a linear combination of the two physical input channels. Other multiports, such as mixed-mode two ports [8], introduce the additional complication of multiple output ports. The widespread use of such components in cell phones and other applications makes it desirable to have a convenient standard description of their noise characteristics. Such a description should be simple, have a physical basis or interpretation, and reduce to a familiar form for the two-port case.

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This paper suggests a description of noise in differential amplifiers and other multiports based on a wave description of the noise matrix [4], [5]. Our interest is in multiple (especially two) input and output ports, at a single frequency, with all ports at the same frequency. A definition of the noise figure for a given port is suggested, and that noise figure is expressed in terms of the noise matrix and the S-parameters of the amplifier, as well as the reflection coefficients of the terminations and the correlation matrix among the incident noise waves. The degradation of the signal-to-noise ratio for the general multiport case is expressed in terms of this general framework. Two simple examples are considered in detail: a three-port differential amplifier with reflectionless terminations and uncorrelated incident noise, and a four-port mixed-mode amplifier, also with reflectionless terminations and uncorrelated incident noise. For these cases, the noise figures, effective input temperatures, gains, and degradations of signal-to-noise ratio are related to the results of a series of hot–cold measurements, as in the familiar two-port case. Section II reviews the noise matrix formalism, as applied to multiports, and goes on to develop a definition of the noise figure for each output port of a linear multiport device. Section III treats the special case of a three-port differential amplifier with reflectionless terminations. Section IV contains the four-port example, and Section V is devoted to a summary. An earlier abbreviated version of this work can be found in [9].

II. Formalism

A. Noise Matrix

Throughout this paper, the term "noise temperature" denotes the available noise power spectral density divided by the Boltzmann constant k_{B}. A port will refer to a single mode in a single physical port. An M-port will refer to a multiport with M ports, N will be reserved to denote the noise matrix, and n will be used for noise powers. Multiple modes in a single physical port are treated as multiple ports. Thus, a four-port may refer to an amplifier with two input and two output ports or to an amplifier with two different modes in a single input port and two in an output port (or to some combination of these). Our primary interest is in three- and four-port amplifiers, but in principle, the formalism applies to any M greater than one. All the work in this paper is in terms of wave amplitudes. They may be defined in terms of voltages and currents [4], [8], [10] or they may be introduced and used with no reference to voltage and current [5], [11]. Details of the modes and waves are not of concern. What are important are two general properties. In order for the formalism of this paper to be valid, the modes must be power orthogonal, i.e., the total power across a reference plane must be the sum of the powers in each of the individual modes or ports. If this is not the case, and the total power contains cross

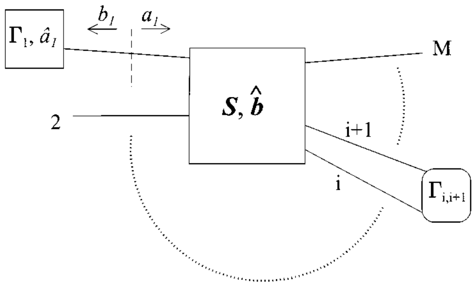

Fig. 1. Illustration of notation.

terms between the modes, it is possible to regain power orthogonality by a linear transformation, at least for lossless or low-loss lines [12]. The second general property that we assume about the waves is that they can be physically generated in practical applications, otherwise the discussion of measurements based on these waves is purely academic.

A linear M-port amplifier can be represented by its $M \times M$ scattering matrix ($\boldsymbol{S}$) and an M vector of internal (noise) sources ($\hat{\boldsymbol{b}}$)

$$\boldsymbol{b} = \boldsymbol{S}\boldsymbol{a} + \hat{\boldsymbol{b}} \tag{1}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are M vectors of the usual incident and outgoing wave amplitudes. The ith element of $\hat{\boldsymbol{b}}$, $\hat{b}_i$, is the amplitude of the generator wave at port i, which would be the output noise amplitude for reflectionless terminations and no input noise. The normalization is such that the spectral power density is given by the square of the absolute value of the wave amplitude. The noise amplitudes are assumed to be approximately constant in a small bandwidth (e.g., 1 Hz) around the frequency of interest, and we have divided out that bandwidth. Bold characters indicate vectors or matrices. The incident wave vector can be written as

$$\boldsymbol{a} = \boldsymbol{\Gamma}\boldsymbol{b} + \hat{\boldsymbol{a}} \tag{2}$$

where $\hat{\boldsymbol{a}}$ is the vector of generator waves of the sources connected to the M ports, and $\boldsymbol{\Gamma}$ is the $M \times M$ matrix of reflection coefficients. In simple cases $\boldsymbol{\Gamma}$ is diagonal, and $\Gamma_{\rm ii}$ is the reflection coefficient from the termination on port i. More generally, $\boldsymbol{\Gamma}$ has off-diagonal elements $\Gamma_{\rm ij}$ corresponding to a wave emerging from port or mode j and being reflected back (at least in part) in port or mode i. The general configuration and notation is illustrated in Fig. 1. Combining (1) and (2) in the usual manner yields the expression for the outgoing wave vector in terms of the generator waves

$$\boldsymbol{b} = [\boldsymbol{1} - \boldsymbol{S}\boldsymbol{\Gamma}]^{-1}[\boldsymbol{S}\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}]. \tag{3}$$

The noise matrix (or noise correlation matrix) can then be defined and expressed in terms of the intrinsic parameters of the M-port and the properties of the M terminations on its ports. We can define two distinct noise matrices: an intrinsic noise matrix, which depends only on the properties of the amplifier itself, and an *in situ* noise matrix, which depends on the characteristics of the circuit in which the amplifier is embedded. The full *in situ* noise matrix $\boldsymbol{N}$ is defined as

$$\boldsymbol{N} \equiv \overline{\boldsymbol{b}\boldsymbol{b}^\dagger} \tag{4}$$

or

$$N_{ij} = \overline{b_i b_j^*} \tag{5}$$

where the bar indicates either an ensemble or time average (assumed equal) and the † indicates a Hermitian conjugate. Diagonal elements of the noise matrix give the power spectral density of the output noise in the respective port, while off-diagonal elements are the correlations between the output noise in different ports. We can use (3) to write the noise matrix in terms of the generator waves

$$\boldsymbol{N} = \overline{\boldsymbol{b}\boldsymbol{b}^\dagger} = [\boldsymbol{1} - \boldsymbol{S}\boldsymbol{\Gamma}]^{-1}\boldsymbol{S}\overline{\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^\dagger}\boldsymbol{S}^\dagger[\boldsymbol{1} - \boldsymbol{S}\boldsymbol{\Gamma}]^{-1\dagger} + [\boldsymbol{1} - \boldsymbol{S}\boldsymbol{\Gamma}]^{-1}\overline{\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}^\dagger}[\boldsymbol{1} - \boldsymbol{S}\boldsymbol{\Gamma}]^{-1\dagger} \tag{6}$$

where we have used the fact that the generator waves from the amplifier are uncorrelated with those from the terminations. The first term is due to the noise (and signal) from the sources and terminations connected to the ports, and the second term is due to the noise generated by the amplifier itself, suitably modified by reflections from the terminations. In the absence of any external noise ($\hat{\boldsymbol{a}} = \boldsymbol{0}$), the first term vanishes, and we are left with only the amplifier noise represented by the second term. Conversely, for a noiseless amplifier $\hat{\boldsymbol{b}} = \boldsymbol{0}$, and only the first term is present.

The *intrinsic* noise matrix is defined by

$$\hat{\boldsymbol{N}} = \overline{\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}^\dagger}. \tag{7}$$

It is the noise matrix that would occur if all terminations of the amplifier were reflectionless and noiseless. Since the intrinsic noise matrix (supplemented by the scattering matrix) contains full information on the intrinsic noise parameters of the amplifier, we find it useful to introduce a more physical parameterization of it. It will also be convenient to have a more compact notation. For the diagonal elements, we associate a characteristic or reduced noise temperature with each port

$$\overline{|\hat{b}_i|^2} = k_B\hat{T}_i \tag{8}$$

where k_B is Boltzmann's constant. The quantity $k_B\hat{T}_i$ is the noise power per unit bandwidth that would be *delivered* to a noiseless, reflectionless load attached to port i if all other ports were also terminated in noiseless reflectionless loads. The actual noise temperature of port i is related to the *available* noise power, and for the case of all noiseless reflectionless terminations, the noise temperature is given by $T_{i,0} = \hat{T}_i/(1 - |S_{ii}|^2)$.

The off-diagonal elements of the intrinsic noise matrix are appropriately scaled correlation functions

$$\overline{\hat{b}_i\hat{b}_j^*} = k_B\sqrt{\hat{T}_i\hat{T}_j}\rho_{ij}. \tag{9}$$

The absolute values of the ρ_{ij} can range from 0 to 1, as befits a correlation function. For the discussion that follows, it is also useful to define a correlation matrix for the incident noise waves. Let

$$\overline{\hat{a}_i\hat{a}_j^*} = k_BT_0A_{ij} \tag{10}$$

where T_0 is a reference temperature (290 K). Diagonal elements of the matrix $\boldsymbol{A}$ are then the ratios of the noise temperatures incident on the different ports divided by T_0.

With multiple ports, there may be several different useful choices for the set of basis waves (e.g., common and differential modes or ports 1 and 2). The noise matrix (*in situ* or intrinsic) will be different for different choices of basis waves. Under a change of basis represented by the matrix $\boldsymbol{L}$, $\boldsymbol{b} \rightarrow \boldsymbol{b}' = \boldsymbol{Lb}$, the noise matrix transforms according to

$$\boldsymbol{N} \rightarrow \boldsymbol{N}' = \boldsymbol{LNL^{-1}}. \quad (11)$$

B. Noise-Figure Definition

We are now prepared to consider the definition of the noise figure and its expression in terms of the intrinsic parameters of the amplifier and its terminations. We deal only with the noise figure at a single frequency, i.e., the spot noise figure. The IEEE definition of operating noise temperature and effective noise temperature for multiple input ports [7] stops short of defining a noise figure for multiple inputs. For a two-port, the IEEE definition [1] is that the noise figure (or noise factor) at a given frequency is the ratio of total output noise power per unit bandwidth to the portion of the output noise power that is due to the input noise, evaluated for the case where the input noise power is $k_{\rm B}T_0$ ($T_0 = 290$ K). Equivalently, it is one plus the ratio of the output noise due to the amplifier to the output noise due to T_0 input noise. We wish to extend this definition to the M-port case. In principle, we can define a noise figure$F_{\rm i}$ for every port i, but, in practice, we will consider noise figures only for output ports. As in the two-port case, the noise figure of a given port should be the ratio of the total output noise in that port to the output noise power that is due to the input noise for the case when the input noise is T_0.

Complications and ambiguities arise immediately however. Is T_0 input to all the input ports and, if so, is the input noise to different ports correlated? How are the other output ports terminated; is T_0 input to them as well? For differential amplifiers, is the noise input to the physical ports 1 and 2 or to the differential and common modes?

The most significant complicating factor is the correlation matrix of the incident noise waves. In actual use, the input to the amplifier will come from other parts of the circuit, and the noise incident on different ports may well be correlated to some degree. Also, each input port may have a different incident noise temperature. Both these complications are contained in the incident noise correlation matrix (10). The output noise powers will be given by the appropriate elements of the noise matrix (6), which depend on the incident noise waves $\overline{a_i a_j^*}$ and, therefore, on $\boldsymbol{A}$. There are two possible strategies for dealing with the correlations between incident noise waves. The noise figure can be defined to be a function of the incident noise correlation matrix, just as it is a function of the reflection coefficients of the terminations; or a reference value (e.g., the identity matrix) can be chosen for the incident noise correlation matrix, much as we choose a reference noise temperature T_0. However, unlike the case with the reference noise temperature T_0, it would not be possible to compute the noise figure for some other value of $\boldsymbol{A}$ from the noise figure with the reference value $\boldsymbol{A}_0$. Consequently, we choose to treat the noise figure as a function of $\boldsymbol{A}$, the (complex) correlation matrix of the input noise waves.

For the terminations of the output ports, we follow the spirit of the two-port definition, namely, that the noise figure measures the noise added by the amplifier for a given choice of reflection coefficients for the input terminations, but it should not include noise contributions from the various output loads. (Note, however, that the IEEE definition of the operating temperature does include such contributions [7].) We will (tentatively) adopt the convention that no noise source is connected to the output ports. In practice, it should make little difference since the isolation between the different output ports should be great enough that the output of a given port would be insensitive to whether T_0 is applied to some other output port, especially considering that T_0 is applied to the input channels, which are being amplified.

The definition of the noise figure for a given output channel is then complete. In terms of the notation introduced in the preceding section, it takes the form

$$F_i(\boldsymbol{\Gamma},\, \boldsymbol{A}) = 1 + \frac{1}{k_B T_0}\, \frac{\left\{[\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}} \hat{\boldsymbol{N}} [\mathbf{1} - \boldsymbol{S\Gamma}]^{-1\dagger}\right\}_{\boldsymbol{ii}}}{\left\{[\mathbf{1} - \boldsymbol{S\Gamma}]^{-1} \boldsymbol{SAS}^{\dagger} [\mathbf{1} - \boldsymbol{S\Gamma}]^{-1\dagger}\right\}_{\boldsymbol{ii}}} \quad (12)$$

where the subscript i's indicate the element of the matrix within the braces.

The definition of (12) reduces to the usual definition for the two-port case and embodies the intuitive idea that the noise figure measures how much noise the amplifier adds to a 290-K reference signal. If it seems rather formal at this point, it should become clearer in the following sections, when we work through two simple examples in detail.

Besides defining the noise figure, (12) also constitutes a parameterization of its dependence on the reflection coefficients of the sources and loads and on the incident noise correlation matrix. The noise parameters of the amplifier are the independent elements of the intrinsic noise matrix. For a three-port amplifier, there would be nine real parameters: three characteristic noise temperatures and three complex correlation functions. In principle, one could develop a parameterization analogous to the IEEE parameterization for two-port noise figure or effective noise temperature. This would also require nine real parameters: a minimum noise figure, optimal complex values for the reflection coefficients of the two sources, and four parameters describing the rate of variation of the noise figure as the reflection coefficients deviated from their optimal values. This set of nine parameters could be expressed in terms of the elements of the noise matrix and the S-parameters of the amplifier, but that is well beyond the scope of this paper.

C. Signal-to-Noise-Ratio Degradation

For a two-port amplifier, the noise figure is a direct measure of the degradation of the signal-to-noise ratio ($\rm s/n$). In general the noise figure of (12) is not the ratio of input $\rm s/n$ to output $\rm s/n$, but it is not difficult to obtain the expression for that ratio. Let the wave amplitude of the signal be $a_{\rm sig}$, and assume that the signal channel is port 1. The input signal power density is $s_{\rm in} = |a_{\rm sig}|^2$, and the output power density in port i is given by

$$s_{\rm out} = \left\{ [\mathbf{1} - \boldsymbol{S\Gamma}]^{-1} \boldsymbol{A}_{\rm sig} [\mathbf{1} - \boldsymbol{S\Gamma}]^{-1\dagger} \right\}_{\boldsymbol{ii}} s_{\rm in}$$
$$\boldsymbol{A}_{\rm sig} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ . & . & . & \cdots \end{pmatrix}. \quad (13)$$

The input noise power density is taken to be $k_{\rm B}$ times the reference temperature $n_{\rm in} = k_{\rm B} T_0$ and the output noise power density in port i is

$$n_{\rm out} = \Big\{ k_B T_0 [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}} \boldsymbol{SAS}^\dagger [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}\dagger} + [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}} \hat{\boldsymbol{N}} [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}\dagger} \Big\}_{\boldsymbol{ii}} \quad (14)$$

where $\boldsymbol{A}$ is the incident noise correlation matrix for the actual configuration, except that $A_{11} = 1$. The degradation of the signal-to-noise ratio for output port i, which will be denoted $F_i^{\rm s/n}$, is then given by

$$F_i^{\rm s/n}(\boldsymbol{\Gamma}, \boldsymbol{A}) \equiv \frac{({\rm s/n})_{\rm in}}{({\rm s/n})_{\rm out}} = \frac{\Big\{ [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}} [k_B T_0 \boldsymbol{SAS}^\dagger + \hat{\boldsymbol{N}}] [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}\dagger} \Big\}_{\boldsymbol{ii}}}{k_B T_0 \Big\{ [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}} \boldsymbol{SA}_{\rm sig} \boldsymbol{S}^\dagger [\mathbf{1} - \boldsymbol{S\Gamma}]^{-\mathbf{1}\dagger} \Big\}_{\boldsymbol{ii}}}. \quad (15)$$

The difference between this form and (12) is that (12) has the full $\boldsymbol{A}$ in the denominator, whereas the $F^{\rm s/n}$ of (15) has only $\boldsymbol{A}_{\rm sig}$. The noise figure of (12) takes the total noise out and divides it by the noise out due to all the incident noise, whereas $F^{\rm s/n}$ divides by the noise out due to the incident noise *in the signal channel only*. For the $\boldsymbol{\Gamma} = 0$ case, (15) reduces to

$$F_i^{\rm s/n}(\boldsymbol{\Gamma} = \mathbf{0}, \boldsymbol{A}) = \frac{\Big\{ [k_B T_0 \boldsymbol{SAS}^\dagger + \hat{\boldsymbol{N}}] \Big\}_{ii}}{k_B T_0 |S_{1i}|^2} \quad (16)$$

and $|S_{1{\rm i}}|^2$ is the gain from the incident signal channel (1) to the output channel i.

III. Example—Differential Amplifier $\boldsymbol{\Gamma} = \mathbf{0}$

A. Characteristic Noise Temperature, Gains, and Effective Input Temperature

In order to completely characterize the noise properties of a multiport amplifier, or to determine its noise figures for general terminations and input noise correlations, it is necessary to determine the complete intrinsic noise matrix. There are, however, specific configurations or choices of terminations that are of interest in their own right. In particular, the case in which all ports have reflectionless terminations is often a useful approximation to the exact actual configuration. Also, it is often useful to have a single number, or a figure-of-merit, that summarizes or represents an amplifier's noise properties. The noise figure with reflectionless terminations is often used for this purpose in the two-port case, and we expect that the noise figure with reflectionless terminations and uncorrelated incident noise can serve a similar purpose for multiport amplifiers. Consequently, the $\boldsymbol{\Gamma} = 0$ examples considered in this and the following section should be of some practical use, as well as clarifying the multiport noise-figure definition.

A differential amplifier is a three-port device with a single output port whose signal (ideally) is proportional to the difference between the signals at the two input ports. Let the output port be port 3, and define input waves and S-parameters to describe the common (+) and differential (−) modes

$$a_\pm \equiv (a_1 \pm a_2)/\sqrt{2}$$
$$S_{3\pm} = (S_{31} \pm S_{32})/\sqrt{2}. \quad (17)$$

We can then write the output amplitude at port 3 as

$$b_3 = S_{3-} a_- + S_{3+} a_+ + \hat{b}_3 \quad (18)$$

where ideally $S_{3+} = 0$. One immediate, important consequence of the definitions of (17) is that if the noise waves represented by a_1 and a_2 are uncorrelated, then the noise temperatures input to the common and differential modes are equal ($\hat{T}_+ = |a_+|^2/k_B = |a_-|^2/k_B = \hat{T}_-$). Therefore, to obtain different input noise temperatures for the common and differential modes requires correlated noise sources for ports 1 and 2.

We consider the simple case of all ports terminated with matched (reflectionless) loads or sources. Since there are no reflections from the terminations, the off-diagonal elements of the noise matrix do not contribute to the output noise at port 3, nor do the characteristic noise temperatures of the input ports $\hat{T}_1$ and $\hat{T}_2$. Only $\hat{T}_3$, the characteristic noise temperature of port 3, contributes to the output noise, just as in the case of a two-port amplifier with reflectionless terminations.

The average noise power per unit bandwidth emerging from port 3 is given by

$$n_3 = \overline{|S_{31} a_1 + S_{32} a_2 + \hat{b}_3|^2}. \quad (19)$$

If two uncorrelated noise sources with noise temperatures $T_1^{(i)}$ and $T_2^{(i)}$ are input to ports 1 and 2, (19) becomes

$$n_3 = k_B \left[G_{31} T_1^{(i)} + G_{32} T_2^{(i)} + \hat{T}_3 \right] \quad (20)$$

where $G_{31} = |S_{31}|^2$.

The unknown parameters in (20) can be determined from a series of hot–cold measurements similar to the two-port case. Let T_{h1} denote the noise temperature of the hot source connected to port 1, etc. In principle, T_{h1} and T_{h2} could be equal, and T_{c1} and T_{c2} probably will be equal to the ambient temperature and, therefore, to each other, but we begin with the general case. There are then four different measurements that can be performed. Let $n_{3,hc}$ be the output noise power per unit frequency measured at port 3, for a hot source on port 1 and a cold source on port 2. $n_{3,\,hh}$, $n_{3,\,ch}$, and $n_{3,\,cc}$ are defined in a similar manner. The results of the four measurements are then given by

$$\begin{aligned} n_{3,\,hh} &= k_B[G_{31}T_{h1} + G_{32}T_{h2} + \hat{T}_3] \\ n_{3,\,hc} &= k_B[G_{31}T_{h1} + G_{32}T_{c2} + \hat{T}_3] \\ n_{3,\,ch} &= k_B[G_{31}T_{c1} + G_{32}T_{h2} + \hat{T}_3] \\ n_{3,\,cc} &= k_B[G_{31}T_{c1} + G_{32}T_{c2} + \hat{T}_3]. \end{aligned} \tag{21}$$

There are only three unknowns in (21), i.e., G_{31}, G_{32}, and $\hat{T}_3$; and, consequently, the equations are not all independent. Indeed, one notes that

$$n_{3,\,hh} + n_{3,\,cc} = n_{3,\,hc} + n_{3,\,ch}. \tag{22}$$

Therefore, it is sufficient to measure only three of the four hot–cold combinations to determine the gains and $\hat{T}_3$. (In practice, it may be preferable to measure all four combinations and fit for a best solution.) The set of hc, ch, and cc may give slightly better accuracy, and it requires only one hot noise source; thus, we begin with that set. The measured values for the gains are then

$$\begin{aligned} G_{31} &= \frac{n_{3,\,hc} - n_{3,\,cc}}{k_B(T_{h1} - T_{c1})} \\ G_{32} &= \frac{n_{3,\,ch} - n_{3,\,cc}}{k_B(T_{h2} - T_{c2})} \end{aligned} \tag{23}$$

and the intrinsic output noise temperature for port 3 is given by

$$k_B\hat{T}_3 = \frac{(T_{h1}T_{h2} - T_{c1}T_{c2})}{(T_{h1} - T_{c1})(T_{h2} - T_{c2})}\, n_{3,\,cc} - \frac{T_{c1}}{(T_{h1} - T_{c1})}\, n_{3,\,hc} - \frac{T_{c2}}{(T_{h2} - T_{c2})}\, n_{3,\,ch}. \tag{24}$$

The equivalent input temperature, which is equal for the two input ports [7], is given by

$$T_e = \frac{\hat{T}_3}{G_{31} + G_{32}}. \tag{25}$$

Assuming the two cold temperatures are equal, this can be written as

$$T_e = \frac{T_{h1}T_{h2} - (T_{h1}Y_{ch} + T_{h2}Y_{hc})T_c + Y_{hh}T_c^2}{(Y_{ch} - 1)T_{h1} + (Y_{hc} - 1)T_{h2} - (Y_{hh} - 1)T_c} \tag{26}$$

where $Y_{hc} = n_{3,\,hc}/n_{3,\,cc}$, etc. Although $n_{3,\,hh}$ may not have been measured, Y_{hh} can be determined from (22), $Y_{hh} = Y_{hc} + Y_{ch} - 1$. If we further assume that only one hot noise source is used, so that $T_{h1} = T_{h2} = T_h$, (26) reduces to

$$T_e = \frac{T_h - Y_{hh}T_c}{Y_{hh} - 1} \tag{27}$$

which is the familiar two-port result, with Y_{hh} playing the role of the two-port Y.

Equation (27) indicates that T_e can be determined either from the set of three measurements (hc, ch, and cc) or from just two measurements (hh and cc) if $T_{h1} = T_{h2}$. If only hh and cc are measured, we can still determine the sum of the gains and $\hat{T}_3$

$$\begin{aligned} G_{31} + G_{32} &= \frac{n_{hh} - n_{cc}}{k_B(T_h - T_c)} \\ \hat{T}_3 &= \frac{k_B(T_h - Y_{hh}T_c)(T_h - T_c)}{(Y_{hh} - 1)(n_{hh} - n_{cc})} \end{aligned} \tag{28}$$

but we cannot determine either gain separately, as in (23).

The discussion in this section has not yet treated the differential or common mode, nor has it mentioned noise figure. From $G_{3\pm} = |S_{3\pm}|^2$ and (17) and (18), it follows that

$$G_{3+} + G_{3-} = G_{31} + G_{32}. \tag{29}$$

Since $\hat{T}_3$ is the same no matter how we describe the input ports and since the sum of the gains is the same, the effective input noise temperature in the differential and common modes is the same as for ports 1 and 2. The hot–cold measurements with uncorrelated sources, described above, are therefore sufficient to determine $\hat{T}_3$, T_e, and $G_{3+} + G_{3-}$ for the differential and common modes, but not G_{3+} or G_{3-} individually. Since G_{3-} is designed to be much larger than G_{3+}, we might use the approximation $G_{3-} \approx G_{31} + G_{32}$, but it would be useful to measure G_{3+} or G_{3-} independently. Using noise to measure G_{3+} or G_{3-} requires correlated noise input to ports 1 and 2. If $a_1 = a_2$, then $a_- = 0$ and $a_+ = \sqrt{2}a_1$, which, in turn, leads to $T_- = 0$ and $T_+ = 2T_1$. If the measured noise power out of port 3 in such a measurement is called $n_{3,+}$, then

$$n_{3,\,+} = k_B\left[2G_{3+}T_1 + \hat{T}_3\right] \tag{30}$$

from which it follows that

$$G_{3+} = \frac{(G_{3+} + G_{3-})}{2T_1}\left[T_cY_+ + T_e(Y_+ - 1)\right] \tag{31}$$

where $Y_+ = n_{3,\,+}/n_{3,\,cc}$. All the quantities on the right-hand side, except Y_+, can be determined from the uncorrelated measurements described above, and therefore measurement of Y_+ determines G_{3+}.

To summarize the matched case, with just one hot source and two equal-temperature cold sources, a set of three measurements (hc, ch, and cc) with uncorrelated input noise will determine $\hat{T}_3$, T_e, G_{31}, G_{32}, and $G_{3+} + G_{3-}$. (Obviously, if a second hot source is available, hh could be done as a consistency check or to reduce the measurement uncertainty.) If two equal-temperature hot sources and two equal-temperature cold sources are available, then just two measurements (hh and cc) suffice to determine $\hat{T}_3$, T_e, $G_{31} + G_{32}$, and $G_{3+} + G_{3-}$, but not any individual gain. To determine G_{3-} or G_{3+} individually (in a noise measurement) requires the noise input to ports 1 and 2 to be correlated.

B. Noise Figure

Once all the relevant parameters have been measured, as in the preceding subsection, we can compute the noise figure of the differential amplifier. For two input ports and one output port, all with reflectionless terminations, and no correlation between the incident noise waves ($\boldsymbol{A} = \boldsymbol{1}$), (12) reduces to

$$\begin{aligned} F_3 &= 1 + \frac{\overline{|\hat{b}_3|^2}}{|S_{31}|^2\overline{|\hat{a}_1|^2} + |S_{32}|^2\overline{|\hat{a}_2|^2}} \\ &= 1 + \frac{\hat{T}_3}{(G_{31} + G_{32})T_0} \\ &= 1 + \frac{T_e}{T_0}. \end{aligned} \tag{32}$$

Note that this noise factor does not require separate measurement of G_{3+} or G_{3-} and, thus, does not require any measurements with correlated noise input.

Equation (32) gives the ratio of total output noise to output noise due to all input noise for the particular case considered, $\boldsymbol{\Gamma} = 0$, $\boldsymbol{A} = \mathbf{1}$. As discussed in Section II-C, however, it does not give the degradation of the signal-to-noise ratio. That is given by (15), or (16) for $\boldsymbol{\Gamma} = 0$. If the input channel is the differential mode, (16) takes the form

$$F^{\mathrm{s/n}}(\boldsymbol{\Gamma} = 0,\, \boldsymbol{A} = \mathbf{1}) = \frac{(G_{3-} + G_{3+})T_0 + \hat{T}_3}{G_{3-}T_0} = \left(1 + \frac{G_{3+}}{G_{3-}}\right)\left(1 + \frac{T_e}{T_0}\right). \quad (33)$$

This differs from the F of (32) by the factor $(1 + G_{3+}/G_{3-})$, which makes $F^{\mathrm{s/n}}$ more difficult to measure. It does, however, provide a better measure of the amplifier's signal-to-noise performance. The gains G_{3+} and G_{3-} can be determined from noise measurements with correlated noise incident on the two input ports or from vector-network-analyzer measurements.

It is also interesting to consider $F^{\mathrm{s/n}}$ for the case of $\boldsymbol{A} \neq \mathbf{1}$, which would be appropriate for a circuit configuration in which the noise incident on the different ports of the differential amplifier was correlated. In this case, (16) takes the form

$$F^{\mathrm{s/n}}(\boldsymbol{\Gamma} = \mathbf{0},\, \boldsymbol{A}) = 1 + \frac{\hat{T}_3 + T_0\left[G_{3+}A_{++} + 2\mathrm{Re}(S_{3+}S_{3-}^{*}A_{+-})\right]}{T_0 G_{3-}}. \quad (34)$$

The only noise parameter of the amplifier that enters (34) is $\hat{T}_3$. The other parameters needed are the elements of the correlation matrix of the incident noise, coming from other parts of the circuit in which the amplifier is embedded, and the scattering parameters of the amplifier, not just the gains. The incident noise correlation matrix element A_{11} is taken equal to one, as prescribed by the convention for defining the signal-to-noise noise factor.

IV. Four-Port Example

As a further example of the formalism, we consider an amplifier with two input and two output ports (or modes), such as the mixed-mode two-port treated in [8]. To make the example concrete, we take port 3 to be the differential output mode and port 4 to be the common output mode. Ports 1 and 2 are taken to correspond to two physically separate input ports, though they could just as well be the differential and common input modes. Again, we treat only the reflectionless case with uncorrelated incident noise, $\boldsymbol{\Gamma} = 0$, $\boldsymbol{A} = 0$. Equation (12) then reduces to

$$F_3 = 1 + \frac{\overline{|\hat{b}_3|^2}}{|S_{31}|^2\overline{|\hat{a}_1|^2} + |S_{32}|^2\overline{|\hat{a}_2|^2}} = 1 + \frac{\hat{T}_3}{(G_{31} + G_{32})T_0} \quad (35)$$

for the noise figure of port 3. Port 4 has its own noise figure, given by a similar equation, but with $3 \rightarrow 4$. Determination of F_3 then requires determination of $\hat{T}_3$, the characteristic noise temperature for port 3, as well as the sum of the two gains $G_{31} + G_{32}$ or $G_{3-} + G_{3+}$.

The characteristic noise temperatures and sums of gains can be determined by a series of hot–cold measurements, as in the preceding section. If we first measure the output noise in port 3 while connecting hot and cold loads to the input ports, we again obtain the set of equations in (21) and (22), and solving for $\hat{T}_3$ again yields (24). Similarly, if we measure the output noise in port 4 while connecting hot and cold loads to the input ports, we obtain

$$k_B\hat{T}_4 = \frac{(T_{h1}T_{h2} - T_{c1}T_{c2})}{(T_{h1} - T_{c1})(T_{h2} - T_{c2})}\, n_{4,\,cc} - \frac{T_{c1}}{(T_{h1} - T_{c1})}\, n_{4,\,hc} - \frac{T_{c2}}{(T_{h2} - T_{c2})}\, n_{4,\,ch} \quad (36)$$

and the gains are given by

$$G_{41} = \frac{n_{4,\,hc} - n_{4,\,cc}}{k_B(T_{h1} - T_{c1})}$$
$$G_{42} = \frac{n_{4,\,ch} - n_{4,\,cc}}{k_B(T_{h2} - T_{c2})} \quad (37)$$

Thus far, everything is essentially the same as in the three-port case. However, a nuance arises when we attempt to compute the effective input temperature. When there are two output ports, the two input ports, in general, cannot have the same effective input temperature. The equations that define T_{1e} and T_{2e} are

$$\hat{T}_3 = G_{31}T_{1e} + G_{32}T_{2e}$$
$$\hat{T}_4 = G_{41}T_{1e} + G_{42}T_{2e}. \quad (38)$$

Solving, we obtain

$$T_{1e} = \frac{G_{42}\hat{T}_3 - G_{32}\hat{T}_4}{\Delta_G}$$
$$T_{2e} = \frac{-G_{41}\hat{T}_3 + G_{31}\hat{T}_4}{\Delta_G}$$
$$\Delta_G = G_{31}G_{42} - G_{32}G_{41}. \quad (39)$$

For the signal-to-noise degradation, the expressions are similar to the three-port case

$$F_3^{\mathrm{s/n}}(\boldsymbol{\Gamma} = \mathbf{0},\, \boldsymbol{A}) = 1 + \frac{\hat{T}_3 + T_0\left[G_{3+}A_{++} + 2\mathrm{Re}(S_{3+}S_{3-}^{*}A_{+-})\right]}{T_0 G_{3-}}$$
$$F_3^{\mathrm{s/n}}(\boldsymbol{\Gamma} = 0,\, \boldsymbol{A} = \mathbf{1}) = \frac{(G_{3-} + G_{3+})T_0 + \hat{T}_3}{G_{3-}T_0}. \quad (40)$$

The equations for port 4 are obtained by $3 \rightarrow 4$ in (40).

In summary, the four-port case introduces two complications not present for three-ports. The obvious complication is that there are two noise figures, one for each output port. Each noise figure can be measured in a manner similar to the three-port (or two-port) case, with a series of hot–cold measurements. (In fact, the two noise figures could, in principle, be measured simultaneously.) The second complication, which might not have been expected, is that there must be two different effective input noise temperatures. These are given in (39) in terms of the gains and characteristic noise temperatures of the output ports. The expressions for the signal-to-noise noise figure are similar to the three-port case.

Relegated to the indefinite future is the definition of the effective input noise temperature when there are more output than input ports. In that case, the generalization of (38) will usually not admit a solution for the T_{ie}'s, and we would have to define a different effective input noise temperature for each output port.

V. Summary

A formalism based on the wave-amplitude form of the noise matrix has been presented for multiport amplifiers, particularly differential amplifiers. The noise figure for an output channel was defined and written in terms of the noise matrix and scattering parameters of the amplifier, the reflection coefficients of the terminations, and the correlation matrix of the incident noise waves in the actual configuration of use of the amplifier. The noise figure corresponding to the degradation of the signal-to-noise ratio was also defined and expressed in terms of the same quantities for the general case. Two special cases were considered, a three-port differential amplifier with reflectionless terminations and uncorrelated incident noise and a four-port mixed-mode amplifier, also with reflectionless terminations and uncorrelated incident noise. For each case, the noise figures, effective input temperatures, and gains were related to the results of a series of hot–cold measurements, as in the familiar two-port case. In both examples, the off-diagonal elements of the intrinsic noise matrix, the correlation coefficients ρ_{ij}, were not determined, since they do not affect the noise figure in the $\mathbf{\Gamma} = 0$ case. To characterize a multiport amplifier for nonzero reflection coefficients, additional measurements would be required to measure the ρ_{ij}. That more general case is left for future work.

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