Bifurcations, Center Manifolds, and Periodic Solutions

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Summary. Nonlinear time delay differential equations are well known to have arisen in models in physiology, biology and population dynamics. These delay differential equations usually have parameters in their formulation. How the nature of the solutions change as the parameters vary is crucial to understanding the underlying physical processes. When the delay differential equation is reduced, at an equilibrium point, to leading linear terms and the remaining nonlinear terms, the eigenvalues of the leading coefficients indicate the nature of the solutions in the neighborhood of the equilibrium point. If there are any eigenvalues with zero real parts, periodic solutions can arise. One way in which this can happen is through a bifurcation process called a Hopf bifurcation in which a parameter passes through a critical value and the solutions change from equilibrium solutions to periodic solutions. This chapter describes a method of decomposing the delay differential equation into a form that isolates the study of the periodic solutions arising from a Hopf bifurcation to the study of a reduced size differential equation on a surface, called a center manifold. The method will be illustrated by a Hopf bifurcation that arises in machine tool dynamics, which leads to a machining instability called regenerative chatter.

Keywords: Center manifolds, delay differential equations, exponential polynomials, Hopf bifurcation, limit cycle, machine tool chatter, normal form, semigroup of operators, subcritical bifurcation.

1 Background

With the advent of new technologies for measurement instrumentation, it has become possible to detect time delays in feedback signals that affect a physical system's performance. System models in a number of fields have been investigated in which time delays have been introduced in order to have the output of the models more closely reflect the measured performance. These fields have included physiology, biology, and population dynamics (see an der Heiden [2], Kuang [23], and MacDonald [24]).

In recent years, time delays have arisen in models of machine tool dynamics. In particular, a phenomenon, called regenerative chatter, is being heavily studied. Regenerative chatter can be recognized on a manufacturing plant floor by a characteristic high pitched squeeling sound, distinctive marks on the workpiece, and by undulated or dissected chips (see Tlusty [32]). It is a self-excited oscillation of the cutting tool relative to the workpiece during machining. Self-excited oscillations, mathematically called limit cycles or isolated periodic solutions, reflect the fact that there are non-linearities in the physical system being modeled that have to be taken into account. For further reading on delay differential equations in turning or numerically controlled lathe operations the reader is referred to Kalmár-Nagy et al. in [20] and [21]. For problems in drilling see Stone and Askari [29] and Stone and Campbell [30]. Finally for problems in milling operations see Balachandran [5] and Balachandran and Zhao [6]. The modeling of regenerative chatter arose in work at the National Institute of Standards and Technology (NIST) in conjunction work related to error control and measurement for numeric control machining.

Along with the nonlinearities, the differential equations, from ordinary, partial, and delay equations, that model the physical processes, usually depend on parameters that have physical significance, such as mass, fundamental system frequency, nonlinear gains, and levels of external excitation. Changes, even small ones, in many of these system parameters can drastically change the qualitative nature of the the system model solutions. In this chapter we will examine the effect that variations of these parameters have on the nature and number of solutions to a class of nonlinear delay differential equations. The changing nature of solutions to a differential equation is often referred to as a bifurcation, although formally the concept of bifurcation refers to parameter space analysis. The term bifurcation means a qualitative change in the number and types of solutions of a system depending on the variation of one or more parameters on which the system depends. In this chapter we will be concerned with bifurcations in the nature of solutions to a delay differential equation (DDE) that occur at certain points in the space of parameters, called Hopf bifurcation points. The bifurcations that arise will be called Hopf bifurcations.

From an assumed earlier course in differential equations, it should be clear to the reader that the eigenvalues of the linear portion of the state equations are an indicator of the nature of the solutions. For example, if all of the eigenvalues have negative real parts then we can expect the solutions to be stable in some sense and if any of them have positive real parts then we can expect some instabilities in the system. What happens if any of the eigenvalues have zero real parts? This is where, one might say, the mathematical fun begins, because these eigenvalues indicate that there is likely to be some oscillatory affects showing up in the solutions. The game then is to first determine those system parameters that lead to eigenvalues with zero real parts. The next step in analyzing a system of differential equations, that depends on parameters, is to write the system in terms of its linear part and the remaining nonlinear part and then to decompose it in order to isolate those equation components most directly affected by the eigenvalues with zero real parts and those equations affected by the eigenvalues with nonzero real parts. This same approach applies to problems both in ordinary differential equations as well as in delay differential equations. Once this decomposition has been developed we can then concentrate our effort on studying the component equations related to the eigenvalues with zero real parts and apply methods to simplify them.

We will assume at this point that a time delay differential equation modeling a physical phenomenon has been written with linear and nonlinear terms. Although delay differential equations come in many forms, in this chapter we will only consider delay equations of the form

$$\frac{dz}{dt}(t,\mu) = U(\mu)z(t,\mu) + V(\mu)z(t-\sigma,\mu) + f(z(t,\mu), z(t-\sigma), \mu), \qquad (1)$$

where $z \in R^n$, the space of n-dimensional real numbers, U and V, the coefficient matrices of the linear terms, are $n \times n$ matrices, $f \in R^n$, $f(0,0,\mu) = 0$, is a nonlinear function, μ is a system parameter, and $\sigma, s \in R$. For most practical problems we can assume the f function in equation (1) is sufficiently differentiable with respect to the first and second variables and with respect to the parameter μ. Let $\mathcal{C}_0$ be the class of continuous functions on $[-\sigma, 0]$ and let $z(0) = z_0 \in R^n$. Then, there exists, at least locally, a unique solution to (1) that is not only continuous but differentiable with respect to μ. For a full discussion of the existence, uniqueness, and continuity questions for delay differential equations the reader is referred to Hale and Lunel [15]. For ordinary differential equations see the comparable results in Cronin [10]. For the rest of this chapter we will assume that, for both ordinary and delay differential equations, unique solutions exist and that they are continuous with respect to parameters.

Although the results discussed in this chapter can be extended to higher dimension spaces we will mainly be interested in problems with $n = 2$ that dependent on a single parameter. We will also concentrate on bifurcations in the neighborhood of the $z(t) \equiv 0$ solution. This is clearly an equilibrium point of (1) and is referred to as a local bifurcation point. For a discussion of various classes of bifurcations see Nayfeh and Balachandran [26]. Since machine tool chatter occurs when self oscillating solutions emanate from equilibrium points, i. e. stable cutting, we will concentrate in this chapter on a class of bifurcations called Hopf (more properly referenced as Poincaré-Andronov-Hopf in Wiggins [34]) bifurcations. These are bifurcations in which a family of isolated periodic solutions arises as the system parameters change and the eigenvalues cross the imaginary axis. As earlier noted, the equilibrium points at which Hopf bifurcations occur are sometimes referred to as Hopf points. The occurrence of Hopf bifurcations depend on the eigenvalues of the linear portion of (1), given by

$$\frac{dz}{dt}(t,\mu) = U(\mu)z(t,\mu) + V(\mu)z(t-\sigma,\mu), \qquad (2)$$

in which at least one of the eigenvalues of this problem has a zero real part.

As in ordinary differential equations the eigenvalues of the linear system tell the nature of the stability of solutions of both (1) and (2). Whereas in ordinary differential equations the eigenvalues are computed from the characteristic polynomial, in delay differential equations the eigenvalues arise from an equation called the *characteristic equation* associated with the linear equation (2). This equation is a transcendental equation with an infinite number of solutions called the eigenvalues of (2). We will assume that $z(t) \equiv 0$ is an equilibrium point of (1) and that, in the neighborhood of this equilibrium point, (2) has a family of pairs of eigenvalues, $\lambda(\mu),\ \overline{\lambda}(\mu)$, of (2) such that $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, where $\alpha,\ \omega$ are real, $\alpha(0) = 0,\ \omega(\mu) > 0,\ \alpha'(0) \neq 0$. This last condition is called a transversality condition and implies that the family of eigenvalues $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ is passing across the imaginary axis. Under these conditions a Hopf bifurcation occurs, where periodic solutions arise from the equilibrium point. These conditions also apply in the case of ordinary differential equations.

In both ordinary and delay differential equations, the nature of the stability of these bifurcating periodic solutions at Hopf points can be more easily studied if equation (1), in the case of delay differential equations, can be reduced to a simpler form in the vicinity of the bifurcation point. We will develop a simplification technique comparable to that used in ordinary differential equations. It reduces the study of the stability of the bifurcating periodic solutions to the study of periodic solutions of a simplified system, called a normal form, on a surface, called a center manifold. Center manifolds arise when the real parts of some eigenvalues are zero. A center manifold is an invariant manifold in that, if solutions to equation (1) begin on the manifold, they remain on the manifold. This manifold is usually of lower dimension than the space and the nature of the stability of the equilibrium point depends on the projected form of equation (1) on the manifold. The projection of equation (1) onto the center manifold usually leads to a lower order system and the conversion of that system to a normal form provides a means of studying the stability of the bifurcating periodic solutions. In fact we will see that, for the example machining problem, the analysis will reduce to a problem of solving a simplified approximate ordinary differential equation on a center manifold of dimension two. The reduction process is carried out in a number of steps based on the approaches of Hassard et al. [17] and Wiggins [34].

The bifurcation analysis developed in this chapter depends on three simplification steps. In the first step we will show how the delay differential equation (1) can be transformed into three equations, where the eigenvalues of the first two have zero real parts and the eigenvalues of the third have negative real parts. In the process of developing these equations we will show how the transformations involved are analogous to those used in ordinary differential equations. In the second simplification step we will show the form that (1) takes on the center manifold and finally, in the third step, we will see what the normal form looks like and, from this, how the bifurcating periodic solutions are developed. We will develop the first decomposition more thoroughly because it exemplifies the analogies between ordinary and delay differential equations. The other simplifications will be given as formulas, but an example will be given at the end in which the details of the transformations in a particular case are developed and used to predict stability of the bifurcating solutions. For a more automated approach to computing a center manifold, using a symbolic manipulation program, the reader is referred to Campbell [8].

This is a note to the reader. In this chapter we will be dealing with differential equations, both ordinary and delay, that have complex eigenvalues and eigenvectors. Although the differential equations associated with real world models are most often formulated in the real space R^n, these differential equations can be viewed as being embedded in the complex space C^n. This process is called *complexification* and allows the differential equation to be studied or solved in the complex space. At the end, appropriate parts of the complex solution are identified with solutions to the original differential equation in the real domain. A thorough discussion of this process is given in Hirsch and Smale [19]. We will not be concerned with the formal embedding of the real differential equations in the complex domain, but just to note that it is possible and causes no difficulties with what we will be talking about in this chapter. We only bring this up so that the reader may understand why we seem to flip back and forth between real and complex equations. We are simply working with this complexification process behind the scenes without writing out the complete details. As we work through the examples the reader will see that it is quite natural to exist in both domains and see that the final result leads to the desired approximate solution of the original delay differential equation.

The chapter is divided as follows. In Section 2 we will show how the adjoint to a linear ordinary differential equation can be used to naturally generate a bilinear form that acts as an inner product substitute and introduces geometry to a function space. This bilinear form is then used to define an orthogonality property that is used to decompose the differential equations into those equations that have eigenvalues with zero real parts and those that have non-zero real parts. We first introduce these ideas for ordinary differential equations in order to show that the ideas are models for the analogous ideas for delay differential equations. In Section 3 we will show how this decomposition works for an ordinary differential equation example. In Section 4 we will show how a delay differential equation can be formulated as an operator equation that has a form similar to an ordinary differential equation. Within that section we will also introduce a bilinear form by way of an adjoint equation that is analogous to the one in ordinary differential equations and is used in a similar manner to decompose the delay differential equation into a system of operator equations that have components dependent on the eigenvalues of the characteristic equation with zero real parts and those with non-zero real parts. In Section 5 we will start introducing the main example of a delay differential equation that we will consider in this chapter. We will show how it is reduced to an operator equation and decomposed into components dependent on eigenvalues with zero real parts and those dependent on non-zero real parts. In Section 6 we will introduce the general formulas needed in order to compute the center manifold, the normal form of the delay differential equation on the center manifold, and the bifurcated periodic solution for the delay differential equation on the manifold. In Section 7 we will continue with the main example and develop the form of the center manifold, the normal form for the example on the center manifold, and finally the resulting periodic solution on the center manifold. In the last Section 8 we show the results of numerical simulations of the example delay differential equation in the vicinity of the bifurcation points and show that there is a possibility of unstable behavior for values of system parameters that extend into parameter regions that would otherwise be considered stable.

2 Decomposing Ordinary Differential Equations Using Adjoints

Many results for delay differential equations are direct analogies of results in ordinary differential equations. In this section we will review some properties of ordinary differential equations that motivate analogous properties in delay differential equations. We will break the decomposition process down into five basic steps. Later we will show that analogous five steps can be used to decompose a delay differential equation.

Step 1: Form the Vector Equation

We begin by considering the differential equation

$$\frac{dz}{dt}(t) = Az(t) + f(z(t), \mu), \tag{3}$$

$z \in R^n$, A an $n \times n$ real matrix, $f \in R^n$ with locally bounded derivatives, $-\infty < t < \infty$. The homogeneous part is given by

$$\frac{dz}{dt}(t) = Az(t). \tag{4}$$

We will stay as much as we can with real variables since many of the problems leading to delay differential equations are formulated with real variables.

The solution of the linear system (4) with constant coefficients can be represented as a parametric operator of the form

$$T(t)q = z_t(q) = e^{At}q, \tag{5}$$

acting on a vector $q \in R^n$. $T(t)$ is said to be a **group** of operators since $T(t_1 + t_2) = T(t_1)T(t_2)$ and $T(t)T(-t) = I$, the identity. The family of operators would be called a **semigroup** if an identity exists but there are no inverse elements. In Figure 1 we show two points of view about solutions to differential

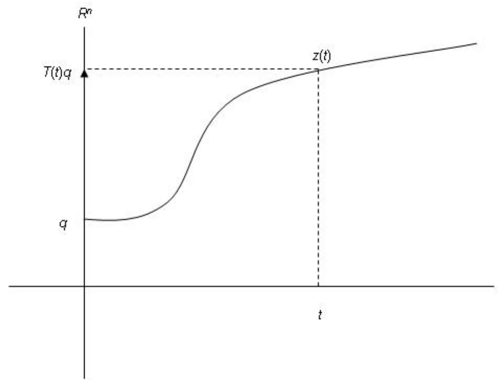

Fig. 1. The solution operator maps functions in R^n to functions in R^n.

equations. Traditionally we look at solutions to ordinary differential equations as trajectories begining with an initial vector q and taking on the value $z(t)$ at some time $t > 0$, say. There is a sense of physical meaning here such as the trajectory of a ball. However, the solutions can also be viewed as t dependent maps $T(t)$ in R^n of initial vectors q. They can then be thought of as a flow in R^n space. This is a point of view taken by Arnold [3]. It will be this point of view that will turn out more fruitful when we get to delay differential equations.

In most ordinary differential equation textbooks a linear transformation is used to write the matrix A in a form that separates it into blocks with eigenvalues having positive, zero, or negative real parts, called a Jordan normal form. However, in this section we will introduce a change of coordinates procedure that accomplishes the decomposition by way of the adjoint equation since this translates to an analogous method for delay differential equations. This process allows us to introduce a geometric point of view to the decomposition of equation (3). In particular, we can use the geometric property of orthogonality as a tool to decompose equation (3).

Step 2: Define the Adjoint Equation

The adjoint equation to equation (23) is given by

$$\frac{dy}{dt}(t) = -A^T y(t). \tag{6}$$

Step 3: Define a Natural Inner Product by way of an Adjoint

Equations (4) and (6) are related by the Lagrange identity

$$y^T \Theta z + \Omega y^T z = \frac{d}{dt}\left(y^T z\right), \tag{7}$$

where $\Theta z = \dot{z} - Az$, $\Omega y = \dot{y} + A^T y$. If z and y are solutions of (4) and (6), respectively, then it is clear that $(d/dt)\left(y^T z\right) = 0$ which implies $y^T z$ is constant and is the natural inner product of R^n. We note here that in the case that the inner product is taken, where y and z are complex, then the inner product would be $\overline{y}^T z$. Thus, the use of an adjoint equation leads naturally to an inner product definition. It might then seem reasonable that using adjoint properties could geometrically lead to some form of *orthogonal* decomposition of a system of differential equations in a similar manner to the process of orthogonal decomposition of a vector in R^n. In fact this is what we will show in this section and in Section 4 we will show the same idea extends to delay differential equations.

We begin by stating some results on eigenvalues and eigenvectors from linear algebra that have direct analogs in the delay case. To be general, we will state them in the complex case. Let A be an $n \times n$ matrix with elements in C^n and A^* the usual conjugate transpose matrix. Let $(\cdot, \cdot)$ be the ordinary inner product in C^n. Then the following hold.

1. $\left(\overline{\psi},\, A\phi\right) = \left(A^*\overline{\psi},\, \phi\right)$.
2. λ is an eigenvalue of A if and only if $\overline{\lambda}$ is and eigenvalue of A^*.
3. The dimensions of the eigenspaces of A and A^* are equal.
4. Let $\phi_1, \cdots, \phi_d$ be a basis for the right eigenspace of A associated with eigenvalues $\lambda_1, \cdots, \lambda_d$ and let $\psi_1, \cdots, \psi_d$ be a basis for the right eigenspace of A^* asociated with the eigenvalues $\overline{\lambda}_1, \cdots, \overline{\lambda}_d$. Construct the matrices $\Phi = (\phi_1, \cdots, \phi_d)$, $\Psi = (\psi_1, \cdots, \psi_d)$. The matrices Φ and Ψ are $n \times d$. If we define the bilinear form
$$\langle \Psi, \Phi \rangle = \begin{pmatrix} (\psi_1, \phi_1) \cdots (\psi_1, \phi_d) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\psi_d, \phi_1) \cdots (\psi_d, \phi_d) \end{pmatrix}, \tag{8}$$
then $\langle \Psi, \Phi \rangle$ is non-singular and can be chosen so that $\langle \Psi, \Phi \rangle = I$.

Although the $\langle \cdot, \cdot \rangle$ is defined in terms of Ψ and Φ, the definition is general and can be applied to any two matrices U and Z. Thus, $\langle U, Z \rangle$, where U and Z are matrices, is a bilinear form that satisfies properties of an inner product. In particular

$$\begin{aligned} \langle U, \alpha Z_1 + \beta Z_2 \rangle &= \alpha \langle U, Z_1 \rangle + \beta \langle U, Z_2 \rangle, \\ \langle \alpha U_1 + \beta U_2, Z \rangle &= \overline{\alpha} \langle U_1, Z \rangle + \overline{\beta} \langle U_2, Z \rangle, \\ \langle UM, Z \rangle &= M^* \langle U, Z \rangle, \\ \langle U, ZM \rangle &= \langle U, Z \rangle M, \end{aligned} \tag{9}$$

where α, β are complex constants and M is a compatible matrix.

Step 4: Get the critical Eigenvalues

Although A in equation (3) is real, it can have multiple real and complex eigenvalues. The complex ones will appear in pairs. To reduce computational complexity, we will assume that A has two eigenvalues $i\omega,\ -i\omega$ with associated eigenvectors $\phi,\ \overline{\phi}$. We will assume that all other eigenvalues are distinct from these and have negative real parts. The associated eigenvalues with zero real parts of A^* are $-i\omega,\ i\omega$ with right eigenvectors $\overline{\psi},\ \psi$. We will use properties of the adjoint to decompose (3) into three equations in which the first two will have eigenvalues with zero real parts.

Step 5: Apply Orthogonal Decomposition

We begin by defining the matrices

$$\Phi = (\phi, \overline{\phi}), \quad \Psi = \left(\overline{\psi}, \psi\right)^T. \tag{10}$$

Here we take $d = 2$ in equation (8) and note that Φ and Ψ are $n \times 2$ matrices.

Let $z(t, \mu)$ be the unique family of solutions of equation (3). This is possible due to the standard existence, uniqueness, and continuity theorems. Define

$$Y(t, \mu) = \langle \Psi,\ z(t, \mu)\rangle = \begin{pmatrix} (\overline{\psi},\ z(t, \mu)) \\ (\psi,\ z(t, \mu)) \end{pmatrix}, \tag{11}$$

where $Y(t, \mu) \in C^2$. Set $Y(t, \mu) = (y_1(t, \mu), y_2(t, \mu))^T$, where $y_1(t, \mu) = \left(\overline{\psi},\ z(t, \mu)\right)$ and $y_2(t, \mu) = (\psi,\ z(t, \mu))$.

Define the matrices

$$B = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \quad B^* = \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix}. \tag{12}$$

They satisfy $A\Phi = \Phi B,\ A^*\Psi = \Psi B^*$. If we join equations (9), (11), and (12) with the fact that $B^{**} = B$ then we have

$$\frac{dY}{dt}(t, \mu) = BY(t, \mu) + \langle \Psi,\ f(z(t, \mu), \mu)\rangle. \tag{13}$$

This can be written as

$$\begin{aligned} \frac{dy}{dt}(t, \mu) &= i\omega y(t, \mu) + F(t, \mu), \\ \frac{d\overline{y}}{dt}(t, \mu) &= -i\omega \overline{y}(t, \mu) + \overline{F}(t, \mu), \end{aligned} \tag{14}$$

where $F(t, \mu) = \left(\overline{\psi},\ f(z(t, \mu), \mu)\right)$. These are the first two of the three equations.

In order to develop the third equation we will use a notion involved with decomposing a vector into two orthogonal components. Here is where geometry enters the picture. This decomposition process arises, for example, when one applies the Gramm-Schmidt orthogonalization method numerically to vectors. The general idea is that the difference between a vector and its orthogonal projection on a linear space, formed from previous orthogonalized vectors, generates an orthogonal decomposition of the original vector. This idea will be generalized here to function spaces, but the basic methodology holds true.

Begin by defining the difference between $z(t,\mu)$ and its projection onto the linear space formed by the columns of Φ as

$$w(t,\mu) = z(t,\mu) - \Phi Y(t,\mu), \tag{15}$$

which makes sense in terms of dimensionality, since Φ is an $n \times 2$ matrix, $Y(t,\mu) \in C^2$, and $z \in R^n$. This is a vector orthogonal to $\Phi Y(t,\mu)$ in function space. We note that $w(t,\mu)$ is real because $\Phi Y(t,\mu) = \left(\overline{\psi},\, z(t,\mu)\right)\phi + \left(\psi,\, z(t,\mu)\right)\overline{\phi} = 2\mathrm{Re}\{\left(\overline{\psi},\, z(t,\mu)\right)\phi\}$.

To show that this function is orthogonal to the space formed by the columns of Φ we note that

$$\begin{aligned} w(t,\mu) &= z(t,\mu) - \Phi\langle\Psi, z(t,\mu)\rangle, \\ \langle\Psi, w(t,\mu)\rangle &= \langle\Psi, z(t,\mu)\rangle - \langle\Psi, \Phi\langle\Psi, z(t,\mu)\rangle\rangle, \\ &= \langle\Psi, z(t,\mu)\rangle - \langle\Psi, \Phi\rangle\langle\Psi, z(t,\mu)\rangle, \\ &= \langle\Psi, z(t,\mu)\rangle - \langle\Psi, z(t,\mu)\rangle = 0, \end{aligned} \tag{16}$$

where we have used a property from equation (9) and the fact that $\langle\Psi,\Phi\rangle = I$.

Now let $f(z,\mu) = f(z(t,\mu),\mu)$, and use equations (3), (11), and (13) to show

$$\begin{aligned} \frac{dw}{dt}(t,\mu) &= \frac{dz}{dt}(t,\mu) - \Phi\frac{dY}{dt}(t,\mu), \\ &= Az(t,\mu) + f(z,\mu) - \Phi BY(t,\mu) - \Phi\langle\Psi, f(z,\mu)\rangle, \end{aligned} \tag{17}$$

Substitute $z(t,\mu) = w(t,\mu) + \Phi Y(t,\mu)$ into equation (17) and use $A\Phi = \Phi B$ to get

$$\frac{dw}{dt}(t,\mu) = Aw(t,\mu) + f(z,\mu) - \Phi\langle\Psi, f(z,\mu)\rangle. \tag{18}$$

Therefore we have the final decomposition as

$$\begin{aligned} \frac{dy}{dt}(t,\mu) &= i\omega y(t,\mu) + F(t,\mu), \\ \frac{d\overline{y}}{dt}(t,\mu) &= -i\omega\overline{y}(t,\mu) + \overline{F}(t,\mu), \\ \frac{dw}{dt}(t,\mu) &= Aw(t,\mu) - \Phi\langle\Psi, f(z,\mu)\rangle + f(z,\mu). \end{aligned} \tag{19}$$

After defining some operators in Section 4, that will take the place of the matrices used here, we will see that there is an analogous decomposition for delay differential equations.

3 An Example Application in Ordinary Differential Equations

In this example we will consider a simple ordinary differential equation and work through the details of the decomposition described in this section. In Section 4 we will begin working out a more extensive example in delay differential equations and show how the decomposition in the time delay case is analogous to the decomposition in the ordinary differential equations case. In this section we will follow the decomposition steps give in Section 2.

Start with the equation

$$\ddot{x} + x = \mu x^2. \tag{20}$$

From the existence and uniqueness theorem we know there exists a unique solution $x(t,\mu)$, given the initial conditions $x(0,\mu) = x_0,\ \dot{x}(0,\mu) = x_1$, that is continuous with respect to μ.

Step 1: Form the Vector Equation

If we let $z_1 = x, z_2 = \dot{x}$ then equation (20) can be written in vector form as

$$\dot{z}(t,\mu) = Az(t,\mu) + f(z(t,\mu),\mu), \tag{21}$$

where $z(t,\mu) = (z_1(t,\mu), z_2(t,\mu))^T$, $f(z(t,\mu),\mu) = \mu\left(0, z_1(t,\mu)^2\right)^T$, and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{22}$$

The linear part of equation (21) is

$$\dot{z}(t,\mu) = Az(t,\mu). \tag{23}$$

Step 2: Define the Adjoint Equation

The adjoint equation of (23) is given by

$$\dot{y}(t,\mu) = -A^T z(t,\mu). \tag{24}$$

Step 3: Define a Natural Inner Product by way of an Adjoint

The inner product is developed, as in Section 2, as the natural inner product of vectors. We will go directly to forming the basis vectors.

A basis for the right eigenspace of A can easily be computed as

$$\phi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \quad \overline{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{-\sqrt{2}} \end{pmatrix}, \tag{25}$$

These are associated, respectively, with the eigenvalues $\lambda = i$ and $\lambda = -i$. The related eigenvalues and eigenvectors for A^T are $\lambda = -i$ and $\lambda = i$ with the respective eigenvectors $\overline{\phi},\ \ \phi$. The factor $1/\sqrt{2}$ is a normalization factor. Now define

$$\Phi = \left(\phi,\ \overline{\phi}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}, \tag{26}$$

and

$$\Psi = \left(\overline{\phi},\ \phi\right)^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}. \tag{27}$$

Then $\langle \Psi, \Phi \rangle = \Psi\Phi = I$.

Step 4: Get the Critical Eigenvalues

The eigenvalues of A are $\pm i$.

Step 5: Apply Orthogonal Decomposition

Let $z(t,\mu)$ be a unique family of solutions of equation(21), where we will write $z(t,\mu) = (z_1(t,\mu),\ z_2(t,\mu))^T$. Now define

$$Y(t,\mu) = \langle \Psi,\ z(t,\mu) \rangle = \begin{pmatrix} \frac{z_1(t,\mu)}{\sqrt{2}} - \frac{iz_2(t,\mu)}{\sqrt{2}} \\ \frac{z_1(t,\mu)}{\sqrt{2}} + \frac{iz_2(t,\mu)}{\sqrt{2}} \end{pmatrix}. \tag{28}$$

If we let

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B^* = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{29}$$

then

$$\frac{dY}{dt}(t,\mu) = BY(t,\mu) + \langle \Psi, f(z(t,\mu),\mu) \rangle, \tag{30}$$

or in an equivalent form

$$\begin{aligned} \frac{dy}{dt}(t,\mu) &= iy(t,\mu) + \left(\overline{\phi}, f(z(t,\mu),\mu)\right), \\ \frac{d\overline{y}}{dt}(t,\mu) &= -i\overline{y}(t,\mu) + \left(\phi, f(z(t,\mu),\mu)\right), \end{aligned} \tag{31}$$

where $f(z(t,\mu),\mu) = \left(0, z_1(t,\mu)^2\right)^T$.

We now develop the orthogonal function $w(t,\mu)$ as

$$w(t,\mu) = z(t,\mu) - \Phi Y(t,\mu). \tag{32}$$

However, if we form $\Phi Y(t,\mu)$ from equations (26) and (28) it is clear that $\Phi Y(t,\mu) = (z_1(t,\mu), z_2(t,\mu))^T = z(t,\mu)$ and therefore from equation (32) that $w(t,\mu) = 0$ as expected, since there are no other eigenvalues of A than i and $-i$. Thus equation (31) is the decomposed form of equation (21).

4 Delay Differential Equations as Operator Equations

As discussed earlier, the solutions to differential equations can be thought of in terms of trajectories or in terms of mappings of initial conditions. This same dichotomous point of view can be applied to delay differential equations. But, in the case of delay differential equations, the mapping or operator approach provides very fruitful qualitative results and, therefore, will remain that approach for this chapter.

Step 1: Form the Operator Equation

The principal difference between ordinary and delay differential equations is that, in ordinary differential equations, the initial condition space is finite dimensional and in delay differential equations it is infinite dimensional. The delay differential equations (1) and (2) can be thought of as maps of entire functions. In particular, we will start with the class of continuous functions defined on the interval $[-\sigma,\ 0]$ with values in R^n and refer to this as class C_0. The maps are constructed by defining a family of solution operators for the linear delay differential equation (2) by

$$\left(T(t)\phi\right)(\theta) = \left(z_t(\phi)\right)(\theta) = z(t+\theta) \tag{33}$$

for $\phi \in C_0,\ \theta \in [-\sigma, 0], s \geq 0$. This is a mapping of a function in C_0 to another function in C_0. Then equations (1) and (2) can be thought of as maps from C_0 to C_0. The norm on the space is taken as

$$\| \phi \| = \max_{-\sigma \leq t \leq 0} |\phi(t)|, \tag{34}$$

where $|\cdot|$ is the ordinary Euclidean 2-norm.

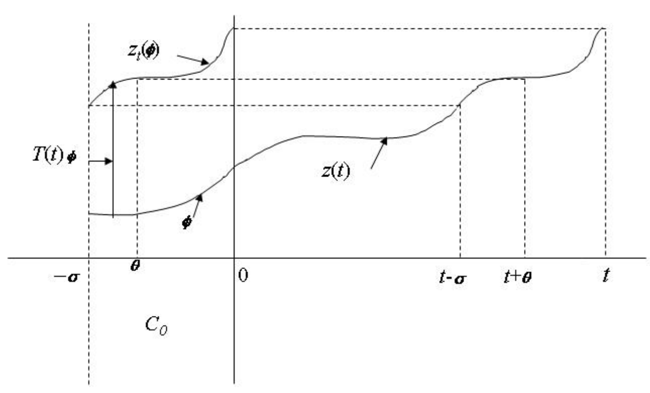

Fig. 2. The solution operator maps functions in C_0 to functions in C_0

Figure 2 shows the mapping between an initial function in C_0, ϕ, to another function, $z_t(\phi)$ in C_0. $z_t(\phi)$ is the projection of the portion of the trajectory $z(t)$ from $t-\sigma$ to t back to C_0. This figure exhibits two approaches to looking at solutions of delay differential equations. One way is to consider a solution as a trajectory of z as a function of t with an initial condition function in C_0. The value of the solution at t on the trajectory graph depends on the values of the function on the trajectory from $t-\sigma$ to t. Another way to look at the solutions of a delay differential equation is to consider them as parameterized mappings $T(t)\phi$ of functions ϕ in C_0. From Figure 2 this would be the portion of the trajectory from $t-\sigma$ to t projected back to C_0. In the figure it is represented as $z_t(\phi)$, which is the function in C_0 to which ϕ is mapped under $T(t)$. The mapped function relates to the trajectory as follows. The value of $z_t(\phi)(\theta)$ for $\theta \in [-\sigma, 0]$ is given as the trajectory value $z(t+\theta)$. This idea is not new, in that, in ordinary differential equations solutions can be thought of in terms of either trajectories or maps of initial condition vectors in R^n, say. Traditionally, we usually think of solving ordinary differential equations in terms of trajectories. For some qualitative analyses, however, the mapping approach is useful.

To determine what properties this operator must satisfy we look at what basic properties equation (5) satisfies. In particular, for each t, $T(t)$ is a bounded linear transformation for $\phi \in R^n$. The boundedness comes from $\|T(t)\phi\| \leq \|e^{At}\||\phi|$. For $t > 0$, $T(0)\phi = \phi$, i.e. $T(0) = I$. Finally,

$$\lim_{t \to t_0} \| T(t)\phi - T(t_0)\phi \| = 0, \tag{35}$$

since $\| T(t)\phi - T(t_0)\phi \| \leq \|e^{A(t-t_0)}\||\phi|$.

Based on these properties we formulate the following definition for a family of operators. A *Strongly Continuous Semigroup* satisfies

$$\begin{aligned} &T(t) \text{is bounded and linear for } t \geq 0, \\ &T(0)\phi = \phi \text{ or } T(0) = I, \\ &\lim_{t \to t_0} \| T(t)\phi - T(t_0)\phi \| = 0. \end{aligned} \tag{36}$$

where $\| \cdot \|$ is an appropriate operator norm and $\phi \in C_0$. The family of operators, $T(t)$, $t \geq 0$, is called a semigroup since the inverse property does not hold (see [18] and [35]).

If we take the derivative with respect to t in equation (5) we see that $T(t)\phi$ satisfies equation (4). It is also easy to see that

$$A = \lim_{t \to 0} \frac{1}{t} \left(e^{At} - I \right). \tag{37}$$

We can call the matrix A the infinitesimal generator of the family $T(t)$ in equation (5). The term infinitesimal generator can be thought of as arising from the formulation

$$dz = Az\,dt, \tag{38}$$

where for each infinitesimal increment, dt, in t, A produces an infinitesimal increment, dz, in z at the point z.

In terms of operators we define an operator called the infinitesimal generator. An *infinitesimal generator* of a semigroup $T(t)$ is defined by

$$A\phi = \lim_{t\to 0^+} \frac{1}{t}\left[T(t)\phi - \phi\right], \tag{39}$$

for $\phi \in C_0$.

We will state properties of the family of operators (33) without proofs, since the proofs require extensive knowledge of operator theory and they are not essential for the developments in this chapter. To begin with, the mapping (33) satisfies the semigroup properties on C_0. In the case of the linear system (2) the infinitesimal generator can be constructed as

$$(\mathrm{A}(\mu)\phi) = \begin{cases} \frac{d\phi}{d\theta}(\theta) & -\sigma \le \theta < 0, \\ U(\mu)\phi(0) + V(\mu)\phi(-\sigma) & \theta = 0, \end{cases} \tag{40}$$

where the parameter μ is included in the definition of A. Then $\mathrm{T(t)}\phi$ satisfies

$$\frac{d}{dt}\mathrm{T(t)}\phi = \mathrm{A}(\mu)\mathrm{T(t)}\phi, \tag{41}$$

where

$$\frac{d}{dt}\mathrm{T(t)}\phi = \lim_{\mathrm{h}\to 0} \frac{1}{\mathrm{h}}\left(\mathrm{T(t+h)} - \mathrm{T(t)}\right)\phi. \tag{42}$$

Finally, the operator form for the nonlinear delay differential equation (1) can be written as

$$\frac{d}{dt}z_t(\phi) = \mathrm{A}(\mu)z_t(\phi) + \mathrm{F}(z_t(\phi), \mu) \tag{43}$$

where

$$\left(\mathrm{F}(\phi,\mu)\right)(\theta) = \begin{cases} 0 & -\sigma \le \theta < 0, \\ f(\phi,\mu) & \theta = 0. \end{cases} \tag{44}$$

For $\mu = 0$, write $f(\phi) = f(\phi, 0)$, $F(\phi) = F(\phi, 0)$. We note the analogy between the ordinary differential equation notation and the operator notation for the delay differential equation.

Step 2: Define an Adjoint Operator

We can now construct a formal adjoint operator associated with equation (40). Let $C_0^* = C\left([0,\sigma], R^n\right)$ be the space of continuous functions from $[0,\sigma]$ to R^n with $\parallel \psi \parallel = \max_{0\le\theta\le\sigma} |\psi(\theta)|$ for $\psi \in C_0^*$. The formal adjoint equation associated with the linear delay differential equation (2) is given by

$$\frac{du}{dt}(t,\mu) = -U(\mu)^T u(t,\mu) - V(\mu)^T u(t+\sigma,\mu). \tag{45}$$

If we define

$$\left(T^*(t)\psi\right)(\theta) = \left(u_t(\psi)\right)(\theta) = u(t+\theta), \tag{46}$$

for $\theta \in [0,\sigma]$, $t \le 0$, and $u_t \in C_0^*$, $u_t(\psi)$ the image of $T^*(t)\psi$, then (46) defines a strongly continuous semigroup with infinitesimal generator

$$(\mathrm{A}^*(\mu)\psi) = \begin{cases} -\frac{d\psi}{d\theta}(\theta) & 0 < \theta \le \sigma, \\ -\frac{d\psi}{d\theta}(0) = U(\mu)^T\psi(0) + V(\mu)^T\psi(\sigma) & \theta = 0. \end{cases} \tag{47}$$

Note that, although the formal infinitesimal generator for (46) is defined as

$$A_0^*\psi = \lim_{t\to 0^-} \frac{1}{t}\left[T^*(t)\psi - \psi\right], \tag{48}$$

Hale [12], for convenience, takes $A^* = -A_0^*$ in (47) as the formal adjoint to (40). This family of operators (46) satisfies

$$\frac{d}{ds}\mathrm{T}^*(\mathrm{t})\psi = -\mathrm{A}^*\mathrm{T}^*(\mathrm{t})\psi. \tag{49}$$

Step 3: Define a Natural Inner Product by way of an Adjoint Operator

In contrast to R^n, the space C_0 does not have a natural inner product associated with its norm. However, following Hale [12], one can introduce a substitute device that acts like an inner product in C_0. This is an approach that is often taken when a function space does not have a natural inner product associated with its norm. Spaces of functions that have natural inner products are called Hilbert spaces. Throughout we will be assuming the complexification of the spaces so that we can work with complex eignvalues and eigenvectors.

In analogy to equation (7) we start by constructing a Lagrange identity as follows. If

$$\begin{aligned} \Theta z(t) &= z'(t) - U(\mu)z(t) - V(\mu)z(t-\sigma), \\ \Omega u(t) &= u'(t) + U(\mu)^T u(t) + V(\mu)^T u(t+\sigma), \end{aligned} \tag{50}$$

then

$$\overline{u}^T(t)\Theta z(t) + \overline{\Omega u}^T(t)z(t) = \frac{d}{dt}\langle u, z\rangle(t), \tag{51}$$

where

$$\langle u, z\rangle(t) = \overline{u}^T(t)z(t) + \int_{t-\sigma}^{t} \overline{u}^T(s+\sigma)V(\mu)z(s)ds. \tag{52}$$

Deriving the natural inner product for R^n from the Lagrange identity (7) motivates the derivation of equation (52). Again, if z and u satisfy $\Theta z(t) = 0$

and $\Omega u(t) = 0$ then, from equation (52), $(d/dt)\langle u, z\rangle(t) = 0$, which implies $\langle u, z\rangle(t)$ is constant and one can set $t = 0$ in equation (52) and define the form

$$\langle u, z\rangle = \overline{u}^T(0)z(0) + \int_{-\sigma}^{0} \overline{u}^T(s+\sigma)V(\mu)z(s)ds. \tag{53}$$

One can now state some properties of equations (40), (47), and (53) that are analogs of the properties given in Section 2 for ordinary differential equations.

1. For $\phi \in C_0$, $\psi \in C_0^*$,
$$\langle \psi, \mathrm{A}(\mu)\phi\rangle = \langle \mathrm{A}^*(\mu)\psi, \phi\rangle. \tag{54}$$
2. λ is an eigenvalue of $\mathrm{A}(\mu)$ if and only if $\overline{\lambda}$ is an eigenvalue of $\mathrm{A}^*(\mu)$.
3. The dimensions of the eigenspaces of $\mathrm{A}(\mu)$ and $\mathrm{A}^*(\mu)$ are finite and equal.
4. If $\psi_1, \cdots, \psi_d$ is a basis for the right eigenspace of $\mathrm{A}^*(\mu)$ and the associated $\phi_1, \cdots, \phi_d$ is a basis for the right eigenspace of $\mathrm{A}(\mu)$, construct the matrices $\Psi = (\psi_1, \cdots, \psi_d)$ and $\Phi = (\phi_1, \cdots, \phi_d)$. Define the bilinear form between Ψ and Φ by
$$\langle \Psi, \Phi\rangle = \begin{pmatrix} \langle \psi_1, \phi_1\rangle & \dots & \langle \psi_1, \phi_d\rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_d, \phi_1\rangle & \dots & \langle \psi_d, \phi_d\rangle \end{pmatrix}. \tag{55}$$

This matrix is nonsingular and can be chosen so that $\langle \Psi, \Phi\rangle = I$. Note that if (55) is not the identity then a change of coordinates can be performed by setting $K = \langle \Psi, \Phi\rangle^{-1}$ and $\Phi' = \Phi K$. Then $\langle \Psi, \Phi'\rangle = \langle \Psi, \Phi K\rangle = \langle \Psi, \Phi\rangle K = I$. (55) also satisfies the inner product properties (9).

Step 4: Get the Critical Eigenvalues

The eigenvalues for equation (40) are given by the λ solutions of the transcendental equation

$$\det\left(\lambda \mathrm{I} - \mathrm{U}(\mu) - e^{-\lambda\sigma}\mathrm{V}(\mu)\right) = 0. \tag{56}$$

This form of characteristic equation, sometimes called an *exponential polynomial*, has been studied in Avellar and Hale [4], Bellman and Cooke [7], Hale and Lunel [15], Kuang [23], and Pinney [27]. The solutions are called the *eigenvalues* of equation (2) and, in general, there are an infinite number of them. For a discussion of the general expansion of solutions of equation (2) in terms of the eigenvalues see Bellman and Cooke [7] or Pinney [27]. The actual computation of these eigenvalues can become very involved as the reader will see in the example that will be considered later in this chapter. Here, though, we will only be concerned with conditions for the existence of eigenvalues of the form $i\omega$ and $-i\omega$ and we further limit ourselves to the case in which there are only two eigenvalues $i\omega$ and $-i\omega$ and all other eigenvalues have negative real parts. The significance of this is that we will be looking for conditions for which the family of eigenvalues, as a function of the parameter μ, passes across the imaginary axis. These conditions will be the Hopf conditions referred to earlier in this chapter. The value of ω is related to the natural frequency of oscillation of the linear part of the delay differential equation system.

Step 5: Apply Orthogonal Decomposition

For the sake of notation, let $A = A(\mu)$, $A^* = A^*(\mu)$, $U = U(\mu)$, $V = V(\mu)$, $\omega = \omega(\mu)$. The basis eigenvectors for A and A^* associated with the eigenvectors $\lambda = i\omega$, $\overline{\lambda} = -i\omega$ will be denoted as ϕ_C, $\overline{\phi}_C$ and ϕ_D, $\overline{\phi}_D$, respectively, where the subscripts C and D refer to parameters defining the basis vectors and depend on U and V.

We define the matrix

$$\Phi = (\phi_C, \overline{\phi}_C). \tag{57}$$

The two eigenvectors for A, associated with the eigenvalues $\lambda = i\omega$, $\overline{\lambda} = -i\omega$, are given by

$$\begin{aligned} \phi_C(\theta) &= \mathrm{e}^{i\omega\theta} C, \\ \overline{\phi}_C(\theta) &= \mathrm{e}^{-i\omega\theta} \overline{C}, \end{aligned} \tag{58}$$

where C is a 2×1 vector. With these functions defined, it is clear that Φ is a function of θ and should formally be written as $\Phi(\theta)$. Note that $\Phi(0) = (C, \overline{C})$. However, in order to simplify notation we will write $\Phi = \Phi(\theta)$ but we will sometimes refer to $\Phi(0)$. These functions follow from equation (40). If $-\sigma \leq \theta < 0$ then $d\phi/d\theta = i\omega\phi$ implies $\phi(\theta) = \exp(i\omega\theta)C$ where $C = (c_1, c_2)^T$. For $\theta = 0$, equation (40) implies $(U + V\,\exp(-i\omega\sigma))\,C = i\omega C$ or $(U - i\omega I + V\,\exp(-i\omega\sigma))\,C = 0$. Since $i\omega$ is an eigenvalue, equation (56) implies that there is a nonzero solution C.

Similarly, the eigenvectors for A^* associated with the eigenvalues $-i\omega, i\omega$ are also given by

$$\begin{aligned} \phi_D(\theta) &= \mathrm{e}^{i\omega\theta} D, \\ \overline{\phi}_D(\theta) &= \mathrm{e}^{-i\omega\theta} \overline{D}, \end{aligned} \tag{59}$$

where $D = (d_1, d_2)^T$. Again, this follows from (47) since, from $0 < \theta \leq \sigma$,

$$-\frac{d\phi}{d\theta} = -i\omega\phi, \tag{60}$$

we can compute the solutions given in equation (59). Define the matrix

$$\Psi = (\phi_D, \overline{\phi}_D) \tag{61}$$

where D is computed as follows. At $\theta = 0$ we have from (47) that

$$\left(U^T + V^T \mathrm{e}^{i\omega\sigma} + i\omega I\right) D = 0. \tag{62}$$

The determinant of the matrix on the left is the characteristic equation so that there is a nonzero D.

From equations (57), (58), (59), and (61) one seeks to solve for D so that

$$\langle \Psi, \Phi \rangle = \begin{pmatrix} \langle \phi_D, \phi_C \rangle & \langle \phi_D, \overline{\phi}_C \rangle \\ \langle \overline{\phi}_D, \phi_C \rangle & \langle \overline{\phi}_D, \overline{\phi}_C \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{63}$$

Due to symmetry we only need to satisfy $\langle \phi_D, \phi_D \rangle = 1$ and $\langle \phi_D, \overline{\phi}_D \rangle = 0$.

On eigenspaces, the infinitesimal generators can be represented by matrices. In fact A and A^* satisfy $AQ = QB$, $A^*\Psi = \Psi B^*$ where Φ, Ψ are given by (57) and (61) and the matrices B, B^* are also given by (12).

Now that we have constructed the adjoint and given some of its properties we can decompose the nonlinear operator equation (43) into a two dimensional system with eigenvalues $i\omega$ and $-i\omega$ and another operator equation with eigenvalues having negative real parts. The procedure is based on Hale [12] and is similar to Step 5 of Section 2.

We will decompose the nonlinear system (43) for the case $\mu = 0$, since we will not need to develop approximations for $\mu \neq 0$ in this chapter. High order approximations have been developed in Hassard and Wan [16], but these will not be needed in order to develop the approximate bifurcating periodic solution studied here.

Based on standard existence and uniqueness theorems for delay differential equations, let $z_t \in C_0$ be the unique family of solutions of (43), where the μ notation has been dropped since we are only working with $\mu = 0$. Define

$$Y(t) = \langle \Psi,\ z_t \rangle = \begin{pmatrix} \langle \phi_D,\ z_t \rangle \\ \langle \overline{\phi}_D,\ z_t \rangle \end{pmatrix}, \tag{64}$$

where $Y(t) \in C^2$ for $t \geq 0$, and set $Y(t) = (y(t),\ \overline{y}(t))^T$ where $y(t) = \langle \phi_D,\ z_t \rangle$ and $\overline{y}(t) = \langle \overline{\phi}_D,\ z_t \rangle$. The reader should note the similarity to equation (11).

By differentiating (53), and using $z_t(0) = z(t),\ z_t(\theta) = z(t + \theta)$, we have

$$\frac{d}{dt}\langle \Psi,\ z_t \rangle = \langle \Psi,\ \frac{dz_t}{dt} \rangle. \tag{65}$$

Use (9), (43), (54), (65), and $A^*\Psi = \Psi B^*$ to write

$$\frac{d}{dt} Y(t) = BY(t) + \langle \Psi,\ F(z_t) \rangle. \tag{66}$$

Using (44) and (53), compute $\langle \phi_D,\ F(z_t) \rangle = \overline{\phi}_D^{\,T}(0)(F(z_t))(0) = \overline{D}^T f(z_t)$. Similarly $\langle \overline{\phi}_D,\ F(z_t) \rangle = D^T f(z_t)$. Then

$$\langle \Psi,\ F(z_t) \rangle = \begin{pmatrix} \langle \phi_D,\ F(z_t) \rangle \\ \langle \overline{\phi}_D,\ F(z_t) \rangle \end{pmatrix} = \begin{pmatrix} \overline{D}^T f(z_t) \\ D^T f(z_t) \end{pmatrix}, \tag{67}$$

which yields the first two equations in (83) below. They can be written as

$$\begin{aligned} \frac{d}{dt}y(t) &= i\omega y(t) + \overline{D}^T f(z_t), \\ \frac{d}{dt}\overline{y}(t) &= -i\omega \overline{y}(t) + D^T f(z_t). \end{aligned} \tag{68}$$

If one defines the orthogonal family of functions

$$w_t = z_t - \Phi Y(t), \tag{69}$$

where Q is given by (57) and $Y(t)$ is given by (64), then $\langle \Psi,\ w_t\rangle = 0$, where Ψ is given by (61). Now apply the infinitesimal generator (40) to

$$z_t = w_t + \Phi Y(t), \tag{70}$$

to get

$$(Az_t)(\theta) = (Aw_t)(\theta) + (A\Phi)(\theta)Y(t) = (Aw_t)(\theta) + \Phi BY(t). \tag{71}$$

We will need this relation below.

One can now construct the third equation. There are two cases: $\theta = 0$ and $\theta \in [-\sigma, 0)$. From (69), for the case with $\theta = 0$,

$$w(t) = w_t(0) = z_t(0) - \Phi(0)Y(t) = z(t) - \Phi(0)Y(t). \tag{72}$$

The reader is reminded here of the notation $w_t(\theta) = w(t+\theta)$. It is easy to show that $w(t) \in R^2$, since $x(t) \in R^2$ and $\Phi(0)Y(t) = \langle \phi_D, z_t\rangle C + \langle \overline{\phi}_D, z_t\rangle \overline{C} = 2\mathrm{Re}\left\{\langle \phi_D, z_t\rangle C\right\} \in R^2$.

From (40) and (43)

$$\frac{d}{dt}z(t) = \frac{dz_t}{dt}(0) = (Az_t)(0) + (F(z_t))(0) = Uz(t) + Vz(t-\sigma) + f(z_t). \tag{73}$$

Differentiate (72) and combine it with (66) and (73) to give

$$\frac{d}{dt}w(t) = \left\{Uz(t) + Vz(t-\sigma) + f(z_t)\right\} - \Phi(0)\left\{BY(t) - \langle \Psi, F(z_t)\rangle\right\}. \tag{74}$$

If $\theta = 0$ in (71) then, from (40),

$$Uz(t) + Vz(t-\sigma) = Uw(t) + Vw(t-\sigma) + \Phi(0)BY(t). \tag{75}$$

Now substitute (75) into (74) to get

$$\begin{aligned} \frac{d}{dt}w(t) &= Uw(t) + Vw(t-\sigma) + f(z_t) - \Phi(0)\langle \Psi, F(z_t)\rangle, \\ &= Uw(t) + Vw(t-\sigma) + f(z_t) - 2\mathrm{Re}\left\{\langle \phi_D, z_t\rangle C\right\}. \end{aligned} \tag{76}$$

For the case with $\theta \neq 0$ we can apply a similar argument to that used to create (76). We start by differentiating (69) to get

$$\begin{aligned}\frac{dw_t}{dt} &= \frac{dz_t}{dt} - \Phi Y'(t),\\ &= \frac{dz_t}{dt} - \Phi\{BY(t) + \langle\Psi, F(z_t)\rangle\},\\ &= \frac{dz_t}{dt} - \Phi BY(t) - \Phi\langle\Psi, F(z_t)\rangle\}.\end{aligned} \tag{77}$$

For $\theta \neq 0$ in (43) and (44)

$$\frac{dz_t}{dt} = Az_t. \tag{78}$$

Then, using (71) and (78), we have

$$\begin{aligned}\frac{dw_t}{dt} &= Az_t - \Phi BY(t) - \Phi\langle\Psi, F(z_t)\rangle\},\\ &= Aw_t + \Phi BY(t) - \Phi BY(t) - \Phi\langle\Psi, F(z_t)\rangle\}, && (79)\\ &= Aw_t - \Phi\langle\Psi, F(z_t)\rangle\}. && (80)\end{aligned}$$

This can then be written as

$$\frac{dw_t}{ds} = Aw_t - 2Re\{\langle\phi_D, z_t\rangle\phi_C\}. \tag{81}$$

Use (44) to finally write the equation

$$\frac{dw_t}{dt} = Aw_t - 2Re\{\langle\phi_D, z_t\rangle\phi_C\} + F(z_t). \tag{82}$$

Equation (43) has now been decomposed as

$$\begin{aligned}\frac{d}{dt}y(t) &= i\omega y(t) + \overline{D}^T f(z_t),\\ \frac{d}{dt}\overline{y}(t) &= -i\omega\overline{y}(t) + D^T f(z_t),\\ \frac{d}{dt}w_t(\theta) &= \begin{cases}(Aw_t)(\theta) - 2Re\{\langle\phi_D, z_t\rangle\phi_C(\theta)\} & -\sigma \leq \theta < 0,\\ (Aw_t)(0) - 2Re\{\langle\phi_D, z_t\rangle\phi_C(0)\} + f(z_t) & \theta = 0.\end{cases}\end{aligned} \tag{83}$$

In order to simplify the notation write (83) in the form

$$\begin{aligned}\frac{dy}{dt}(t) &= i\omega y(t) + F_1(Y,\ w_t),\\ \frac{d\overline{y}}{dt}(t) &= -i\omega\overline{y}(t) + \overline{F}_1(Y,\ w_t),\\ \frac{dw_t}{dt} &= Aw_t + F_2(Y,\ w_t),\end{aligned} \tag{84}$$

where

$$\begin{aligned}F_1(Y,\ w_t) &= \overline{D}^T f(z_t),\\ F_2(Y,\ w_t) &= \begin{cases}-2Re\{\langle\phi_D, z_t\rangle\phi_C(\theta)\} & -\sigma \leq \theta < 0,\\ -2Re\{\langle\phi_D, z_t\rangle\phi_C(0)\} + f(z_t) & \theta = 0.\end{cases}\end{aligned} \tag{85}$$

Note that (84) is a coupled system. The center manifold and normal forms will be used as a tool to partially decouple this system.

5 A Machine Tool DDE Example: Part 1

This example will be discussed in multiple parts. In the first part we will formulate the operator form for the example delay differential equation, describe a process of determining the critical eigenvalues for the problem, and formulate the adjoint operator equation.

Step 1: Form the Operator Equation

The example we will consider involves a turning center and workpiece combination. For readers unfamiliar with turning centers they can be thought of as numerically controlled lathes. The machining tool model, used only for illustration in this paper, is taken from Kalmár-Nagy et al. [20] and can be written as

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{kf_0}{m\alpha}\left(1 - \left(\frac{f}{f_0}\right)^{\alpha}\right), \tag{86}$$

where $\omega_n = \sqrt{r/m}$ is the natural frequency of the undamped free oscillating system and $\xi = c/2m\omega_n$ is the relative damping factor and k is the cutting force coefficient that is related to the slope of the power-law curve used to define the right hand side of (86). The parameters m, r, c and α are taken as m = 10 kg, r = 3.35 MN/m, c = 156 kg/s, and $\alpha = 0.41$ and were obtained from measurements of the machine-tool response function (see Kalmár-Nagy et al. [20]). The parameter α was obtained from a cutting force model in Taylor [31]. These then imply that $\omega_n = 578.791/s$, $\xi = 0.0135$. The nominal chip width is taken as f_0 and the time varyin chip width is

$$f = f_0 + x(t) - x(t - \tau), \tag{87}$$

where the delay $\tau = 2\pi/\Omega_\tau$ is the time for one revolution of the turning center spindle. The cutting force parameter k will be taken as the bifurcation parameter since we will be interested in the qualitative change in the displacement, $x(t)$, as the cutting force changes. The parameters m, r, c and f are shown in Figure 3. The displacement $x(t)$ is directed positively into the workpiece and the tool is assumed not to leave the workpiece.

The model is simplified by introducing a nondimensional time s and displacement z by

$$\begin{aligned} s &= \omega_n t, \\ z &= \frac{x}{A} \end{aligned} \tag{88}$$

where the length scale is computed as

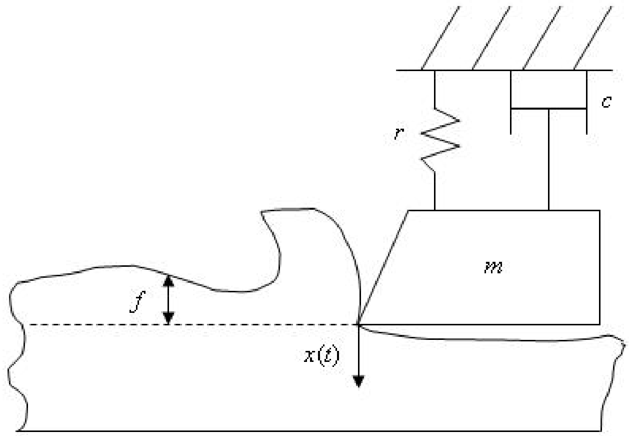

Fig. 3. One Degree-of-Freedom-Model for Single-Point Turning

$$A = \frac{3f_0}{2-\alpha}, \tag{89}$$

a new bifurcation parameter p is set to

$$p = \frac{k}{m\omega_n^2}, \tag{90}$$

and the delay parameter becomes

$$\sigma = \omega_n \tau. \tag{91}$$

The dimensionless model then becomes, after expanding the right hand side of (86) to the third order,

$$\frac{d^2x}{ds^2} + 2\xi\frac{dx}{ds} + x = p\left(\Delta x + E(\Delta x^2 + \Delta x^3)\right), \tag{92}$$

where

$$\begin{aligned} \Delta x &= x(s-\sigma) - x(s), \\ E &= \frac{3(1-\alpha)}{2(2-\alpha)}. \end{aligned} \tag{93}$$

We will now consider x as a function of the dimensionless s instead of t. The linear part of the model is given by

$$\frac{d^2x}{ds^2} + 2\xi\frac{dx}{ds} + x = p\Delta x, \tag{94}$$

Since the Hopf bifurcation studied in this paper is local, the bifurcation parameter will be written as

$$p = \mu + p_c, \tag{95}$$

where p_c is a critical value at which bifurcation occurs. Then (92) can be put into vector form (1) by letting $z_1(s) = x(s)$, $z_2(s) = x'(s)$. Then

$$\frac{dz}{ds}(s) = U(\mu)z(s) + V(\mu)z(s-\sigma) + f(z(s), z(s-\sigma), \mu), \tag{96}$$

where

$$\begin{aligned} z(s) &= \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix}, \\ U(\mu) &= \begin{pmatrix} 0 & 1 \\ -1-(\mu+p_c) & -2\xi \end{pmatrix}, \\ V(\mu) &= \begin{pmatrix} 0 & 0 \\ \mu+p_c & 0 \end{pmatrix}, \end{aligned} \tag{97}$$

and

$$f(z(s)\ , z(s-\sigma), \mu) = \begin{pmatrix} 0 \\ (\mu+p_c)E\left(z_1(s-\sigma)-z_1(s)\right)^2 + (\mu+p_c)E\left(z_1(s-\sigma)-z_1(s)\right)^3 \end{pmatrix}. \tag{98}$$

The linear portion of this equation is given by

$$\frac{dz}{ds}(s) = U(\mu)z(s) + V(\mu)z(s-\sigma). \tag{99}$$

and the infinitesimal generator is given by

$$(\mathrm{A}(\mu)\phi) = \begin{cases} \frac{d\phi}{d\theta}(\theta) & -\sigma \le \theta < 0 \\ U(\mu)\phi(0) + V(\mu)\phi(-\sigma) & \theta = 0 \end{cases}. \tag{100}$$

Then (96) can easily be put into the operator form (43).

Step 2: Define the Adjoint Operator

Here we can follow the lead of Step 2 of Section 4 and define the formal adjoint as

$$\frac{dz}{ds}(s,\mu) = -U(\mu)^T z(s,\mu) - V(\mu)^T z(s+\sigma,\mu). \tag{101}$$

As in Section 4, if we define

$$\left(T^*(s)\psi\right)(\theta) = \left(z_s(\psi)\right)(\theta) = z(s+\theta) \tag{102}$$

for $\theta \in [0,\sigma]$, $s \le 0$, $u_s \in C_0^*$, and $z_s(\psi)$ as the image of $T^*(s)\psi$, then (102) defines a strongly continuous semigroup with infinitesimal generator

$$(\mathrm{A}^*(\mu)\psi) = \begin{cases} -\frac{d\psi}{d\theta}(\theta) & 0 < \theta \leq \sigma, \\ -\frac{d\psi}{d\theta}(0) = U(\mu)^T\psi(0) + V(\mu)^T\psi(\sigma) & \theta = 0. \end{cases} \tag{103}$$

Note that, although, as before, the formal infinitesimal generator for (102) is defined as

$$A_0^*\psi = \lim_{s \to 0^-} \frac{1}{s}\left[T^*(s)\psi - \psi\right]. \tag{104}$$

Hale [12], for convenience, takes $A^* = -A_0^*$ in (103) as the formal adjoint to (100).

Step 3: Define a Natural Inner Product by way of an Adjoint

This step follows simply by defining the inner product in the same manner as in equation (53).

Step 4: Get the Critical Eigenvalues

In this step the reader will begin to see some of the complexity of dealing with the transcendental characteristic equation. The eigenvalues will depend on the parameters in equation (94) and only certain parameter combinations will lead to eigenvalues of the form $i\omega$ and $-i\omega$. We will also establish the connection of the critical eigenvalues with the Hopf bifurcation conditions.

Following Hale [14], introduce the trial solution

$$z(s) = ce^{\lambda s}, \tag{105}$$

where $c \in C^2$, and $U(\mu)$, and $V(\mu)$ are given by (97), into the linear system (94) and set the determinant of the resulting system to zero. This yields the transcendental characteristic equation

$$\chi(\lambda) = \lambda^2 + 2\xi\lambda + (1+p) - pe^{-\lambda\sigma} = 0. \tag{106}$$

Before developing the families of conjugate eigenvalues, we wish to characterize certain critical eigenvalues of (106) of the form $\lambda = i\omega$. However, the eigenvalues for (106) of the form $\lambda = i\omega$ exist only for special combinations of p and σ. We will say that a triple $(\omega,\ \sigma,\ p)$, where $\omega,\ \sigma,\ p$ are real, will be called a *critical eigen triple* of (106) if $\lambda = i\omega,\ \sigma,\ p$ simultaneously satisfy (106). The discussion below points out the significant computational difficulties involved with estimating the eigenvalues for a characteristic equation or exponential polynomial related to a linear delay differential equation.

The following properties characterize the critical eigen triples for linear delay equations of the form (2) with coefficients from (97).

1. $(\omega,\ \sigma,\ p)$ is a critical eigen triple of (106) if and only if $(-\omega,\ \sigma,\ p)$ also is a critical eigen triple of (106).

2. For $\omega > 1$ there is a uniquely defined sequence $\sigma_r = \sigma_r(\omega),\ r = 0,\ 1,\ 2, \ldots$, and a uniquely defined $p = p(\omega)$ such that $(\omega,\ \sigma_r,\ p),\ r = 0,\ 1,\ 2, \ldots$, are critical eigen triples.
3. If $(\omega,\ \sigma,\ p)$ is a critical eigen triple, with $\omega > 1$, then $p \geq 2\xi(1+\xi)$. That is, no critical eigen triple for (106) exists for $p < 2\xi(1+\xi)$.
4. For
$$p_m = 2\xi(1+\xi), \tag{107}$$
the minimum p value, there is a unique $\omega > 1$ and a unique sequence $\sigma_r,\ r = 0,\ 1,\ 2, \ldots$, such that $(\omega_m,\ \sigma_r,\ p_m)$ is a critical eigen triple for (106) for $r = 0,\ 1,\ 2, \ldots$. The frequency at the minimum is
$$\omega_m = \sqrt{1+2\xi}. \tag{108}$$
5. For $p > 2\xi(1+\xi)$ there exist two ω's, $\omega > 1$, designated $\omega_+,\ \omega_-$ and uniquely associated sequences $\sigma_r^+ = \sigma_r(\omega_+),\ \sigma_r^- = \sigma_r(\omega_-),\ r = 0,\ 1,\ 2, \ldots$ such that $(\omega_+,\ \sigma_r^+,\ p),\ (\omega_-,\ \sigma_r^-,\ p)$ are critical eigen triples for (106) for $r = 0,\ 1,\ 2, \ldots$. $\omega_+,\ \omega_-$ are given by
$$\omega_+^2 = (1+p-2\xi^2) + \sqrt{p^2 - 4\xi^2 p + (4\xi^4 - 4\xi^2)}, \tag{109}$$
$$\omega_-^2 = (1+p-2\xi^2) - \sqrt{p^2 - 4\xi^2 p + (4\xi^4 - 4\xi^2)}. \tag{110}$$
$\sigma_r^+,\ \sigma_r^-$ are given by
$$\sigma_r^+ = \frac{2(\psi_+ + r\pi) + 3\pi}{\omega_+} \tag{111}$$
$$\sigma_r^- = \frac{2(\psi_- + r\pi) + 3\pi}{\omega_-} \tag{112}$$
where
$$\psi_+ = -\pi + \tan^{-1}\left(\frac{2\xi\omega_+}{\omega_+^2 - 1}\right), \tag{113}$$
$$\psi_- = -\pi + \tan^{-1}\left(\frac{2\xi\omega_-}{\omega_-^2 - 1}\right), \tag{114}$$
6. There do not exist critical eigen triples for $0 \leq \omega \leq 1$

We will not prove these results (for proofs see Gilsinn [11]) but we briefly discuss their significance graphically by examining Figure 4 where the plots are based on the value of $\xi = 0.0135$. The entire development of the periodic solutions on the center manifold depends on knowing the critical bifurcation parameter p in (92). This parameter is linked to the rotation rate, Ω_r, of the turning center spindle. One can plot p against $\Omega_r = 1/\sigma_r$, where Ω_r is the rotation rate of the turning center spindle, for $r = 0,\ 1,\ 2, \ldots$ where each r indexes a lobe in Figure 4 moving from right to left in the figure. Call the right most lobe, lobe 0, the next on the left lobe 1, etc. For each r the pairs (Ω_r, p)

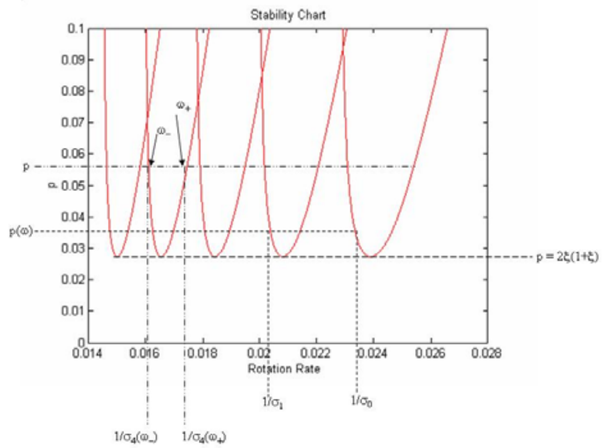

Fig. 4. Stability Chart with Sample Critical Eigen Triples Identified

are computed for a vector of ω values. When these families of pairs are plotted they form a family of N lobes. Each lobe is parameterized by the same vector of ω's so that each point on a lobe boundary represents an eigenvalue of (106) for a given p and $\sigma_r = 1/\Omega_r$. The minimum of each lobe is asymptotic to a line often called the *stability limit*. Te second property above states that for a given ω there is associated a unique value on the vertical axis, called $p(\omega)$, but an infinite number of $\sigma_r(\omega)$'s, one for each lobe, depicted graphically on the horizontal axis as $1/\omega_r$, for rotation rate. The minimum value on each lobe occurs at $p = 2\xi(1+\xi)$ with an associated unique $\omega = \sqrt{1+2\xi}$. Finally, for each lobe there are two ω's associated with each p, denoted by ω_- and ω_+, where ω_- is the parameter associated with the left side of the lobe and ω_+ with the right side of the lobe. At the minimum $\omega_- = \omega_+$.

The significance of the stability chart is that the lobe boundaries divide the plane into regions of stable and unstable machining. In particular, the regions below the lobes are considered stable and those above are considered unstable in the manufacturing sense. This will be a result of the Hopf bifurcation at the lobe boundaries. Since the parameter p is proportional to material removal, the regions between lobes represent areas that can be exploited for material removal above the stability limit line. This figure, called a *stability chart*, graphically shows the meanings of properties two through five above and was introduced by Tobias and Fishwick [33]. The structure of stability charts for delay differential equation can be very complex. The current machine tool example exhibits one of the simpler ones. To see some examples of different stability charts the reader is referred to the book by Stépán [28].

We will use an argument modeled after Altintas and Budak [1] to develop necessary conditions for σ_r and p and show how they relate to ω. These conditions are in fact used to graphically display the stability lobes. Set

$$\Phi(\lambda) = \frac{1}{\lambda^2 + 2\xi\lambda + 1} \tag{115}$$

Then (106) becomes

$$1 + p(1 - e^{-\lambda\sigma})\Phi(\lambda) = 0. \tag{116}$$

Set $\lambda = i\omega$ and write

$$\Phi(i\omega) = G(\omega) + iH(\omega), \tag{117}$$

where

$$G(\omega) = \frac{1-\omega^2}{(1-\omega^2)^2 + (2\xi\omega)^2}, \tag{118}$$

$$H(\omega) = \frac{-2\xi\omega}{(1-\omega^2)^2 + (2\xi\omega)^2}. \tag{119}$$

Substitute (117) into (116) and separate real and imaginary parts to get

$$1 + p[(1 - \cos\omega\sigma)G(\omega) - (\sin\omega\sigma)H(\omega)] = 0, \tag{120}$$

$$p[G(\omega)\sin\omega\sigma + H(\omega)(1 - \cos\omega\sigma)] = 0. \tag{121}$$

From (118) and (119)

$$\frac{H(\omega)}{G(\omega)} = -\frac{\sin\omega\sigma}{1 - \cos\omega\sigma}. \tag{122}$$

From the definition of $G,\ H$ and the fact that $\omega > 1$, (122) falls in the third quadrant so that one can introduce the phase angle for (117), using (118) and (119), as

$$\psi = \tan^{-1}\left(\frac{H(\omega)}{G(\omega)}\right) = -\pi + \tan^{-1}\left(\frac{2\xi\omega}{\omega^2 - 1}\right). \tag{123}$$

Clearly, $-\pi \le \psi \le \pi$. Using half-angle formulas,

$$\begin{aligned}
\tan\psi &= -\frac{\sin\omega\sigma}{1 - \cos\omega\sigma}, \\
&= -\frac{\cos\left(\frac{\omega\sigma}{2}\right)}{\sin\left(\frac{\omega\sigma}{2}\right)}, \\
&= -\cot\left(\frac{\omega\sigma}{2}\right), \\
&= \tan\left(\frac{\pi}{2} + \frac{\omega\sigma}{2} \pm n\pi\right),
\end{aligned} \tag{124}$$

for $n = 0, 1, 2, \ldots$. Therefore

$$\psi = \frac{\pi}{2} + \frac{\omega\sigma}{2} \pm n\pi, \tag{125}$$

where $\omega\sigma > 0$ must be satisfied for all n. In order to satisfy this and the condition that $-\pi \le \psi \le \pi$, select the negative sign and

$$n = 2 + r, \tag{126}$$

for $r = 0, 1, 2, \ldots$. Therefore, from (125), the necessary sequence, σ_r, is given by

$$\sigma_r = \frac{\omega}{2(\psi + r\pi) + 3\pi}, \tag{127}$$

where ψ is given by (123). Finally, substituting (122) into (120), one has the necessary condition for p as

$$p = -\frac{1}{2G(\omega)}, \tag{128}$$

where, $p > 0$ since $\omega > 1$. Therefore (127) and (128) are the necessary conditions for $(\omega,\ \sigma_r,\ p),\ r = 0,\ 1,\ 2, \ldots$, to be critical eigen triples for (106). Note that this also implies uniqueness. Equations (127) and (128) show how $p = p(\omega)$ and $1/\sigma_r$ uniquely relate in Figure 4.

The lobes in Figure 4 are plotted by the following algorithm. Since p must be positive, select any set of values $\omega > 1$ such that $G(\omega) < 0$. Given a set of $\omega > 1$ values, pick $r = 0, 1, 2, \cdots$ for as many lobes as desired. Compute $\frac{1}{\sigma_r}$ from (127) and p from (128), then plot the pairs $\left(\frac{1}{\sigma_r},\ p\right)$.

The following is a consequence of properties one through six for critical eigen triples. If $(\omega_0,\ \sigma,\ p)$ is a critical eigen triple, $\omega_0 > 0$, then there cannot be another critical eigen triple $(\omega_1,\ \sigma,\ p),\ \omega_1 > 0,\ \omega_1 \neq \omega_0$. Furthermore, since $(-\omega_0,\ \sigma,\ p)$ is also a critical eigen triple, there can be no critical eigen triple $(\omega_2,\ \sigma,\ p),\ \omega_2 < 0,\ \omega_2 \neq -\omega_0$. This does not preclude two or more lobes crossing. It only refers to a fixed lobe.

Finally we can state the Hopf criteria. That is, there is a family of simple, conjugate eigenvalues $\lambda(\mu),\ \bar{\lambda}(\mu)$, of (106), such that

$$\lambda(\mu) = \alpha(\mu) + i\omega_0(\mu), \tag{129}$$

where $\alpha,\ \omega_0$ are real, $\omega_0(\mu) = \omega(\mu) + \omega_c$ where $\omega(\mu)$ is a perturbation of a critical frequency ω_c, and

$$\begin{aligned} \alpha(0) &= 0, \\ \omega_0(0) &> 0, \\ \alpha'(0) &> 0, \end{aligned} \tag{130}$$

The proof of this result depends on the Implicit Function Theorem and is given in Gilsinn [11]. As a consequence of the Implicit Function Theorem the conditions $\alpha(0) = 0$ and $\omega(0) = 0$ and thus $\omega_0(0) > 0$ follow. The last condition, that $\alpha'(0) > 0$, follows from following relations, valid for the current machine tool model, that are also shown in Gilsinn [11]

$$\begin{aligned} \alpha'(0) &= \frac{[2\xi - \sigma\omega_c^2 + \sigma(1 + p_c)][1 - \omega_c^2] + [2\omega_c(1 + \sigma\xi)][2\xi\omega_c]}{p_c[2\xi - \sigma\omega_c^2 + \sigma(1 + p_c)]^2 + p_c[2\omega_c(1 + \sigma\xi)]^2}, \\ \omega'(0) &= \frac{[2\xi - \sigma\omega_c^2 + \sigma(1 + p_c)][2\xi\omega_c] - [1 - \omega_c^2][2\omega_c(1 + \sigma\xi)]}{p_c[2\xi - \sigma\omega_c^2 + \sigma(1 + p_c)]^2 + p_c[2\omega_c(1 + \sigma\xi)]^2}, \end{aligned} \tag{131}$$

The numerator of $\alpha'(0)$, divided by p_c, can be expanded to give

$$2\xi(1+\omega_c^2)+\sigma(1-\omega_c^2)^2+4\sigma\omega_c^2\xi^2+\sigma p_c(1-\omega_c^2) \tag{132}$$

The only term in (132) that can potentially cause (132) to become negative is the last one. However, p_c and ω_c are related by (128) which implies that

$$p_c=\frac{(1-\omega_c^2)^2+(2\xi\omega_c)^2}{2(\omega_c^2-1)} \tag{133}$$

If we substitute (133) into (132) one gets

$$2\xi(1+\omega_c^2)+\frac{\sigma}{2}(1-\omega_c^2)^2+2\sigma\omega_c^2\xi^2 \tag{134}$$

which is clearly positive so that $\alpha'(0)>0$. To compute the bifurcating periodic solutions and determine their periods later we will need to use both $\alpha'(0)$ and $\omega'(0)$. Finally, the last Hopf condition is also shown in Gilsinn [11] and that is that all other eigenvalues than the two critical ones have negative real parts. The proof of this part of the Hopf result involves a contour integration.

We are now in a position to determine the nature of the Hopf bifurcation that occurs at a critical eigen triple for the machine tool model. To simplify the calculations only the bifurcation at the minimum points of the lobes, p_m and ω_m, given by (107) and (108), will be examined. Any other point on a lobe would involve more complicated expressions for any p greater than p_m and obscure the essential arguments. We will see that a bifurcation, called a *subcritical bifurcation* occurs at this point, which implies that, depending on the initial amplitude used to integrate (92), the solution can become unstable in a region that otherwise might be considered a stable region.

The rotation rate, Ω_m, at p_m can be computed from (108) and (123) as

$$\begin{aligned}\psi_m&=-\pi+\tan^{-1}\left(\sqrt{1+2\xi}\right),\\ \Omega_m&=\frac{1}{\sigma_m}=\frac{\omega_m}{2(\psi_m+r\pi)+3\pi},\end{aligned} \tag{135}$$

for $r=0,1,2,\cdots$. When $\xi=0.0135$, as computed for the current machining model, one has that $\psi_m=-2.3495$. When $r=0$, $\Omega_m=0.2144$ ($\sigma_m=4.6642$), which is the dimensionless rotation rate at the minimum of the first lobe to the right in Figure 4. This point is selected purely in order to illustrate the calculations. The stability limit in Figure 4 is given by (107) as

$$p_m=2\xi(\xi+1)=0.027365 \tag{136}$$

The frequency at this limit is given by (108) as

$$\omega_m=\sqrt{1+2\xi}=1.01341 \tag{137}$$

Then from (131)

$$\alpha'(0) = \frac{1}{2(1+\xi)^2(1+\xi\sigma_m)},$$
$$\omega'(0) = \frac{\sqrt{1+2\xi}}{2(1+\xi)^2(1+\xi\sigma_m)}. \tag{138}$$

Step 5: Apply Orthogonal Decomposition

We will now follow the steps needed to compute the bifurcating periodic solutions on the center manifold. The first step is to compute the eigenvectors for the infinitesimal generators A and A^*. The general forms for these eigenvectors are given by (58) and (59). We wish to compute the constant vectors C and D. To compute C we note that for $\theta = 0$, (40) implies $(U + V\,\exp(-i\omega\sigma))\,C = i\omega C$ or $(U - i\omega I + V\,\exp(-i\omega\sigma))\,C = 0$. Since $i\omega$ is an eigenvalue, (56), (97), and (106) imply that there is a nonzero solution C. If we set $c_1 = 1$ it is easy to compute $c_2 = i\omega$. The eigenvectors of A are then given by

$$\phi_C(\theta) = \mathrm{e}^{i\omega\theta}\begin{pmatrix}1\\ i\omega\end{pmatrix},$$
$$\overline{\phi}_C(\theta) = \mathrm{e}^{-i\omega\theta}\begin{pmatrix}1\\ -i\omega\end{pmatrix}. \tag{139}$$

To compute D we have, at $\theta = 0$, from (47), that $\left(\mathrm{U}^T + \mathrm{V}^T\mathrm{e}^{i\omega\sigma} + i\omega I\right)D = 0$. The determinant of the matrix on the left is the characteristic equation so that there is a nonzero D. From (58), (59), (10), and (140) one seeks to solve for D so that

$$\langle \Psi, \Phi\rangle = \begin{pmatrix}\langle\phi_D,\phi_C\rangle & \langle\phi_D,\overline{\phi}_C\rangle\\ \langle\overline{\phi}_D,\phi_C\rangle & \langle\overline{\phi}_D,\overline{\phi}_C\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{140}$$

Due to symmetry one only needs to satisfy $\langle\phi_D,\phi_C\rangle = 1$ and $\langle\phi_D,\overline{\phi}_C\rangle = 0$. From (53), (58), and (59) compute d_1 and d_2 to satisfy

$$1 = \overline{d}_1 + \left[\sigma p_c\cos\omega\sigma + i\left(\omega - \sigma p_c\sin\omega\sigma\right)\right]\overline{d}_2,$$
$$0 = \overline{d}_1 + \left[\frac{p_c}{\omega}\sin\omega\sigma - i\omega\right]\overline{d}_2, \tag{141}$$

from which the eigenvectors of A^* associated with the eigenvalues $-i\omega,\ i\omega$ can be computed as

$$\phi_D(\theta) = \mathrm{e}^{i\omega\theta}D,$$
$$\overline{\phi}_D(\theta) = \mathrm{e}^{-i\omega\theta}\overline{D}, \tag{142}$$

where $D = (d_1, d_2)^T$ and

$$d_1 = -\left(\frac{p_c}{\omega}\sin\omega\sigma + i\omega\right) d_2,$$
$$d_2 = \frac{\left(\sigma p_c\omega^2\cos\omega\sigma - p_c\omega\sin\omega\sigma\right) + i\left(2\omega^3 - \sigma p_c\omega^2\sin\omega\sigma\right)}{\left(\sigma p_c\omega\cos\omega\sigma - p_c\sin\omega\sigma\right)^2 + \left(2\omega^2 - \sigma p_c\omega\sin\omega\sigma\right)^2}, \quad (143)$$

From (143) the value of $\overline{d}_2$ can be calculated as

$$\overline{d}_2 = \frac{-\xi - i\sqrt{1+2\xi}}{2(1+\xi\sigma_c)(1+\xi)^2}, \quad (144)$$

where $\sigma = \sigma_m,\ p_c = p_m,\ \omega = \omega_m$ in (143).

Once these eigenvectors have been computed the decomposition then can be written by a straightforward use of (98), (84) and (85).

6 Computing the Bifurcated Periodic Solution on the Center Manifold

This section will be written in such a manner that it could apply to both ordinary and delay differential equations. We know from ordinary differential equations that in the case of the homogeneous portion having two eigenvalues with zero real parts and all of the others negative real parts there are two manifolds of solutions. One manifold, called the stable manifold, is an invariant manifold of solutions that decay to the equilibrium point. The other manifold, called the center manifold, is an invariant manifold on which the essential behavior of the solution in the neighborhood of the equilibrium point is determined.

Step 6: Compute the Center Manifold Form

To begin the construction of a center manifold we start with the equations (84)

$$\begin{aligned}\frac{dy}{dt}(t) &= i\omega y(t) + F_1(Y,\ w_t),\\ \frac{d\overline{y}}{dt}(t) &= -i\omega\overline{y}(t) + \overline{F}_1(Y,\ w_t),\\ \frac{dw_t}{dt} &= Aw_t + F_2(Y,\ w_t),\end{aligned} \quad (145)$$

where

$$\begin{aligned}F_1(Y,\ w_t) &= \overline{\phi}_D{}^T(0) f(z_t),\\ F_2(Y,\ w_t) &= \begin{cases} -2Re\{\langle\phi_D, z_t\rangle C\} & -\sigma \le \theta < 0,\\ -2Re\{\langle\phi_D, z_t\rangle C\} + f(z_t) & \theta = 0.\end{cases}\end{aligned} \quad (146)$$

This system has been decomposed into two equations that have eigenvalues with zero real parts and one that has eigenvalues with negative real parts. For notation, let E^c be the subspace formed by the eigenvectors $\phi_C,\ \overline{\phi_C}$. We can now define a center manifold by a function $w = w(y, \overline{y})$ for $|y|,\ |\overline{y}|$ sufficiently small such that $w(0,0) = 0,\ \mathcal{D}w(0,0) = 0$, where $\mathcal{D}$ is the total derivative operator. Note that $w = w(y, \overline{y})$ is only defined locally and the conditions on w make it tangent to E^c at $(0,0)$. According to Wiggins [34] the center manifold for (84) can then be specified as

$$W^c(0) = \{(y, \overline{y}, w) \in C^3 | w = w(y, \overline{y}), |y|, |\overline{y}| < \delta, w(0,0) = 0, \mathcal{D}w(0,0) = 0\}. \tag{147}$$

for δ sufficiently small.

Since the center manifold is invariant, the dynamics of the first two equations in (145) must be restricted to the center manifold and satisfy

$$\begin{aligned} \frac{dy}{ds} &= i\omega y + F_1(y,\ w(y,\ \overline{y})), \\ \frac{d\overline{y}}{ds} &= -i\omega \overline{y} + \overline{F}_1(y,\ w(y,\ \overline{y})). \end{aligned} \tag{148}$$

There is also one more condition on the dynamics that must be satisfied by $w = w(y, \overline{y})$ and that is, it must satisfy the last equation in (145). Then, by taking appropriate partial derivatives we must have

$$\begin{aligned} \mathcal{D}_y w(y,\ \overline{y})\{i\omega y + F_1(y,\ w(y,\ \overline{y}))\} &+ \mathcal{D}_{\overline{y}} w(y,\ \overline{y})\{-i\omega \overline{y} + \overline{F}_1(y,\ w(y,\ \overline{y}))\} \\ &= Aw(y,\ \overline{y}) + F_2(y,\ w(y,\ \overline{y})) \end{aligned} \tag{149}$$

where $\mathcal{D}$ represents the derivative with respect to the subscripted variable. The argument of Kazarinoff et al. [22] (see also Carr [9]) can be used to look for a center manifold $w(y,\ \overline{y})$ that approximately solves (149). We will not need to reduplicate the argument here but to note that in fact an approximate center manifold, satisfying (149), can be given as a quadratic form in y and $\overline{y}$ with coefficients as functions of θ

$$w(y,\ \overline{y})(\theta) = w_{20}(\theta)\frac{y^2}{2} + w_{11}(\theta) y\overline{y} + w_{02}(\theta)\frac{\overline{y}^2}{2}. \tag{150}$$

Then the projected equation (148) on the center manifold takes the form

$$\begin{aligned} \frac{dy}{ds} &= i\omega y + g(y,\ \overline{y}), \\ \frac{d\overline{y}}{ds} &= -i\omega \overline{y} + \overline{g}(y,\ \overline{y}), \end{aligned} \tag{151}$$

where the $g(y,\ \overline{y})$ is given by

$$g(y,\ \overline{y}) = g_{20}\frac{y^2}{2} + g_{11} y\overline{y} + g_{02}\frac{\overline{y}^2}{2} + g_{21}\frac{y^2\overline{y}}{2}. \tag{152}$$

Since $w_{02} = \overline{w}_{20}$, one only needs to solve for w_{20} and w_{11}. It can be shown that $w_{20},\ w_{11}$ take the form

$$w_{20}(\theta) = c_1\phi(\theta) + c_2\overline{\phi}(\theta) + M\mathrm{e}^{2i\omega\theta},$$
$$w_{11}(\theta) = c_3\phi(\theta) + c_4\overline{\phi}(\theta) + N, \tag{153}$$

where $c_i, i = 1, \ldots, 4$ are constants and $M,\ N$ are vectors. We will show how the coefficients and the vectors can be computed in a specific example below.

Step 7: Develop the Normal Form on the Center Manifold

The equations on the center manifold can be further simplified. Normal form theory, following the argument of Wiggins [34] (see also Nayfeh [25]), can be used to reduce (151) to the simpler form (154) below on the center manifold. In fact, (148) is reduced to a normal form by a transformation of variables, $y \to v$, so that the new system takes the form

$$\frac{dv}{ds} = i\omega v + c_{21}v^2\overline{v},$$
$$\frac{d\overline{v}}{ds} = -i\omega\overline{v} + c_{21}\overline{v}^2 v, \tag{154}$$

where the higher order terms have been dropped. The derivation of this formula is complex and is not needed for this chapter. The interested reader should consult Gilsinn [11] and Wiggins [34] for the essential ideas involved. The formula needed is one that links (154) with (152) and is given by

$$c_{21} = \frac{i}{2\omega}\left\{g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right\} + \frac{g_{21}}{2}. \tag{155}$$

For a more general discussion of the normal form on the center manifold when $\mu \neq 0$ see Hassard et al. [17]. Up to this point one has only needed $\mu = 0$. But, to compute the periodic solution we will reintroduce $\mu \neq 0$. A formula for the periodic solutions of (1) can be computed. The argument is based on that of Hassard et al. [17] and will not be given but the references Gilsinn [11] and Hassard et al. [17] can be consulted by the interested reader. What is important here is a formula that can be used in specific examples.

Step 8: Form the Periodic Solution on the Center Manifold

To begin with let $\epsilon > 0$ and an initial condition for a periodic solution of (154) be given as

$$v(0;\ \epsilon) = \epsilon. \tag{156}$$

Then, there exists a family of periodic solutions $v(s,\ \mu(\epsilon))$ of (194) with

$$\begin{aligned}\mu(\epsilon) &= \mu_2\epsilon^2 + \cdots, \\ \beta(\epsilon) &= \beta_2\epsilon^2 + \cdots, \\ \mathrm{T}(\epsilon) &= \mathrm{T}_0(1 + \tau_2\epsilon^2 + \cdots),\end{aligned} \tag{157}$$

where $\mathrm{T}(\epsilon)$ is the period of $v(s,\ \mu(\epsilon))$, $\beta(\epsilon)$ is the nonzero characteristic exponent, and

$$\begin{aligned}\mu_2 &= -\frac{\mathrm{Re}\left\{c_{21}\right\}}{\alpha'(0)}, \\ \beta_2 &= 2\mathrm{Re}\left\{c_{21}(0)\right\}, \\ \tau_2 &= -\frac{1}{\omega}\left(\mu_2\omega'(0) + \mathrm{Im}\left\{c_{21}\right\}\right), \\ \mathrm{T}_0 &= \frac{2\pi}{\omega}.\end{aligned} \tag{158}$$

Furthermore, $v(s,\ \mu(\epsilon))$ can be transformed into a family of periodic solutions for (1) given by

$$z(s) = \mathcal{P}(s, \mu(\epsilon)) = 2\epsilon\mathrm{Re}\left\{\phi(0)\mathrm{e}^{i\omega s}\right\} + \epsilon^2\mathrm{Re}\left\{M\mathrm{e}^{2i\omega s} + N\right\}. \tag{159}$$

with $\epsilon = (\mu/\mu_2)^{1/2}$. For $\mu_2 > 0$ the Hopf bifurcation is called supercritical and for $\mu_2 < 0$ it is called subcritical. Finally, note that since $\mu \approx \mu_2\epsilon^2$ one can take $\epsilon = (\mu/\mu_2)^{1/2}$ which allows one to associate $Z(s)$ with the parameter $p = \mu + p_c$. From Floquet theory if $\beta(\epsilon) < 0$ the periodic solution is stable and if $\beta(\epsilon) > 0$ it is unstable.

7 A Machine Tool DDE Example: Part 2

In this section we will show how the formulas in the previous section are computed for the specific example of the turning center model. We will conclude this section with a formula approximating the periodic solution of (96)

Step 6: Compute the Center Manifold Form

With the eigenvectors computed we can proceed to approximate the center manifold and the projected equations on the center manifold. We begin by assuming that an approximate center manifold, satisfying (149), can be given as a quadratic form in y and $\overline{y}$ with coefficients as functions of θ

$$w(y,\ \overline{y})(\theta) = w_{20}(\theta)\frac{y^2}{2} + w_{11}(\theta)y\overline{y} + w_{02}(\theta)\frac{\overline{y}^2}{2}. \tag{160}$$

The object of this section is to compute the constants $c_1,\ c_2,\ c_3,\ c_4$, and the vectors $M,\ N$ in (153) in order to create the coefficients for (160).

Using (70) we can introduce coordinates on the center manifold by

$$z = w + \phi y + \overline{\phi}\overline{y}. \tag{161}$$

From (83), (84), (148), and (161) we have on the center manifold

$$\frac{dy}{ds} = i\omega y + \overline{\phi}^{*T}(0) f(w(y,\ \overline{y}) + \phi y + \overline{\phi}\overline{y}). \tag{162}$$

Define

$$g(y,\ \overline{y}) = \overline{\phi}^{*T}(0) f(w(y,\ \overline{y}) + \phi y + \overline{\phi}\overline{y}), \tag{163}$$

where $\overline{\phi}^{*T}(0) = \left(\overline{d}_1,\ \overline{d}_2\right)$ from (142) and

$$f(w(y,\ \overline{y}) + \phi y + \overline{\phi}\overline{y}) = \begin{pmatrix} 0 \\ p_c \left(E\left[w(y,\ \overline{y})_1(-\sigma) + y\phi_1(-\sigma) + \overline{y}\overline{\phi}_1(\sigma) \right.\right. \\ \left. -w(y,\ \overline{y})_1(0) - y\phi_1(0) - \overline{y}\overline{\phi}_1(0)\right]^2 \\ +E\left[w(y,\ \overline{y})_1(-\sigma) + y\phi_1(-\sigma) + \overline{y}\overline{\phi}_1(\sigma) \right. \\ \left.\left. -w(y,\ \overline{y})_1(0) - y\phi_1(0) - \overline{y}\overline{\phi}_1(0)\right]^3\right) \end{pmatrix}. \tag{164}$$

From (160)

$$\begin{aligned} w(y,\ \overline{y})(0) &= w_{20}(0)\frac{y^2}{2} + w_{11}(0)y\overline{y} + w_{02}(0)\frac{\overline{y}^2}{2}, \\ w(y,\ \overline{y})(-\sigma) &= w_{20}(-\sigma)\frac{y^2}{2} + w_{11}(-\sigma)y\overline{y} + w_{02}(-\sigma)\frac{\overline{y}^2}{2}, \end{aligned} \tag{165}$$

where $w_{ij}(\theta) = \left(w_{ij}^1(\theta),\ w_{ij}^2(\theta)\right)^T$.

Note here that in order to compute $\mu_2,\ \tau_2,\ \beta_2$ one need only determine $g(y,\ \overline{y})$ in the form (152). To find the coefficients for (152) begin by expanding the nonlinear terms of (164) up to cubic order, keeping only the cubic term $y^2\overline{y}$. To help simplify the notation let

$$\gamma = \mathrm{e}^{-i\omega\sigma} - 1. \tag{166}$$

Then, using (58), (165), and (166),

$$\begin{aligned} &E\left[w(y,\ \overline{y})_1(-\sigma) + y\phi_1(-\sigma) + \overline{y}\overline{\phi}_1(\sigma) - w(y,\ \overline{y})_1(0) - y\phi_1(0) - \overline{y}\overline{\phi}_1(0)\right]^2 \\ &= Ey^2\gamma^2 + 2Ey\overline{y}\gamma\overline{\gamma} \\ &+E\overline{y}^2\overline{\gamma}^2 + E\left\{\left[w_{20}^1(-\sigma) - w_{20}^1(0)\right]\overline{\gamma} + 2\left[w_{11}^1(-\sigma) - w_{11}^1(0)\right]\gamma\right\}y^2\overline{y}, \quad (167) \\ &E\left[w(y,\ \overline{y})_1(-\sigma) + y\phi_1(-\sigma) + \overline{y}\overline{\phi}_1(\sigma) - w(y,\ \overline{y})_1(0) - y\phi_1(0) - \overline{y}\overline{\phi}_1(0)\right]^3 \\ &= 3E\gamma^2\overline{\gamma}y^2\overline{y}. \end{aligned}$$

If we define

$$\begin{aligned} g_{20} &= 2E\gamma^2\overline{d}_2 p_c, \\ g_{11} &= 2E\gamma\overline{\gamma}\overline{d}_2 p_c, \\ g_{02} &= 2E\overline{\gamma}^2\overline{d}_2 p_c, \end{aligned} \tag{168}$$

and

$$g_{21} = 2p_c \left\{ E\left[w^1_{20}(-\sigma) - w^1_{20}(0)\right]\overline{\gamma} + 2E\left[w^1_{11}(-\sigma) - w^1_{11}(0)\right]\gamma + 3E\gamma^2\overline{\gamma}\right\}\overline{d}_2, \tag{169}$$

we can use (163) through (169) to write

$$f(w(y,\ \overline{y}) + \phi y + \overline{\phi}\overline{y}) = \left\{ \frac{g_{20}y^2}{2\overline{d}_2} + \frac{g_{11}y\overline{y}}{\overline{d}_2} + \frac{g_{02}\overline{y}^2}{2\overline{d}_2} + \frac{g_{21}y^2\overline{y}}{2\overline{d}_2} \right\} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{170}$$

In order to complete the computation of g_{21} one needs to compute the center manifold coefficients $w_{20},\ w_{11}$.

Since we are looking for the center manifold as a quadratic form, we need only expand functions in terms of $y^2,\ y\overline{y},\ \overline{y}^2$. From the definition of F_2 in (85), (152), and (163) write F_2 as

$$\begin{aligned} F_2(y,\ \overline{y})(\theta) = &- \left\{ g_{20}\phi(\theta) + \overline{g}_{02}\overline{\phi}(\theta)\right\}\frac{y^2}{2} \\ &- \left\{ g_{11}\phi(\theta) + \overline{g}_{11}\overline{\phi}(\theta)\right\} y\overline{y} \\ &- \left\{ g_{02}\phi(\theta) + \overline{g}_{20}\overline{\phi}(\theta)\right\}\frac{\overline{y}^2}{2}, \end{aligned} \tag{171}$$

for $-\sigma \le \theta < 0$ and for $\theta = 0$

$$\begin{aligned} F_2(y,\ \overline{y})(0) = &- \left\{ g_{20}\phi(0) + \overline{g}_{02}\overline{\phi}(0) - \frac{g_{20}}{\overline{d}_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\frac{y^2}{2} \\ &- \left\{ g_{11}\phi(0) + \overline{g}_{11}\overline{\phi}(0) - \frac{g_{11}}{\overline{d}_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} y\overline{y} \\ &- \left\{ g_{02}\phi(0) + \overline{g}_{20}\overline{\phi}(0) - \frac{g_{02}}{\overline{d}_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\frac{\overline{y}^2}{2}. \end{aligned} \tag{172}$$

Note that, to compute the coefficients of the center manifold, one only needs to work to the second order.

Since $g_{02}/\overline{d}_2 = \overline{g}_{20}/d_2$ write the coefficients of $F_2(y,\ \overline{y})$ as

$$\begin{aligned} F^2_{20}(\theta) &= \begin{cases} -\left(g_{20}\phi(\theta) + \overline{g}_{02}\overline{\phi}(\theta)\right) & -\sigma \le \theta < 0, \\ -\left(g_{20}\phi(0) + \overline{g}_{02}\overline{\phi}(0) - \frac{g_{20}}{\overline{d}_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) & \theta = 0, \end{cases} \\ F^2_{11}(\theta) &= \begin{cases} -\left(g_{11}\phi(\theta) + \overline{g}_{11}\overline{\phi}(\theta)\right) & -\sigma \le \theta < 0, \\ -\left(g_{11}\phi(0) + \overline{g}_{11}\overline{\phi}(0) - \frac{g_{11}}{\overline{d}_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) & \theta = 0, \end{cases} \\ F^2_{02}(\theta) &= \overline{F}^2_{20}(\theta). \end{aligned} \tag{173}$$

One can now set up equation (149) to approximate the center manifold. On this manifold one must have

$$W(s) = w(y(s),\ \overline{y}(s)). \tag{174}$$

By taking derivatives, the equation for the manifold becomes

$$w_y(y,\ \overline{y})y'(s) + w_{\overline{y}}(y,\ \overline{y})\overline{y}'(s) = \mathrm{A}w(y(s),\ \overline{y}(s)) + F_2(y(s),\ \overline{y}(s)), \quad (175)$$

where $w(y,\ \overline{y})$ is given by (160). The partial derivatives are given by

$$\begin{aligned} w_y(y,\ \overline{y}) &= w_{20}y + w_{11}\overline{y}, \\ w_{\overline{y}}(y,\ \overline{y}) &= w_{11}y + w_{02}\overline{y}. \end{aligned} \quad (176)$$

Using (151) and (176) expand the terms of (175) to second order as

$$\begin{aligned} w_y(y,\ \overline{y})(\theta)y'(s) &= i\omega w_{20}(\theta)y^2(s) + i\omega w_{11}(\theta)\overline{y}(s)y(s), \\ w_{\overline{y}}(y,\ \overline{y})(\theta)\overline{y}'(s) &= -i\omega w_{11}(\theta)y(s)\overline{y}(s) - i\omega w_{02}(\theta)\overline{y}^2(s), \\ \mathrm{A}w(y,\ \overline{y})(\theta) &= (\mathrm{A}w_{20})(\theta)\frac{y^2}{2} + (\mathrm{A}w_{11})(\theta)y\overline{y} + (\mathrm{A}w_{02})(\theta)\frac{\overline{y}^2}{2}, \\ F_2(y,\ \overline{y})(\theta) &= F_{20}^2(\theta)\frac{y^2}{2} + F_{11}^2(\theta)y\overline{y} + F_{20}^2(\theta)\frac{\overline{y}^2}{2}. \end{aligned} \quad (177)$$

Substitute (177) into (175) and equate coefficients to get

$$\begin{aligned} 2i\omega w_{20}(\theta) - \mathrm{A}w_{20} &= F_{20}^2,(\theta) \\ -\mathrm{A}w_{11} &= F_{11}^2(\theta), \\ -2i\omega w_{02}(\theta) - \mathrm{A}w_{02} &= F_{02}^2.(\theta) \end{aligned} \quad (178)$$

Since $F_{02}^2 = \overline{F}_{20}^2$ and $w_{02} = \overline{w}_{20}$, one only needs to solve for w_{20} and w_{11}.

To compute $c_3,\ c_4,\ N$, use the second equation in (178), the definition of A in (40) and (173). Then for $-\sigma \le \theta < 0$

$$\frac{dw_{11}}{d\theta}(\theta) = g_{11}\phi(\theta) + \overline{g}_{11}\overline{\phi}(\theta). \quad (179)$$

Integrate (179) and use (58) to get

$$w_{11}(\theta) = \frac{g_{11}}{i\omega}\phi(\theta) - \frac{\overline{g}_{11}}{i\omega}\overline{\phi}(\theta) + N. \quad (180)$$

Clearly

$$\begin{aligned} c_3 &= \frac{g_{11}}{i\omega}, \\ c_4 &= -\frac{\overline{g}_{11}}{i\omega}. \end{aligned} \quad (181)$$

To determine N we will use (40) for $\theta = 0,\ \mu = 0$ and the fact that $\phi(0),\ \overline{\phi}(0)$ are eigenvectors of A with eigenvalues $i\omega,\ -i\omega$ at $\theta = 0$. The eigenvector property implies

$$\begin{aligned} U\phi(0) + V\phi(-\sigma) &= i\omega\phi(0), \\ U\overline{\phi}(0) + V\overline{\phi}(-\sigma) &= -i\omega\overline{\phi}(0). \end{aligned} \quad (182)$$

If we combine this with (173), then it is straightforward to show that

$$(\mathrm{U}+\mathrm{V})N = -\frac{g_{11}}{\overline{d}_2}\begin{pmatrix}0\\1\end{pmatrix}, \tag{183}$$

which can be solved for

$$N = -\frac{g_{11}}{\overline{d}_2}\begin{pmatrix}-1\\0\end{pmatrix}. \tag{184}$$

To solve for $c_1,\ c_2,\ M$, use the definition of A in (40) for $-\sigma \le \theta < 0$, (173), (178) to get

$$\frac{dw_{20}}{d\theta} = 2i\omega w_{20}(\theta) + g_{20}\phi(\theta) + \overline{g}_{02}\overline{\phi}(\theta). \tag{185}$$

This nonhomogeneous system has the solution

$$w_{20}(\theta) = -\frac{g_{20}}{i\omega}\phi(\theta) - \frac{\overline{g}_{02}}{3i\omega}\overline{\phi}(\theta) + M\mathrm{e}^{2i\omega\theta}. \tag{186}$$

Again, clearly

$$\begin{aligned} c_1 &= -\frac{g_{20}}{i\omega},\\ c_2 &= -\frac{\overline{g}_{02}}{3i\omega}. \end{aligned} \tag{187}$$

To solve for M use the definition of (40) for $\theta = 0,\ \mu = 0$, (173) and again the fact that $\phi(0),\ \overline{\phi}(0)$ are eigenvectors of A with eigenvalues $i\omega,\ -i\omega$ at $\theta = 0$ to show that

$$\left(2i\omega\mathrm{I} - \mathrm{U} - \mathrm{V}\mathrm{e}^{-2i\omega\sigma}\right)M = \frac{g_{20}}{\overline{d}_2}\begin{pmatrix}0\\1\end{pmatrix}, \tag{188}$$

which can be solved for M as

$$M = \frac{g_{20}}{\overline{d}_2\Delta}\begin{pmatrix}1\\2i\omega\end{pmatrix}, \tag{189}$$

where

$$\Delta = -4\omega^2 + 4i\xi\omega + p_c\left(1 - \mathrm{e}^{2i\omega\sigma}\right). \tag{190}$$

One can now return to (169) and use (180) through (184) and (186) through (190) to construct g_{21} in (191), which concludes the construction of (152) and thus the projected equation (151) on the center manifold. Then the projected equation (148) on the center manifold takes the form (151) and (152). The coefficients g_{ij} for (152) are given by

$$\begin{aligned} g_{20} &= 2E\gamma^2\overline{d}_2 p_c,\\ g_{11} &= 2E\gamma\overline{\gamma}\overline{d}_2 p_c,\\ g_{02} &= 2E\overline{\gamma}^2\overline{d}_2 p_c,\\ g_{21} &= 2p_c\left\{E\left[-\frac{g_{20}\gamma}{i\omega} - \frac{\overline{g}_{02}}{3i\omega} + \frac{g_{20}\left(\mathrm{e}^{-2i\omega\sigma}-1\right)}{\overline{d}_2\Delta}\right]\overline{\gamma}\right.\\ &\quad\left. +2E\left[\frac{g_{11}\gamma}{i\omega} - \frac{\overline{g}_{11}\overline{\gamma}}{i\omega}\right]\gamma + 3E\gamma^2\overline{\gamma}\right\}\overline{d}_2, \end{aligned} \tag{191}$$

where

$$\begin{aligned}\gamma &= \mathrm{e}^{-i\omega\sigma} - 1,\\ \Delta &= -4\omega^2 + 4i\xi\omega + p_c\left(1 - \mathrm{e}^{2i\omega\sigma}\right).\end{aligned} \tag{192}$$

Step 7: Develop the Normal Form on the Center Manifold

Once the constants (191) have been developed then the normal form on the center manifold is computed as (154) using (193). Normal form theory, following the argument of Wiggins [34] (see also Nayfeh [25]), can be used to reduce (151) to the simpler form (154) on the center manifold. In fact, the system (151) can be reduced by a near identity transformation to (154) where

$$c_{21} = \frac{i}{2\omega}\left\{g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right\} + \frac{g_{21}}{2}. \tag{193}$$

The proof of the result is detailed and the reader is referred to Gilsinn [11] and Wiggins [34]. As shown in Hassard et al. [17] the general normal form for the case $\mu \neq 0$ is given by

$$\frac{dv}{ds} = \lambda(\mu)v + c_{21}(\mu)v^2, \overline{v} \tag{194}$$

where $\lambda(0) = i\omega$ and $c_{21}(0)$ is given by (193).

One can now compute $g_{20},\ g_{11},\ g_{02}$, and g_{21} from (191). Then from (193) one computes c_{21} and finally, from (158), one can compute $\mu_2,\ \tau_2,\ \beta_2$ as

$$\begin{aligned}\mu_2 &= -0.09244,\\ \tau_2 &= 0.002330,\\ \beta_2 &= 0.08466.\end{aligned} \tag{195}$$

This implies that at the lobe boundary the DDE bifurcates into a family of unstable periodic solutions in a subcritical manner.

Step 8: Form the Periodic Solution on the Center Manifold

Using (159), one can compute the form of the bifurcating solutions for (96) as

$$z(s) = \begin{pmatrix} 2\epsilon\,\cos 1.01341s + \epsilon^2\ ((-2.5968e-6)\ \cos 2.02682s \\ \quad -0.014632\sin 2.02682s + 0.060113) \\ -2\epsilon\sin 1.01341s + \epsilon^2\ (-0.02966\cos 2.02682s \\ \quad +(5.2632e-6)\ \sin 2.02682s) \end{pmatrix}. \tag{196}$$

As noted at the end of Section 6 one can take as an approximation

$$\epsilon = \left(\frac{\mu}{\mu_2}\right)^{1/2}. \tag{197}$$

It is clear from (195) that μ must be negative. Thus select

$$\epsilon = \frac{(-\mu)^{1/2}}{0.3040395}. \tag{198}$$

The period of the solution can be computed as

$$\mathrm{T}(\epsilon) = \frac{2\pi}{\omega_{\mathrm{m}}}\left(1 + \tau_2 \epsilon^2\right) = 6.2000421\left(1 + 0.002330\,\epsilon^2\right), \tag{199}$$

and the characteristic exponent is given by

$$\beta = 0.08466\,\epsilon^2. \tag{200}$$

8 Simulation Results

To compare with the theorectical results, simulations were performed by direct integration of (96). The first of two sets of simulations was initialized at the points A through F in Figure 5 along the vertical line $\Omega = 0.2144$ (selected for ease of calculation only). This line crosses the minimum of the first lobe in Figure 5. The simulations numerically demonstrate that there are three branches of periodic solutions emanating from the critical bifurcation point $p_m = 0.027365$. The three branches are shown in Figure 6. The amplitudes in this figure were computed using (196) with ϵ given by (197). Two of the branches are unstable and one is stable in the following sense. Solutions initialized below the subcritical branch converge to the zero solution. Those initialized above the subcritical branch grow in amplitude. The solutions initialized above zero for bifurcation parameter values greater than the critical value grow in amplitude. Similar results would be obtained along lines crossing at other critical points on the lobes.

The Hopf bifurcation result is very local around the boundary and only for very small initial amplitudes is it possible to track the unstable limit cycles along the branching amplitude curve. This is shown in Figure 7 where initial simulation locations were selected along the subcritical curve and the delay differential equation was integrated forward over five delay intervals. Note that nearer the critical bifurcation point the solution amplitude remains near the initial value, whereas further along the curve the solution amplitude drops away significantly from the subcritical curve.

Since the subcritical bifurcation curve in Figure 6 is itself an approximation to the true subcritical curve, solutions initialized on the curve tend to decay to zero. This occurs at points A, B, and C in Figure 5.

The decay at point B, when initialized on the subcritical curve is similar to the result at point A, but with a less rapid decay. However, one can show the effect of initializing a solution above the curve at point B. That is shown in Figure 9. Point B is given by $(\Omega, p) = (0.2144, 0.020365)$. For $p = 0.020365$

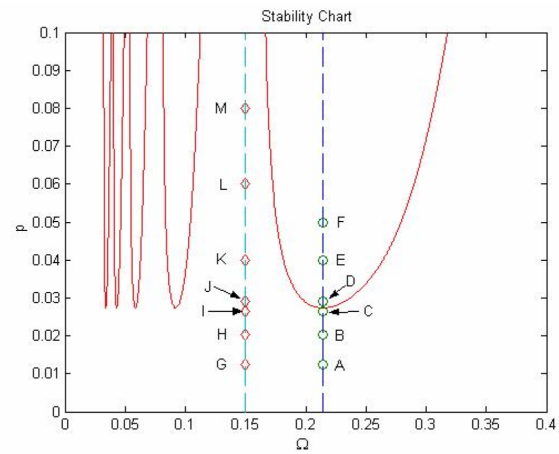

Fig. 5. Locations of Sample Simulated Solutions

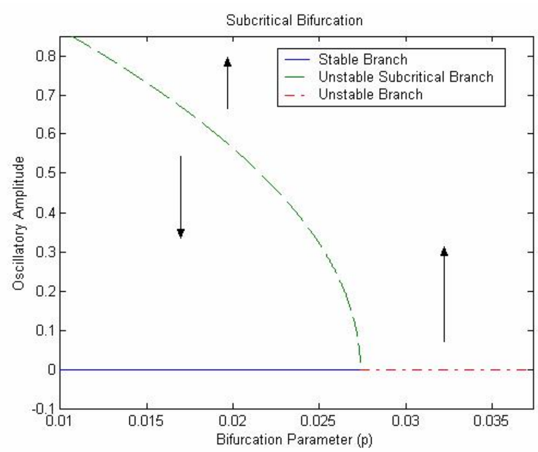

Fig. 6. Amplitudes of Bifurcating Branches of Solutions at Subcritical Point

the subcritical curve amplitude value is 0.550363. Figures 11 and 12 show the solution growing when the simulation amplitude is initialized at 0.8.

At point C, when the solution is initialized on the subcritical bifurcation curve, the phase plot remains very close to a periodic orbit, indicating that the Hopf results are very local in being able to predict the unstable periodic solution.

The behavior at points D, E, and F of Figure 5 are similar in that all of the solutions initialized above zero experience growth and eventually explode numerically.

The second set of simulations, initialized at points G through M along the line $\Omega = 0.15$, in Figure 5, shows the stability of solutions for parameters

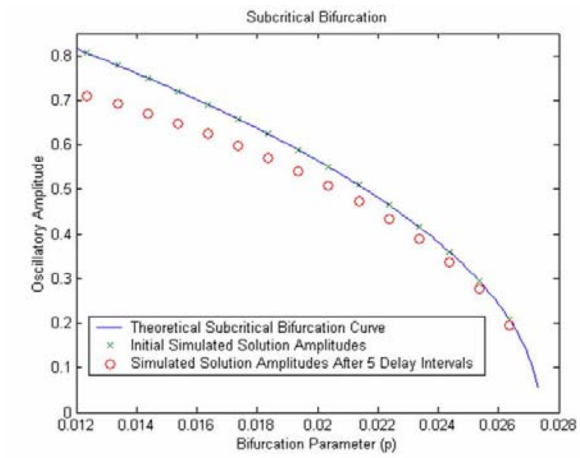

Fig. 7. Theoretical and Simulated Subcritical Solution Amplitudes

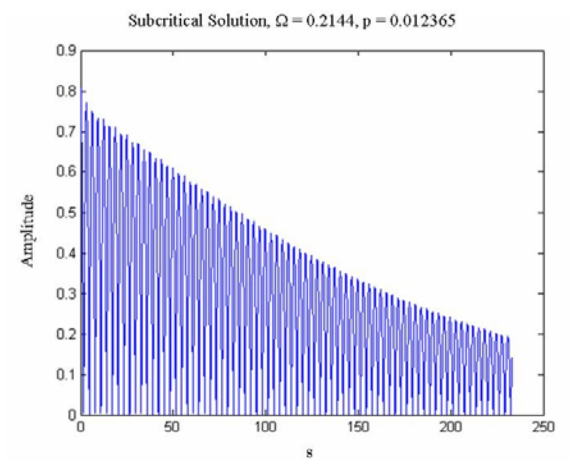

Fig. 8. Stable Amplitude of Solution at Point A

falling between lobes in that the solutions of the delay differential equation (92) all decay to zero. The gaps between lobes are significant for machining. Since the parameter p is proportional to chip width, the larger the p value for which the system is stable the more material can be removed without chatter, where chatter can destroy the surface finish of the workpiece. These large gaps tend to appear between the lobes in high-speed machining with spindle rotation rates of the order of 2094.4 rad/s (20,000 RPM) or greater. Figure 12 illustrates this stability with one time plot for point M, $(\Omega, p) = (0.15, 0.08)$. The solution for this plot is initialized at amplitude 0.08.

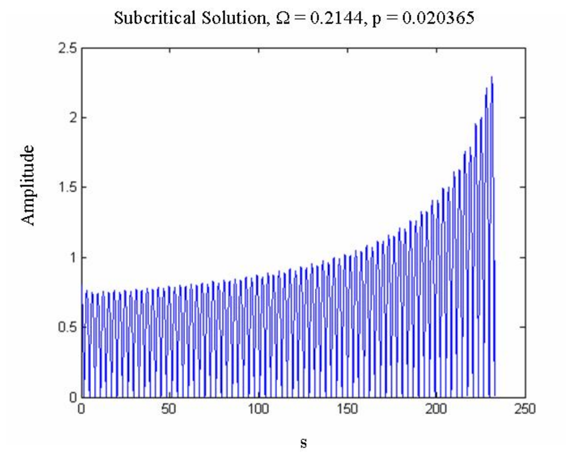

Fig. 9. Unstable Amplitude of Solution at Point B

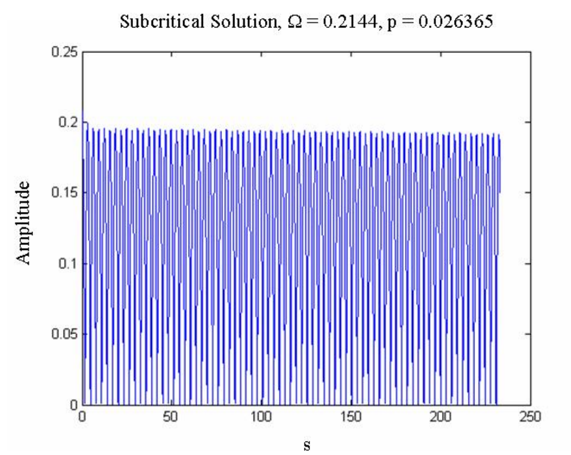

Fig. 10. Nearly Stable Amplitude of Solution at Point C

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References

1. Altintas, Y. and Budak, E., 'Analytic Prediction of Stability Lobes in Milling', *Annals of the CIRP* **44**, 1995, 357-362.

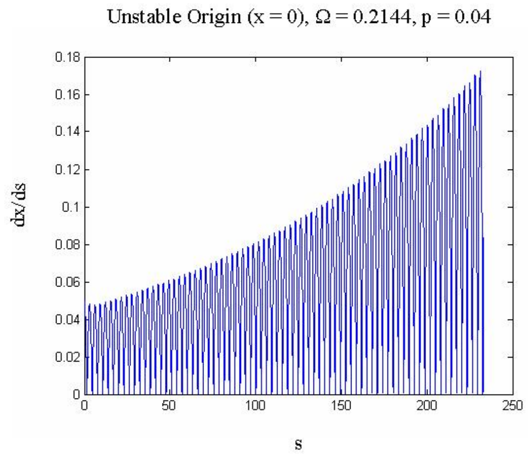

Fig. 11. Unstable Amplitude of Solution at Point E

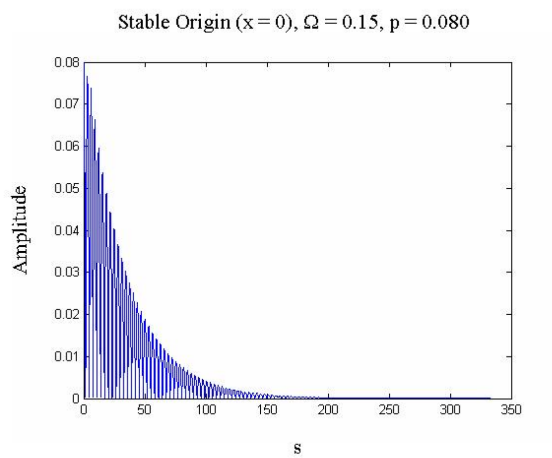

Fig. 12. Stable Amplitude of Solution at Point M

2. an der Heiden, U. 'Delays in physiological systems', *J. Math. Biol.* **8**, 1979, 345-364.
3. Arnold, V. I., *Ordinary Differential Equations*, Springer-Verlag, Berlin, 1992.
4. Avellar, C. E. and Hale, J. K., 'On the Zeros of Exponential Polynomials', *Journal of Mathematical Analysis and Applications* **73**, 1980, 434-452.
5. Balachandran, B., 'Nonlinear dynamics of milling processes', *Phil. Trans. R. Soc. Lond. A* **359**, 2001, 793-819.
6. Balachandran, B. and Zhao, M. X., 'A Mechanics Based Model for Study of Dynamics of Milling Operations', *Meccanica* **35**, 2000, 89-109.
7. Bellman, R. and Cooke, K. L., *Differential-Difference Equations*, Academic Press, New York, 1963.

8. Campbell, S. A., 'Calculating Centre Manifolds for Delay Differential Equations Using Maple', In: Balachandran B, Gilsinn DE, Kalmár-Nagy T (eds), *Delay Differential Equations: Recent Advances and New Directions*. Springer Verlag, New York, p TBD
9. Carr, J., *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.
10. Cronin, J., *Differential Equations: Introduction and Qualitative Theory*, Marcel Dekker, Inc., New York, 1980.
11. Gilsinn, D. E., 'Estimating Critical Hopf Bifurcation Parameters for a Second-Order Differential Equation with Application to Machine Tool Chatter', *Nonlinear Dynamics* **30**, 2002, 103-154.
12. Hale, J. K., 'Linear Functional-Differential Equations with Constant Coefficients', *Contributions To Differential Equations* **II**, 1963, 291-317.
13. Hale, J. K., *Ordinary differential Equations*, Wiley-Interscience, New York, 1969.
14. Hale, J., *Functional Differential Equations*, Springer-Verlag, New York, 1971.
15. Hale, J. K. and Lunel, S. M. V., *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.
16. Hassard, B. and Wan, Y.H., 'Bifurcation Formulae Derived from Center Manifold Theory', *Journal of Mathematical Analysis and Applications* **63**, 1978, 297-312.
17. Hassard, B.D., Kazarinoff, N.D., and Wan, Y.H., *Theory and Applications of Hopf Bifurcations*, Cambridge University Press, Cambridge, 1981.
18. Hille, E. and Phillips, R. S., *Functional Analysis and Semi-Groups*, American Mathematical Society, Providence, 1957.
19. Hirsch, M. W. and Smale, S., *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.
20. Kalmár-Nagy, T, Pratt, J.R., Davies, M.A., and Kennedy, M.D., 'Experimental and Analytical Investigation of the Subcritical Instability in Metal Cutting', *Proceedings of DETC'99*, 17th ASME Biennial Conference on Mechanical Vibration and Noise, Las Vegas, Nevada, Sept. 12-15, 1999, 1-9.
21. Kalmár-Nagy T, Stépán G, Moon FC (2001) Subcritical Hopf bifurcation in the delay equation model for machine tool vibrations. *Nonlinear Dynamics* 26, 121–142
22. Kazarinoff, N. D., Wan, Y., -H., and van den Driessche, P., 'Hopf Bifurcation and Stability of Periodic Solutions of Differential-difference and Integro-differential Equations', *Journal Inst. Maths. Applics.* **21**, 1978, 461-477.
23. Kuang, Y., *Delay Differential Equations With Applications In Population Dynamics*, Academic Press, Inc., Boston, 1993.
24. MacDonald, N. *Biological Delay Systems: Linear Stability Theory*, Cambridge University Press, Cambridge, 1989.
25. Nayfeh, A. H., *Method of Normal Forms* John Wiley & Sons, Inc., New York, 1993.
26. Nayfeh, A. H. and Balachandran, B. *Applied Nonlinear Dynamics*, John Wiley & Sons, Inc., New York, 1995.
27. Pinney, E., *Ordinary Difference-Differential Equations*, University of Clifornia Press, Berkeley, 1958.
28. Stépán, G., *Retarded dynamical systems: stability and characteristic functions*, Longman Scientific & Technical, Harlow, England, 1989.
29. Stone E, Askari A (2002) Nonlinear models of chatter in drilling processes. *Dynamical Systems* 17 (1), 65–85.

30. Stone E, Campbell SA (2004) Stability and bifurcation analysis of a nonlinear DDE model for drilling. *Journal of Nonlinear Science* 14 (1), 27–57.
31. Taylor, J.R., 'On the art of cutting metals', *American Society of Mechanical Engineers*, New York, 1906.
32. Tlusty, J., 'Machine Dynamics', *Handbook of High-speed Machine Technology*, Robert I. King, Editor, Chapman and Hall, New york, 1985, 48-153.
33. Tobias, S.A. and Fishwick, W., 'The Chatter of Lathe Tools Under Orthogonal Cutting Conditions', *Transactions of the ASME* **80**, 1958, 1079-1088.
34. Wiggins, S., *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.
35. Yosida, K., *Functional Analysis*, Springer-Verlag, Berlin, 1965.