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LOCALITY BOUNDS ON HAMILTONIANS FOR STABILIZER CODES

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In this paper, we study the complexity of Hamiltonians whose groundstate is a stabilizer code. We introduce various notions of k-locality of a stabilizer code, inherited from the associated stabilizer group. A choice of generators leads to a Hamiltonian with the code in its groundspace. We establish bounds on the locality of any other Hamiltonian whose groundspace contains such a code, whether or not its Pauli tensor summands commute. Our results provide insight into the cost of creating an energy gap for passive error correction and for adiabatic quantum computing. The results simplify in the cases of XZ-split codes such as Calderbank-Shor-Steane stabilizer codes and topologically-ordered stabilizer codes arising from surface cellulations.

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1 Introduction

A Hamiltonian is realizable only if its complexity is low. We consider in this paper the problem of determining lower bounds on the complexity of Hamiltonians whose groundstate is a stabilizer code. We show that all such Hamiltonians must be at least as complicated as the underlying stabilizer group. Our measure of complexity is *locality*, defined by two parameters: $\delta(G)$, the minimum weight of non-trivial stabilizers of the group G, and $\eta(G)$, the minimum integer k such that stabilizers of weight k generate the entire group.

The notion of k-locality [2] has been introduced to estimate how physically plausible such a Hamiltonian H might be. To describe this, consider a collection of n qubits evolving under a constant Hamiltonian H. Write $\mathcal{H}_1 = \mathbb{C}\{|0\rangle\} \oplus \mathbb{C}\{|1\rangle\}$ and the n-qubit Hilbert space as $\mathcal{H}_n = (\mathcal{H}_1)^{\otimes n} \cong \oplus_{j=0}^{2^n-1} \mathbb{C}\{|j\rangle\}$, so that we might view $H = \sum_{j,k=0}^{2^n-1} h_{jk}|j\rangle\langle k| \in \mathbb{C}^{2^n \times 2^n}$ as a Hermitian matrix. Let J be an n-long list of elements of $\{0,x,y,z\}$. For such a $J = j_1 j_2 \dots j_n$, we create an abbreviation $\sigma_{\otimes J} = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \cdots \otimes \sigma_{j_n}$ for the appropriate tensor product of Pauli matrices. Let $\mathcal{J}$ denote the set of all such indices J. We use $\mathcal{H}(2^n)$ to denote the vector space of Hermitian matrices. This sets notation for the equation

$$\mathcal{H}(2^n) \;=\; \bigoplus_{J \in \mathcal{J}} \mathbb{R}\,\{\sigma_{\otimes J}\}. \tag{1}$$

Containment of the right-hand side follows since tensors of Hermitian matrices are Hermitian, while the equality follows from linear independence given that the Pauli-tensors are orthogonal in the matrix inner product $A \bullet B = \mathrm{Trace}(A\overline{B}^T)$ for $A, B \in \mathbb{C}^{2^n \times 2^n}$. Thus, H may also be written as

$$H \;=\; \sum_{J \in \mathcal{J}} t_J \sigma_{\otimes J}, \quad t_J \in \mathbb{R}. \tag{2}$$

If we use $\#J$ to denote the number of nonzero indices, then any summand $t_J \sigma_{\otimes J}$ of H denotes a $\#J$-body interaction among the qubits. We say H is k-local when $k = \max\{\#J \mid t_J \neq 0\}$.

Much recent work in quantum complexity theory considers the ground states of k-local Hamiltonians. For example, an adiabatic quantum computer [1, 2] must remain in the ground-state of a k-local Hamiltonian at all times. Early works on anyonic excitations of topologically ordered Hamiltonians [3, 4] used Hamiltonians whose addends were based on the local structure of some lattice. These were usually k-local for k small. Square lattices produce four-local Hamiltonians while triangular lattices and their dual hexagonal lattices each produce six-local Hamiltonians. Considerable effort has produced results on quantum states that cannot be groundstates of k-local Hamiltonians Cf. [5]. That work measures the distance of the code from the groundstates of the Hamiltonian, while in the present work we focus on the complexity of Hamiltonians that do have the code as their groundstate. Recent attention has been given to the problem of realizing graph states as groundstates [6]. Realizing a graph state in this way is of interest since (i) the graph state is a *nondegenerate* groundstate which in principle could be obtained from the physical system by cooling, and (ii) any quantum circuit may be emulated using one-qubit rotations and measurements of a suitable state [7]. Thus realizing a Hamiltonian for a large enough graph state, cooling the system, and then applying local control and measurement is equivalent to universal quantum computation. Finally, recent work has considered a constrained family of Hamiltonians in order to produce new results on allowed groundstates [8].

Label $\mathcal{P}_n$ as the group with elements $\{\sigma_{\otimes J}\}_{J \in \mathcal{J}}$ up to phase multiple i, and consider a subgroup $G \subset \mathcal{P}_n$. The stabilizer code space of G is defined as

$$\mathcal{C}(G) \stackrel{\text{def}}{=} \left\{ |\psi\rangle \in \mathcal{H}_n \mid g|\psi\rangle = |\psi\rangle \quad \forall g \in G \right\}. \tag{3}$$

The code space $\mathcal{C}(G)$ is nonzero if and only if G is commutative and $-\mathbf{1} \notin G$ [9]. Now suppose we have a set $S = \{g_a\}$ that generates G, with each generator g_a equal to $\pm\sigma_{\otimes J}$ and at most k-local. Given commutativity, we may equally well think of the code space as the ground eigenspace of the Hamiltonian

$$H_{\mathcal{C}} \;=\; -\sum_a g_a. \tag{4}$$

Even for rather small k, in fact even $k = 3$, engineering such a k-local Hamiltonian is challenging. Hence one wishes to find a Hamiltonian with the same groundstate eigenspace as $H_{\mathcal{C}}$ yet which is ℓ-local for $\ell < k$. We call such a Hamiltonian an *ℓ-equivalent Hamiltonian* for $H_{\mathcal{C}}$. This paper provides conditions under which no such ℓ-equivalent Hamiltonian exists.

Two applications result. The first concerns error correction. The Hamiltonian whose groundstate is a stabilizer code has created an energy gap to leaving the code. This energy gap might be viewed as passive error correction, and our bounds on locality of ℓ-equivalent Hamiltonians become minimum expenses for obtaining such behavior. In particular, these results provide a quantitative argument that the four-local costs for toric codes [3] and analogous codes for cellulated surfaces [4, 10] are the best possible. A second application regards adiabatic computing, where attempts to drive down

the required k-locality of adiabatic algorithms motivates the search for ℓ-equivalent Hamiltonians [2]. In that context, these arguments give lower bounds on the locality of ℓ-equivalent Hamiltonians. However, such bounds are only on ℓ-equivalent Hamiltonians which do not exploit ancillae. Of course, they still apply to systems with ancillae if the ancillae are included in a larger system. Together with results in [2] and related work concerning ground states that can be approximated using ancillae, we have bounds on what is and is not possible with k-local operators.

The manuscript is organized as follows. Two notions of locality of a stabilizer subgroup of the Pauli group are introduced in §2, and each notion leads to a theorem constraining the inclusion of code spaces in the groundstates of Hamiltonians which are excessively local. In particular, in Theorem 2 we show that a Hamiltonian whose groundstate is a stabilizer code $\mathcal{C}(G)$ must be at least $\delta(G)$-local, where $\delta(G)$ is the minimum weight of a non-identity generator of the subgroup G. We then show in Theorem 3 that when the subgroup G can be generated by a set of generators that are at most η-local, then the Hamiltonian need be at most η-local. We analyze the ability of ν-local Hamiltonians, $\nu < \eta(G)$, to distinguish between the code space of G and that of other subgroups (Theorem 4 and Corollary 5). A perturbative result (Theorem 6 and Corollary 7) in §3 shows that if the groundspace of an excessively local Hamiltonian is too near the stabilizer code, then the gap between the groundstate eigenvalue and the next distinct eigenvalue is pinched. Finally, we apply the results to XY-split codes in §4 and consider two particular examples, namely Calderbank-Shor-Steane codes and stabilizer codes arising from cellulations of surfaces. We end the paper with discussion and conclusions.

2 Stabilizer Codes as Exact Groundspaces

We write Pauli tensors as $\sigma_{\otimes J} = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \cdots \otimes \sigma_{j_n}$ for $J = j_1 j_2 \ldots j_n$ and each $j_k \in \{0, x, y, z\}$, where $\sigma_0 = \mathbf{1}$ and the other letters denote the usual Pauli matrices. For $\mathcal{J}$ the set of all such indices J, the Pauli group $\mathcal{P}_n$ is all power of i multiples of $\sigma_{\otimes J}$ for $J \in \mathcal{J}$. Thus $|\mathcal{P}_n| = 4 \cdot 4^n$, and all elements of $\mathcal{P}_n$ commute or anticommute. For G a subgroup of $\mathcal{P}_n$, the stabilizer code of G is the subspace of $\mathcal{H}_n$ which is the intersection of the $+1$ eigenspaces of all $g \in G$. It is known that the code space, say $\mathcal{C}$, is nonzero if and only if G is commutative and $-I \notin G$ [9, p.455]. It is common to refer to G as a stabilizer group of $\mathcal{C}$ when (conversely) $\mathcal{C}$ is the intersection of the $+1$ eigenspaces of $g \in G$. Being less precise, a commutative subgroup $G \subseteq \mathcal{P}_n$ is a stabilizer group (of some nonzero $\mathcal{C}$).

The discussion requires additional background on stabilizer codes. In particular, we highlight the following facts.

Lemma 1: [See [9, §10.5.1].] *(i) Let $G \subseteq \mathcal{P}_n$ and $\Pi_G = (1/|G|)\sum_{g \in G} g$. Then for commutative G, Π_G is a projector onto the code space of G. Else $\Pi_G = \mathbf{0}$. (ii) If σ and $-\sigma$ are both in G, then the code space is trivial.*

Proof: The first item is proven in the citation. For the second, the hypothesis requires $-\mathbf{1} = (\sigma)(-\sigma) \in G$. Thus $\mathrm{Trace}(\Pi) = \mathrm{Trace}(\mathbf{1} - \mathbf{1}) = 0$, since every element of $\mathcal{P}_n$ other than $\pm\mathbf{1}$ is traceless. Since the projector Π is traceless, it is zero. Hence its target, the code space, is trivial. □

Use $\mathrm{wt}(g)$ for $g \in \mathcal{P}_n$ to denote the number of σ_x, σ_y, and σ_z factors of the tensor product. In particular, $\mathrm{wt}(-\mathbf{1}) = 0$. Also, $\mathrm{wt}(g_1 g_2) \leq \mathrm{wt}(g_1) + \mathrm{wt}(g_2)$, since any qubit whose tensor factors are $\mathbf{1}$ in g_1 and g_2 will have tensor factor $\mathbf{1}$ in their product. Finally, for $S \subset \mathcal{P}_n$, we use $\langle S \rangle$ to denote the subgroup generated by the elements of S. This is standard notation from abstract algebra, and we hope that context will make clear that it is not the Dirac notation for the expectation of an operator S.

Recall from Equation 1 that any Hamiltonian on n qubits may be written as a real linear combination of Pauli tensors. The Hamiltonian is k-local if the degree of no monomial summand exceeds k.

This is a measure of complexity of the Hamiltonian and physical systems that realize it, in that k-local Hamiltonians require at most k-qubits to interact during any infinitesimal time.

This section presents two results which argue that Hamiltonians whose groundstate captures a stabilizer code must be at least as complicated as the underlying stabilizer group. The complication of Hamiltonians is measured in k-locality. On the other hand, two reasonable definitions of the k-locality of stabilizer group are considered in separate subsections. These two measures are motivated by earlier work [6] and so are denoted $\delta(G)$, a lower bound on the weight of $g \in G$, and $\eta(G)$, in principle an upper bound. We begin with $\delta(G)$.

2.1 Lower Bound Case

We now define a quantity $\delta(G)$ that may be viewed as a lower bound on the k-locality of a stabilizer group.

Definition: Let $G \subseteq \mathcal{P}_n$ be a subgroup. Then $\delta(G) = \min\{\mathrm{wt}(g) \mid g \in G, g \neq \mathbf{1}\}$.

The next result implies that any Hamiltonian H whose groundstate is the stabilizer code must be at least $\delta(G)$ local. To see this, normalize so that H is traceless by subtracting the appropriate multiple of $\mathbf{1}$. The groundspace of the traceless Hamiltonian is then a negative eigenspace.

Theorem 2: *Let G be a stabilizer group and let H be a traceless Hamiltonian on n-qubits which is k-local for $k < \delta(G)$. Let $\mathcal{V}_- \subset \mathcal{H}_n$ be the direct sum of eigenspaces of H corresponding to negative eigenvalues. Then the code space of G is not contained within $\mathcal{V}_-$.*

Proof: Let $\{|\psi_j\rangle\}_{j=1}^L$ form a basis for the code space. Recall Π from the proof of the Lemma 1:

$$\Pi \;=\; \sum_{j=1}^{L} |\psi_j\rangle\langle\psi_j| \;=\; (1/|G|)\sum_{g\in G} g\,.$$

While the second expression is an orthogonal decomposition of a projector, the third is a well known formula for a projector onto the code space [9, §10.5.1].

Recall the decomposition of the Hamiltonian H according to Equation 2 in the introduction.

$$H \;=\; \sum_{J\in\mathcal{J}} t_J \sigma_{\otimes J}, \quad t_J \in \mathbb{R}.$$

The traceless condition forces $t_{00\ldots0} = 0$, since for $J \neq 00\ldots0$ we have $\mathrm{Trace}(\sigma_{\otimes J}) = \prod_{k=1}^n \mathrm{Trace}(\sigma_{j_k}) = 0$. If some coefficient t_J is nonzero, then by hypothesis $\#J \leq k < \delta(G)$.

The estimate follows by considering $\mathrm{Trace}(\sigma_{\otimes J} g)$ for $g \in G$ and $t_J \neq 0$. Then $\sigma_{\otimes J} g \neq \pm\mathbf{1} \in \mathcal{P}_n$, since the product has weight at least one. For g has weight at least $\delta(G)$ while $\sigma_{\otimes J}$ has weight at most $k < \delta(G)$, and $\sigma_{\otimes J} = \sigma_{\otimes J}^{-1}$, hence $\mathrm{wt}(g) = \mathrm{wt}(\sigma_{\otimes J}^{-1}\sigma_{\otimes J} g) \leq \mathrm{wt}(\sigma_{\otimes J} g) + \mathrm{wt}(\sigma_{\otimes J})$ or $\mathrm{wt}(\sigma_{\otimes J} g) \geq \delta(G) - k$. Therefore $\mathrm{Trace}(\sigma_{\otimes J} g) = 0$ since $\mathrm{Trace}(h) = 0$ for any $h \in \mathcal{P}_n - \{\pm\mathbf{1}\}$. The right hand equality of the equation below follows.

$$\sum_{j=1}^{L} \langle\psi_j|H|\psi_j\rangle \;=\; \mathrm{Trace}\big(\,\Pi H\,\big) \;=\; (1/|G|)\sum_{g\in G} \mathrm{Trace}\big(\,gH\,\big) \;=\; 0\,. \tag{5}$$

Now if $\{|\psi_j\rangle\}_{j=1}^L \subseteq \mathcal{V}_-$, then each term at the far left of Equation 5 would be negative, leading to a contradiction. □

How might one compute $\delta(G)$?

We now sketch how one might compute $\delta(G)$, using the stabilizer check matrix A of the stabilizer code. Thus $A = (A_X|A_Z) \in (\mathbb{F}_2)^{m\times n}$ corresponding to the choice of generators $\{g_j\}_{j=1}^m$, i.e. $G = \langle\{g_j\}_{j=1}^m\rangle$. A 1 in row k of column j of A_X corresponds to a factor of σ_x in qubit position k of generator g_j, and A_Z is similar. (See [9, eqn. (10.112)] or [7, §2.2.3].) Since m is the number of generators for G and $g^2 = \mathbf{1}$ for any $g \in \mathcal{P}_n$, one way to calculate $\delta(G)$ would be to enumerate all 2^m products of generators. A possible optimization of this approach would be to delete generators until the set $\{g_j\}_{j=1}^m$ is *minimal*, i.e. until the number of rows of A is also its rank.

We present a different approach. Namely, suppose that a p-local tensor product $g = \pm\sigma_{\otimes J}$ is in G. The support of g will be given by $\mathrm{supp}(g) = \{k \in \{1,2,\ldots,n\} \mid j_k \neq 0\}$, so that $|\mathrm{supp}(g)| = p$. Then for v an indicator vector of which generators occur in the product for g, $v^T A = (w_X|w_Z)$ has w_X and w_Z zero outside entries indexed by S. Now label A_S as that matrix with the columns of A_X and A_Z corresponding to S replaced by zero entries. Then v^T is a left-null vector of A_S but not of A. On the other hand, any left-null vector of A, say w with $w^T A = 0$, must also satisfy $w^T A_S = 0$. Thus $\mathrm{rank}(A_S) < \mathrm{rank}(A)$. This algorithm is defined more formally below, and we illustrate its use in an example.

Example: Consider $G = \langle X\otimes I\otimes Z, I\otimes Z\otimes X\rangle$, for which $\delta(G) = 2$. Taking a basis for A according to the generating set above yields the equation

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{6}$$

Say $S = \{1,3\}$, since the first row of A recovers the two-local $X\otimes I\otimes Z$ supported on these qubits. Then

$$A_S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

Thus the existence of this two-local element of G has caused $\mathrm{rank}(A_S) < \mathrm{rank}(A)$, which might also be inferred due to the left-null vector $v^T = (10)$.

Algorithm: Computing $\delta(G)$

For $k = 1,\ldots,m$
  For each $S \subseteq \{1,\ldots,n\}$ with $\#S = k$ do:
    Compute A_S by zeroing columns of A corresponding to S.
    If $\mathrm{rank}(A_S) < \mathrm{rank}(A)$ then
      Return $\delta(G) = k$ and exit.
  End for.
End for.

The algorithm above for computing $\delta(G)$ is polynomial in n, at least if $\delta(G) \in O(1)$. (Cf. [6].) Note however that it is not a polynomial time algorithm should $\delta(G) \in \Omega(n)$, since then the loop will loop over (a nonnegligible fraction of) the power set of $\{1,2,\ldots,n\}$.

2.2 Upper Bound Case

This section considers $\eta(G)$, which is an upper bound on the k-locality of G. However, we do not define $\eta(G)$ to be the maximum of weights of $g \in G$. For $|+\rangle^{\otimes n} = [2^{-1/2}(|0\rangle + |1\rangle)]^{\otimes n}$ spans the one-

dimensional code space of the stabilizer group generated by the n Hermitian Pauli tensors $(\sigma_x)_j = \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\sigma_x\otimes\cdots\otimes\mathbf{1}$ with a single Pauli-X on qubit j. Then G contains $\sigma_x^{\otimes n}$ of weight n, yet $|+\rangle^{\otimes n}$ is local. Thus to get a useful definition of an upper bound we resort to a minimax construction, taking the minimum over all generating sets of G of the maximum k-locality in a given set. The following definition (Cf. [6]) is equivalent to that minimax, and it gives us an upper bound on the locality of the Hamiltonian.

Definition: For $S \subseteq \mathcal{P}_n$, let $\langle S\rangle$ denote the subgroup generated by S. Let $G \subseteq \mathcal{P}_n$ be a stabilizer group with nontrivial code space. Then $\eta(G)$ is the minimal ν such that $\langle\{g \in G \mid \mathrm{wt}(g) \leq \nu\}\rangle = G$.

Theorem 3: *There is a Hamiltonian of locality $\eta(G)$ whose groundstate is the stabilizer code of G.*

Proof: This follows from the construction in equation (4). □

Next we define certain stabilizer groups, related to those in [6], that arise after choosing a particular generating set for G.

Definition of $G_\pm$: Let $G \subseteq \mathcal{P}_n$ be a commutative subgroup and $\nu < \eta(G)$. We label $G_\nu = \langle\{g \in G \mid \mathrm{wt}(g) \leq \nu\}\rangle$. Fix a minimal generating set so that (i) $G_\nu = \langle\{g_j\}_{j=1}^s\rangle$ and (ii) $\mathrm{wt}(g_j) \leq \nu$ for each j. Extend this to a minimal generating set so that $G = \langle\{g_j\}_{j=1}^t\rangle$, where the $g_j = \pm\sigma_{\otimes J}$ may be Pauli tensors of any degree if $j > s$. Finally, let $b_{s+1}, \ldots, b_t$ be some choice of $t-s$ bits. Associating these bits to sign conventions on Pauli tensors produces a stabilizer group:

$$G_\pm \;=\; \langle\{g_1, g_2, \ldots, g_s, (-1)^{b_{s+1}} g_{s+1}, \ldots, (-1)^{b_t} g_t\}\rangle. \tag{8}$$

The generating set for $G_\pm$ above is also minimal [9]. The dependence of $G_\pm$ on ν, the (ordered) sequence of generators $\{g_j\}_{j=1}^t$, and $b_{s+1}, \ldots, b_t$ will be left implicit.

Say $\nu < \eta(G)$. We next argue that the eigenspaces of ν-local Hamiltonians may not distinguish G and other extensions $G_\pm$. This might be of independent interest and will also imply a result similar to the previous one.

Theorem 4: *Let $\nu < \eta(G)$. Let Π_G and $\Pi_{G_\pm}$ be projectors on the respective code spaces, for any $G_\pm$ as above. Then for any traceless ν-local Hamiltonian H,*

$$\mathit{Trace}\big(\,\Pi_G H\,\big) \;=\; \mathit{Trace}\big(\,\Pi_{G_\pm} H\,\big). \tag{9}$$

Proof: Let $\sigma \in \mathcal{P}_n$ such that $\mathrm{wt}(\sigma) \leq \nu$. It suffices to show that $\mathrm{Trace}\big(\,\Pi_G\sigma\,\big) = \mathrm{Trace}\big(\,\Pi_{G_\pm}\sigma\,\big)$. We prove this formula using a case study.

Case 1: Suppose either $\sigma \in G$ or $-\sigma \in G$ or both. Each such element is in G_ν due to its weight, hence each such element is also an element of $G_\pm$.

Since by hypothesis G has a nontrivial code space, both σ and $-\sigma$ are not in G. (Else $-\mathbf{1} \in G$ and $\mathrm{Trace}(\Pi_G) = \mathbf{0}$ contradiction.) As a remark, $\mathrm{Trace}\big(\,\Pi_G\sigma\,\big) = \mathrm{Trace}\big(\,\Pi_{G_\pm}\sigma\,\big)$ nonetheless holds in this subcase as $0 = 0$.

Thus say $\sigma \in G$ with $-\sigma \notin G$ or vice-versa. Then each trace is $\pm 2^{t-n}$, since (i) $\mathrm{Trace}(g_1 g_2) = 0$ whenever $g_1, g_2 \in \mathcal{P}_n$ and $g_1 \notin \{g_2, -g_2\}$ and (ii) the size of the minimal generating sets demand $2^{n-t} = \#G = \#G_\pm$ [9].

Case 2: Suppose $\sigma \notin G$ and $-\sigma \notin G$. Then $\mathrm{Trace}\big(\,\Pi_G\sigma\,\big) = 0$. It would suffice to show that $\sigma \notin G_\pm$ and $-\sigma \notin G_\pm$.

Assume by way of contradiction that $\sigma \in G_{\pm}$. Then for a bit-string $c_{s+1}c_{s+2}\ldots c_t$, we have

$$\sigma = \prod_{j=1}^{s} g_j^s \prod_{j=s+1}^{t} \left((-1)^{b_j} g_j \right)^{c_j}. \tag{10}$$

Since $\prod_{j=s+1}^{t}(-1)^{b_j c_j} \in \{1,-1\}$, either $\sigma \in G$ or else $-\sigma \in G$. Contradiction. The case that $-\sigma \in G_{\pm}$ is similar. □

Corollary 5: *Suppose that H is a traceless, ν-local Hamiltonian for $\nu < \eta(G)$.*

- *If the code space of G is is a subspace of the groundstate eigenspace of H, then so is the code space of any $G_{\pm}$.*
- *For fixed j (no more than the number of negative eigenvalues of H), let $\mathcal{V}$ be the direct sum of the eigenspaces of the j smallest eigenvalues of H (counting multiplicities). If the code space of G is a subspace of $\mathcal{V}$, then the same holds for $G_{\pm}$.*

Proof: The first part is a special case of the second part. For the second part, let k be the rank of Π_G and $\Pi_{G_{\pm}}$. One may show that the minimum of $\mathrm{Trace}(\Pi H)$ over all projectors Π of rank k is attained if and only if the range of the projector is the subspace of $\mathcal{V}$ corresponding to eigenspaces of the k lowest eigenvalues (counting multiplicities). The minimum value is the sum of the corresponding eigenvalues of H. Given the hypothesis, Π_G attains this minimum. Thus $\Pi_{G_{\pm}}$ achieves it as well, implying that its image is a subspace of $\mathcal{V}$. □

Remark: During review, a referee pointed out another corollary of the theorem. No traceless ν-local Hamiltonian H as above has the code space as its groundstate eigenspace. For if so, then stabilizer codes of G and $G_{\pm}$ are both contained in this groundstate eigenspace, yet they are orthogonal due to one stabilizer holding some g_{s+j} and the other $-g_{s+j}$ for $1 \le j \le s-t$. Dimension counting then produces a contradiction. Note that to apply this result, one must verify that H is both local in a certain sense *and* has eigenspace degeneracies bounded in terms of the dimension of the code space of G.

How might one compute $\eta(G)$*?*

Recall the earlier algorithm to compute $\delta(G)$ using $A = (A_X|A_Z) \in \mathbb{F}_2^{m\times n}$. This section produces a similar algorithm for $\eta(G)$ using linear algebra. However, we first need some more notation. Namely, although the subset of k-local elements within G do not form a subgroup, those elements which only affect any collection of k-qubits do. The algorithm for $\eta(G)$ represents these subgroups as matrices and then uses algebra to decide whether their union generates G.

Definition: Recall the notation $\sigma_{\otimes J} = \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n}$ for $J = j_1 j_2 \ldots j_n$ and $j_k \in \{0,x,y,z\}$, where $\sigma_0 = \mathbf{1}$ and the other sigmas denote the appropriate Pauli matrices. The *support of* $\pm\sigma_{\otimes J}$, $\mathrm{supp}(\pm\sigma_{\otimes J})$, is $S = \{k \mid j_k \neq 0\} \subseteq \{1,2,\ldots,n\}$. Label the subgroup $\mathcal{P}_S = \{g \in \mathcal{P}_n \mid \mathrm{supp}(g) \subseteq S\}$. Also set $G_S = G \cap \mathcal{P}_S$.

Henceforth, suppose G is fixed with *nontrivial code space*, so that by Lemma 1, $g \in G$ demands $-g \notin G$. This creates a map from the row space of A to G. Indeed, since rows of A represent generators of G, the fact that the row vector $(b_1 b_2 \cdots b_n c_1 c_2 \ldots c_n)$ lies within the row space implies $\pm(\sigma_x^{b_1} \otimes \sigma_x^{b_2} \otimes \cdots \otimes \sigma_x^{b_n})(\sigma_z^{c_1} \otimes \sigma_z^{c_2} \otimes \cdots \otimes \sigma_z^{c_n})$ is an element of G. Furthermore, although the $2n$ bitstring does not make clear the choice of sign, Lemma 1 asserts that it is unique. Now recall that A_S is the matrix A except that columns corresponding to $S \subseteq \{1,2,\ldots,n\}$ have been replaced with zero columns. As a

consequence of the unique sign choice of Lemma 1, any $2n$-bit string in the rowspace supported on positions corresponding to S likewise determines an element of G_S. For $\overline{S}$ the complement of A, such an element of the rowspace might be constructed by creating a left-null vector of $A_{\overline{S}}$.

The following algorithm computes $\eta(G)$. Note the rank of the stabilizer check matrix of a subgroup is equal to that for a group if and only if the subgroup is the group. We check this condition for increasing values of k, by forming the stabilizer check matrix A_k for the subgroup of G generated by k-local Pauli tensors in G (i.e., the group generated by $\{g \in G | \mathrm{wt}(g) \leq k\}$). Once $\mathrm{rank}(A_k) = \mathrm{rank}(A)$, they both generate G, so that value of k is η.

Algorithm: Computing $\eta(G)$.

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for $k = 1 : n$
  $A_k \in \mathbb{F}_2^{0 \times 2n}$ (i.e., an empty matrix);
  for $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = k$
   Compute N_S, a matrix whose rows span the left-null space of $A_{\overline{S}}$.
   Compute $B_S = N_S A$, the matrix encoding G_S.
   Set $A_k = \begin{pmatrix} A_k \\ B_S \end{pmatrix}$.
  End for.
  if $\mathrm{rank}(A_k) = \mathrm{rank}(A)$
   return $\eta(G) = k$
  End if.
End for.

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3 Gap-Pinching when Approximating Stabilizer Codes

Our two earlier results limit those cases in which a stabilizer code lies within the groundstate of a Hamiltonian whose k-locality is less than some measure of the locality of the stabilizer group. The two measures of the group's locality were $\delta(G)$ and $\eta(G)$, where $\delta(G) \leq \eta(G)$. Neither of these results was perturbative. We next present a result which limits those cases in which a stabilizer code is merely close to the groundstate of a Hamiltonian which is more local than the code. More precisely, we argue that the groundstate eigenspace of such a Hamiltonian lacks stability, in that the gap between the lowest two distinct eigenvalues is small when compared to the total energy of the system. Similar results regarding stabilizer codes for graph states are known [6].

The notation below will be fixed while discussing the perturbative result.

- Let $q = \dim_{\mathbb{C}} \mathcal{C}(G)$, with $\mathcal{C}(G)$ the stabilizer code of G. For a graph state, $q = 1$.
- Defining $\nu < \eta(G)$ and G_ν as above, we let $r = \dim_{\mathbb{C}} \mathcal{C}_\nu$ where $\mathcal{C}_\nu = \mathcal{C}(G_\nu)$. Similarly let $\mathcal{C} = \mathcal{C}(G)$. Since $G_\nu \subseteq G$, also $\mathcal{C}_\nu \supseteq \mathcal{C}$ and thus $r \geq q$.
- Consider H a ν-local Hamiltonian with Π_H the projection onto its groundstate eigenspace.
- Π_G and Π_{G_ν} are projections onto the appropriate stabilizer code spaces.
- The trace norm $\| * \|_{\mathrm{tr}}$ on Hermitian matrices is that norm induced by the inner product $H_1 \bullet H_2 = \mathrm{Trace}(H_1 H_2^\dagger) = \mathrm{Trace}(H_1 H_2)$.

In addition to the setup, we should note that one way to quantify the distance between the code of G and the groundstate of H is to compute the trace norm of the difference of the projectors onto each space.

Theorem 6: *Let $G \subseteq \mathcal{P}_n$ have a code space of dimension $q > 0$. Let $\nu < \eta(G)$. Then any traceless ν-local Hamiltonian H whose groundstate eigenspace is q-dimensional satisfies the following inequality on the trace norm distance between the projectors Π_G and Π_H onto the code space of G and the groundstate eigenspace of H respectively.*

$$\| \Pi_G - \Pi_H \|_{\rm tr} \geq \frac{q}{\|\vec{E}\|_2} \left(\frac{E_0 + E_1 + E_2 + \cdots + E_{r-1}}{r} - E_0 \right). \tag{11}$$

Here, $E_0 \leq E_1 \leq \cdots \leq E_{2^n-1}$ is the eigenspectrum of H (with multiplicity) and $\|\vec{E}\|_2 = Trace(H^2)^{1/2} = (E_0^2 + E_1^2 + \cdots + E_{2^n-1}^2)^{1/2}$. Also, r denotes the dimension of the code space of the group $G_\nu \subseteq G$ generated by ν-local elements.

Proof: The first step is to check that due to the locality condition on H, we have $\mathrm{Trace}(\Pi_G H) = (q/r)\mathrm{Trace}(\Pi_{G_\nu} H)$. Since all elements of G and G_ν that are at most ν-local coincide, the following traces are equal.

$$\mathrm{Trace}\left(H \sum_{g \in G} g \right) = \mathrm{Trace}\left(H \sum_{g \in G_\nu} g \right). \tag{12}$$

The projectors should be normalized by $\#G$ and $\#G_\nu$ respectively. If m is the number of rows of a stabilizer check matrix for G arising from a minimal generating set and m_ν is similar for G_ν, then $\#G = 2^m$ and $\#G_\nu = 2^{m_\nu}$ [9]. Furthermore $q = 2^n/2^m$ and $r = 2^n/2^{m_\nu}$. Thus appropriately normalizing the above equation produces the desired equality.

Now let Π be any projection onto an r-dimensional space. Since $E_0 \leq E_1 \leq \cdots \leq E_{r-1}$ are the r least eigenvalues of H, we have the inequality

$$\mathrm{Trace}(\Pi H) \geq (E_0 + E_1 + E_2 + \cdots + E_{r-1}). \tag{13}$$

Recall that the inner product associated to the trace norm has a Schwarz inequality. This is the final fact required for the following sequence of inequalities.

$$\begin{aligned} \|\vec{E}\|_2 \|\Pi_G - \Pi_H\|_{\rm tr} &= \|H\|_{\rm tr} \|\Pi_G - \Pi_H\|_{\rm tr} \\ &\geq \mathrm{Trace}\big((\Pi_G - \Pi_H) H \big) \\ &= (q/r)\mathrm{Trace}(\Pi_{G_\nu} H) - \mathrm{Trace}(\Pi_H H) \\ &\geq (q/r)(E_0 + E_1 + E_2 + \cdots + E_{r-1}) - qE_0. \end{aligned} \tag{14}$$

Appropriate manipulations of the inequality between the first and last expression of the sequence above produces the result. □

Corollary 7: *Let the traceless Hamiltonian H and code G satisfy all hypotheses of Theorem 6, including excessive locality of H as compared to $\eta(G)$. Label the spectral gap of H as $\Delta E = E_q - E_0$, recalling $E_0 = E_1 = E_2 = \cdots = E_{q-1}$. The following estimate holds:*

$$\| \Pi_G - \Pi_H \|_{\rm tr} \geq q \, \|\vec{E}\|_2^{-2} \left((r-q)/r \right) \Delta E. \tag{15}$$

In particular, if $\varepsilon > \| \Pi_G - \Pi_H \|_{\rm tr}$ and r and q are treated as constants, then the gap is pinched in the sense that $\Delta E \in O(\varepsilon \|\vec{E}\|_2^2)$.

Proof: Notice that for $j \geq q$, $E_j \geq E_0 + \Delta E$. The term inside the parentheses of Theorem 6 is bounded below by a multiple of this gap (Cf. [6]):

$$\begin{aligned} \left(\tfrac{E_0+E_1+E_2+\cdots+E_{r-1}}{r}\right) - E_0 &\geq \big((q/r)E_0 + ((r-q)/r)E_q\big) - E_0 \\ &= \big((q-r)/r\big)E_0 + \big((r-q)/r\big)E_q \\ &= \big((r-q)/r\big)\Delta E. \end{aligned} \tag{16}$$

□

The pinching bound of Corollary 7 is weak in the following sense. (Cf. [6].) Effective Hamiltonians are used to approximate lower energy eigenstates while ignoring higher energy eigenstates. Thus the large total energy $\|\vec{E}\|_2^2$ is not a concern. On the other hand, Corollary 7 also argues that the higher energy eigenstates can not be (entirely) irrelevant to such approximations.

4 Application to XZ-Split Codes

This section considers the computation of the quantities $\delta(G)$ and $\eta(G)$ used in the Hamiltonian locality bounds in the special case that the code is XZ-split. This is a broad class of codes that includes CSS codes and also topological orders on surfaces.

Definition: Let $\mathcal{P}_{X,n} = \langle \{\sigma_{x,j}\}_{j=1}^n \cup \{-\sigma_{x,j}\}_{j=1}^n \rangle$ be the subgroup of $\mathcal{P}_n$ containing Pauli tensors with only Pauli X factors, and let $\mathcal{P}_{Z,n}$ be similar. A stabilizer group G is XZ-split if $G = \langle \{g_j\}_{j=1}^m \rangle$ where for each j either $g_j \in \mathcal{P}_{X,n}$ or $g_j \in \mathcal{P}_{Z,n}$. Perhaps upon reordering, this produces a block-diagonal stabilizer check matrix with blocks A_X and A_Z defined by the following equation:

$$A = \begin{pmatrix} A_X & \mathbf{0} \\ \mathbf{0} & A_Z \end{pmatrix}. \tag{17}$$

We also label $G_X = \mathcal{P}_{X,n} \cap G$ and $G_Z = \mathcal{P}_{Z,n} \cap G$.

Next, we study $\delta(G)$ and $\eta(G)$ for XZ-split codes.

Theorem 8: *Suppose that G has nonzero code space. Label $G_X G_Z = \{g_x g_z \mid g_x \in G_X, g_z \in G_Z\}$.*

- *(G is XZ-split) $\Longleftrightarrow$ ($G = G_X G_Z$).*
- *If G is XZ-split, then $\delta(G) = \min\{\delta(G_X), \delta(G_Z)\}$.*
- *If G is XZ-split, then $\eta(G) = \max\{\eta(G_X), \eta(G_Z)\}$.*

Proof: For the first item, since $G_X \subseteq G$ and $G_Z \subseteq G$, we must have $G_X G_Z = \{g_x g_z \,;\, g_x \in G_X, g_z \in G_Z\}$ within G. For the opposite containment, the generators guaranteed by the XZ-split condition show that $G = \langle G_X G_Z \rangle$. On the other hand, finite products of elements in $G_X G_Z$ lie in $G_X G_Z$, since G is commutative.

For the second item, the minimum is greater than $\delta(G)$ since $G_X \subseteq G$ and $G_Z \subseteq G$ imply $\delta(G_X) \leq \delta(G)$ and $\delta(G_Z) \leq \delta(G)$. On the other hand, let $g \in G$. Then $g = g_x g_z$ and $\mathrm{wt}(g) \geq \max\{\mathrm{wt}(g_x), \mathrm{wt}(g_z)\}$ since any qubit on which either g_x or g_z has a nontrivial tensor factor will have a nontrivial factor in the product for g.

For the last item, let $G_X = \langle \{g_{x,j}\}_{j=1}^{m_x} \rangle$ and $G_Z = \langle \{g_{z,j}\}_{j=1}^{m_z} \rangle$ be generating sets chosen to be at most $\eta(G_X)$ local and $\eta(G_Z)$ local. Since $G = G_X G_Z$, we have $G = \langle \{g_{x,j}\}_{j=1}^{m_x} \cup \{g_{z,j}\}_{j=1}^{m_z} \rangle$. Thus $\eta(G) \leq \max\{\eta(G_x), \eta(G_z)\}$.

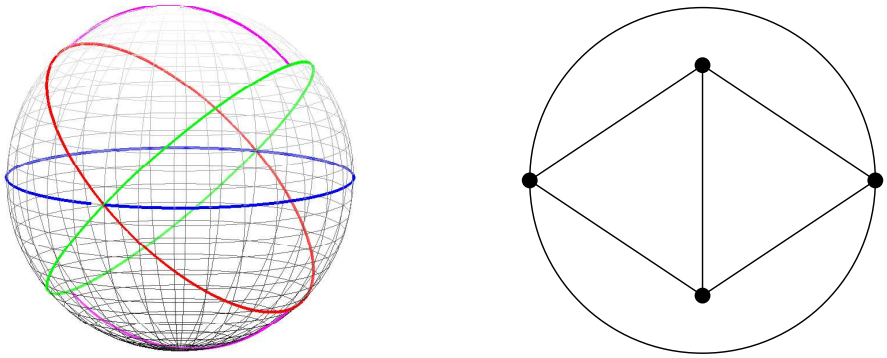

Fig. 1. A counterexample to the conjecture that $\delta(G)$ is the minimum of the valences of the one-skeleta of the cellulation Γ and the dual cellulation Γ^*. Cellulate (partition) the surface of a sphere as shown on the left, where a schematic of the upper hemisphere is shown on the right. Then $\delta(G) = 2$, where the minimal boundary is the two-edge circle around the equator of the sphere. Yet the minimum of the valences is three.

On the other hand, assume by way of contradiction that $G = \langle \{g_j\}_{j=1}^m \rangle$ where every g_j has weight strictly less than $\max\{\eta(G_X), \eta(G_Z)\}$. Writing $g_j = g_{j,x} g_{j,z}$ produces generating sets $G_X = \langle \{g_{j,x}\}_{j=1}^m \rangle$ and $G_Z = \langle \{g_{j,z}\}_{j=1}^m \rangle$, each of which has weight less than the maximum. Contradiction. Thus we have also shown $\eta(G) \geq \max\{\eta(G_x), \eta(G_z)\}$. □

Next we apply this result to Calderbank-Shor-Steane codes and to topological orders from surface cellulations.

4.1 Calderbank-Shor-Steane codes

All Calderbank-Shor-Steane codes [9, §10.4.2] [11, 12] are XZ-split. Indeed, suppose $\mathrm{CSS}(C_1, C_2)$ is the code arising from classical codes C_1 and C_2, where C_1 corrects bit-flips and C_2 phase-flips. Then for A_1 the parity check matrix of C_1 and A_2 the parity check matrix of the dual code $C_2^\perp$, we have a stabilizer check matrix $A = \mathrm{diag}(A_1, A_2)$ for $\mathrm{CSS}(C_1, C_2)$. The converse only holds in a technical sense.[a]

4.2 Topological orders from surface cellulations

This section considers $\delta(G)$ and $\eta(G)$ in the case in which the stabilizer code G results from the cellulation of a surface without boundary [4, 10]. The topological order case requires further attention, in that answers should be computable using only the cellulation of the surface. The codes depend on the cellulation rather than the topology (i.e. genus) of the surface, and $\delta(G)$ and $\eta(G)$ also depend on this cellulation.

We will not review the theory of cellulations or their duals, except to note that the dual cellulation associates a vertex to each face of the original and a face to each vertex (E.g. [13]). The relevant definitions will imply that $\delta(G)$ is the number of edges in the smallest bounding chain in either the cellulation or its dual. We also provide a counterexample to the conjecture that $\delta(G)$ is the minimum of the valences of the one-skeleton and dual one-skeleton, although this is frequently the case in examples.

[a] Any XZ-split code might be associated to $\mathrm{CSS}(C_1, C_2)$ for some classical codes C_1 and C_2, yet the ratio of logical to encoding bits of these classical codes would be arbitrary. Thus we retain XZ-split as a separate concept.

Let S be an oriented surface with no boundary, and let Γ be a two-complex which is a cellulation of S. Let $\mathcal{V}(\Gamma)$, $\mathcal{E}(\Gamma)$, and $\mathcal{F}(\Gamma)$ denote the vertices, edges, and faces of Γ respectively. We also suppose a dual cellulation Γ^* with bijections $\mathcal{V}(\Gamma^*) \leftrightarrow \mathcal{F}(\Gamma)$, $\mathcal{E}(\Gamma^*) \leftrightarrow \mathcal{E}(\Gamma)$, and $\mathcal{F}(\Gamma^*) \leftrightarrow \mathcal{V}(\Gamma)$ (E.g. [13]).

Consider the quantum system which associates a qubit to each edge of Γ (or Γ^*). A well known topologically ordered stabilizer code has a code space whose dimension is $\dim_{\mathbb{F}_2} H_1(\Gamma, \mathbb{F}_2)$, where the latter is a cellular homology with bit coefficients [4, 10]. To review this briefly, the generators are indexed by the unions of the faces and vertices of Γ. Let $q(e)$ denote the qubit of an edge $e \in \mathcal{E}(\Gamma)$ and $\sigma_{x,q}$ denote the Pauli tensor which is an identity except for a single σ_x factor on qubit q. Then the generator associated to a face $f \in \mathcal{F}$ is $\sigma_x^{\otimes f} = \prod_{e \in \partial f} \sigma_{x,q(e)}$. The generator associated to a vertex $v \in \mathcal{V}(\Gamma)$ may be defined in terms of the dual face $v^* \in \mathcal{F}(\Gamma^*)$. Namely, let $\sigma_z^{\otimes v} = \prod_{e^* \in \partial v^*} \sigma_{z,q(e)}$, or equivalently tensor over all qubits on edges incident on the vertex. The stabilizer code is then $G = \langle \{\sigma_x^{\otimes f}\}_{f \in \mathcal{F}(\Gamma)} \cup \{\sigma_z^{\otimes v}\}_{v \in \mathcal{V}(\Gamma)} \rangle$. Such a code is XZ-split. As an aside, the associated Hamiltonian $H = -\sum_{f \in \mathcal{F}(\Gamma)} \sigma_x^{\otimes f} - \sum_{v \in \mathcal{V}(\Gamma)} \sigma_z^{\otimes v}$ is of interest independent of its homologically structured degenerate groundstate, in that the excitations out of this groundstate are abelian anyons with $\mathbb{Z}/2\mathbb{Z}$ gauge [3].

Before considering the topological order as an XZ-split stabilizer code, we set the following notation for homological boundary operators.

$$\begin{array}{lccc} \partial_2 : & \mathrm{span}_{\mathbb{F}_2} \mathcal{F}(\Gamma) & \rightarrow & \mathrm{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma), \\ \partial_2^* : & \mathrm{span}_{\mathbb{F}_2} \mathcal{F}(\Gamma^*) & \rightarrow & \mathrm{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma^*). \end{array} \tag{18}$$

Consider matrices D_X and D_Z for ∂_2 and ∂_2^* respectively. Consider a column of D_X. It contains entries of $1 \in \mathbb{F}_2$ at precisely those positions corresponding to edges $e \in \mathcal{E}$ such that $e \in \partial_2 f$ for $f \in \mathcal{F}$ the column label. A similar comment applies to D_Z, so that the stabilizer check matrix of G has this form:

$$A = \begin{pmatrix} D_X^T & \mathbf{0} \\ \mathbf{0} & D_Z^T \end{pmatrix}. \tag{19}$$

Here, the superscript T denotes transpose. Also, we list face operators before vertex operators when forming the matrix, else an antidiagonal matrix results. Thus, in the special case of a topological order, it is possible to compute $\delta(G)$ and $\eta(G)$ using only homological inputs, namely the matrices of the appropriate boundary maps in the cellulation and cocellulation.

However, $\delta(G)$ and $\eta(G)$ clearly depend on the cellulation rather than the topology of the underlying surface. To emphasize that point, note that for $g \in G_x$ we may associate $|\mathrm{supp}(g)|$ to the size of a boundary in $\mathrm{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma)$, while a similar comment applies to g_z and $\mathrm{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma^*)$. Hence, $\delta(G)$ is the minimum of the smallest number of edges required to support a boundary in either Γ or Γ^*. Since one may always subdivide an edge, this is not a topological invariant.

It is tempting given the last paragraph to conjecture that $\delta(G)$ is the minimum of the valences of the one-skeleta of Γ and Γ^*, i.e. of the graphs which result by ignoring faces (two-cells) in either. In fact, this is incorrect. Figure 1 provides a counterexample, in that the boundary with the least number of edges in Γ does not bound a single face.

5 Discussion and Conclusions

Since the locality of a Hermitian matrix might serve as a crude figure-of-merit for its experimental difficulty, theorists hope to find interactions which are both highly local and have robust stabilizer quantum codes as their groundstate eigenspaces (E.g. [14]). This manuscript constrains such efforts

by arguing that excessively local Hamiltonians with the desired groundstate either do not exist or else have undesirable properties. The three main results argue that (i) no Hamiltonian may be more local than the minimum locality of any element of the corresponding stabilizer group, (ii) Hamiltonians which do not allow for more nonlocal Pauli tensors in the stabilizer group must also be ℓ-equivalent Hamiltonians for many other stabilizer groups, and (iii) approximating a stabilizer code using an excessively local Hamiltonian leads to gap pinching. Nonetheless, the technical statements given here might lead to new examples.

Speculating in a slightly broader context, there are two ways one might attempt to use k-local Hamiltonians to simulate $\ell > k$-local systems of interest. One might (a) exploit crosstalk mediated by an ancilla or (b) pulse noncommuting Hamiltonians (absent ancilla). Our results do not account for clever use of ancilla. Indeed, ancillae have been used successfully to construct Hamiltonians with specified groundstate [2, 15, 16]. On the other hand, our results argue that Hamiltonians whose addends are *noncommuting* Pauli tensors must nonetheless obey certain locality constraints on their groundspaces. Thus, the ancilla-based approach might be preferable.

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