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Computable error bounds for approximate periodic solutions of autonomous delay differential equations

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Abstract In this paper, we prove a result that says: Given an approximate solution and frequency to a periodic solution of an autonomous delay differential equation that satisfies a certain noncriticality condition, there is an exact periodic solution and frequency in a neighborhood of the approximate solution and frequency and, furthermore, numerical estimates of the size of the neighborhood are computed. Methods are outlined for estimating the parameters required to compute the errors. An application to a Van der Pol oscillator with delay in the nonlinear terms is given.

Keywords Adjoint equation · Delay differential equations · Error bounds · Fredholm alternative · Fundamental solution · Monodromy operator · Noncritical solution · Periodic solutions · Van der Pol equation

1 Introduction

Delay differential equations (DDEs) have occurred in many fields from biology [1] to population dynamics [2] to machine tool dynamics [3–6]. The study of machine tool dynamics has led to many problems involving delay differential equations. The problem of regenerative chatter in machining can be traced to a delay problem in dynamics [7–9]. Mathematically, chatter is a stable limit cycle of the delay differential equation that models the particular machining process. The limit cycles representing chatter typically arise from a Hopf bifurcation of a critical parameter in the DDE [10]. Being able to determine the nature of that limit cycle is crucial to a stable machining operation.

Problems in machine tool chatter often fall into the class of DDEs given by

$$\dot{x} = X(x(t), x(t-h)), \tag{1}$$

where $x, X \in \mathbf{C}^n$, $h > 0$, X sufficiently differentiable, $X(0, 0) = 0$, $\mathbf{C}^n$ is the space of n-dimensional vectors of complex numbers. Several approaches can be taken to study the limit cycles for this class. One can numerically integrate the DDE [11] by using amplitude estimates from bifurcation continuation curves. Another approach would be to develop a collocation solution to the DDE [12, 13]. In these cases the usual error estimates are given in terms of "Big O" of the maximum collocation mesh spacing. In this paper, we take the approach that if a "noncritical" (to be defined later) periodic approximate solution of the DDE can be found, by whatever means, such as Galerkin approximation or harmonic balance, with a "sufficiently small residual error," then an exact periodic solution exists in the neighborhood of the approximate solution and it is possible to compute a numerical estimate of the error between the

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approximate and the exact solutions. This result was established in a general manner for functional differential equations by Stokes [14]. To obtain this result, he first extended an earlier result for ordinary differential equations [15], to autonomous differential equations from the results of Urabe [16] for nonautonomous systems. He then extended the results to functional differential equations in Stokes [14]. Urabe [17] also showed that similar results could be applied to multipoint boundary value problems. Earlier results along this line were also established by Cesari [18]. A similar result has been established for abstract operator equations in Banach spaces by Oishi [19].

The result of Stokes [14] for functional differential equations depends on verifying certain conditions that require computing various parameters. Only recently have algorithms been developed to computationally verify these conditions in the fixed delay case. An announcement of algorithms for computing these parameters was given by Gilsinn [20]. In order to apply Stokes' result, a proof in the case of Equation (1) will be given here since certain inequalities that are developed within the proof are necessary for proving the fixed point contraction mapping conditions. The inequalities rely on the specific form of Equation (1).

The notation used in the paper is described in Section 2. The noncriticality condition is defined in Section 3. In Section 4, we construct an exact frequency and 2π-periodic solution of Equation (1) as a perturbation problem. In Sections 5 and 6, we show that this perturbation problem has a uniqe solution. In particular, in Section 5, we define a map that is used to prove, by a contraction argument, the existence of an exact frequency and 2π-periodic solution of Equation (1). The main contraction theorem is proven in Section 6. In Sections 7 through 10, the necessary algorithms needed to compute the critical parameters for verifying the existence of a 2π-periodic solution of Equation (1) will be given. An algorithm for computing the characteristic multipliers of the variational equation of Equation (1) with respect to the approximate 2π-periodic solution, is outlined in Section 8. An algorithm to determine the solution to the formal adjoint equation with respect to the variational equation of Equation (1) with respect to the approximate 2π-periodic solution, is outlined in Section 9. An algorithm for estimating a critical parameter, M, is given in Section 10. An application of these algorithms to the Van der Pol equation with delay is given in Section 11. Some conclusions are given in Section 12 and acknowledgments in Section 13. Finally, certain bounds and Lipschitz conditions used in the fixed point theorem are proven in Appendices A.1 and A.2.

2 Notation

Let C_ω denote the space of continuous functions from $[-\omega, 0]$ to $\mathbf{C}^n$ with norm in C_ω given by $|\phi| = \max|\phi(s)|$ for $-\omega \le s \le 0$, where

$$|\phi(s)| = \left(\sum_{i=1}^{n} |\phi_i(s)|^2\right)^{1/2}. \tag{2}$$

C_ω is a Banach space with respect to this norm. Let $\mathcal{P}$ be the space of continuous 2π-periodic functions with sup norm, $|\cdot|$, on $(-\infty, \infty)$. Let $\mathcal{P}_1 \subset \mathcal{P}$ be the subspace of continuously differentiable 2π-periodic functions with the sup norm. Let $X(x, y)$, the right-hand side of Equation (1), be continuously differentiable in some domain $\Omega_n \subset \mathbf{C}^n \times \mathbf{C}^n$ with bounded partial derivatives, where

$$|X_i(x, y)| \le \mathcal{B}, \tag{3}$$

for $i = 1, 2, (x, y) \in \Omega_n$. The subscripts of X indicate derivatives with respect to the first and second variables of X, respectively. We further assume that the first partial derivatives satisfy Lipschitz conditions given by

$$|X_i(x_1, y_1) - X_i(x_2, y_2)| \le \mathcal{K}(|x_1 - x_2| + |y_1 - y_2|), \tag{4}$$

for $(x_1, y_1),\ (x_2, y_2) \in \Omega_n$.

Since the period $T = 2\pi/\omega$ of a periodic solution for Equation (1) is unknown we can normalize the period to $[0, 2\pi]$ by introducing the substitution of th/ω for t and rewriting Equation (1) in the form

$$\omega\,\dot{x} = hX(x(t), x(t-\omega)). \tag{5}$$

Since h is simply a rescaling of the system, we will drop it for the rest of the paper.

The problem addressed in this paper is to find an exact 2π-periodic solution, $x(t)$, and an exact frequency ω for Equation (5), given only an approximate 2π-periodic solution, $\hat{x}(t)$, and an approximate frequency

$\hat{\omega}$ for Equation (5) and to compute an error estimate between the two.

For $\psi_1, \psi_2 \in \mathcal{P}$ we denote the total derivative of $X(x, y)$ by

$$dX(x, y; \psi_1, \psi_2) = X_1(x, y)\psi_1 + X_2(x, y)\psi_2. \tag{6}$$

Let $A(t),\ B(t)$ be continuous 2π-periodic matrices. Then a characteristic multiplier is defined as follows.

Definition 1. ρ is a *characteristic multiplier* of

$$\dot{y} = A(t)y(t) + B(t)y(t - \omega), \tag{7}$$

if there is a nontrivial solution $y(t)$ of Equation (7) such that $y(t + 2\pi) = \rho\ y(t)$. Note that if $\rho = 1$ then $y(t)$ is 2π-periodic.

To simplify some of the notation, we will suppress the t and write, for example, $x = x(t),\ x_\omega = x(t - \omega)$, but in other cases we will maintain the t, especially when describing computational steps. We will also at times use the notation

$$|x|_2 = \left[\int_0^{2\pi} |x(t)|^2\, dt\right]^{1/2}. \tag{8}$$

3 Noncriticality condition

Galerkin and harmonic balance methods can be used to develop 2π-periodic approximate solutions for Equation (5). A fast discrete Fourier series algorithm for computing an approximate series solution and frequency, $(\hat{\omega}, \hat{x})$, has been given by Gilsinn [21]. See Section 7 later for a brief discussion of a Galerkin method for approximating a solution. At this point, then, we assume that there exists a 2π-periodic approximate solution and frequency, $(\hat{\omega}, \hat{x})$ for Equation (5), where $\hat{x}$ is 2π-periodic,

$$\hat{\omega}\dot{\hat{x}} = X(\hat{x}, \hat{x}_{\hat{\omega}}) + k, \tag{9}$$

and $k(t)$ is a 2π-periodic residual bounded as follows:

$$|k| \le r. \tag{10}$$

The required size of the residual error, r, will become clear based upon estimates later in this paper. These estimates will indicate, in particular situations, how good an approximate solution and frequency would need to be computed to satisfy the conditions of the main existence theorem in Section 6 later.

The variational equation with respect to the approximate solution and frequency is given by

$$\hat{\omega}\dot{z} = dX\left(\hat{x}, \hat{x}_{\hat{\omega}}; z, z_{\hat{\omega}}\right). \tag{11}$$

Let $\hat{A} = X_1\left(\hat{x}, \hat{x}_{\hat{\omega}}\right),\ \hat{B} = X_2\left(\hat{x}, \hat{x}_{\hat{\omega}}\right)$. The formal adjoint of Equation (11) is given in row form by

$$\hat{\omega}\dot{v} = -v\hat{A} - v_{-\hat{\omega}}\hat{B}. \tag{12}$$

For a more thorough discussion of the adjoint and its properties the reader is referred to Hale [22], Hale and Verduyn Lunel [23], or Diekmann et al. [24].

The next lemma, proven in Halanay [25], establishes the relationship between the number of independent 2π-periodic solutions of Equations (11) and (12).

Lemma 1. *The system, represented by Equations (11) and (12), have the same finite number of independent 2π-periodic solutions.*

We will not give the proof of the next lemma, since it is stated in Hale [22] and in Halanay [25]. The result, however, motivates the definition of a noncritical approximate solution.

Lemma 2. *Let $\rho_0 = 1$ be a simple characteristic multiplier of Equation (11). Let p be a nontrivial solution of Equation (11) associated with ρ_0. Define*

$$J(p, \hat{\omega}) = p + \hat{B}p_{\hat{\omega}}, \tag{13}$$

then

$$\int_0^{2\pi} v_0^{\mathrm{T}} J(p, \hat{\omega})dt \neq 0 \tag{14}$$

for the simple solution, v_0, of the adjoint Equation (12) associated with the adjoint multiplier $1/\rho_0$.

Note that a simple characteristic multiplier $\rho_0 = 1$ implies a single 2π-periodic solution associated with this multiplier and hence a single associated solution to the formal adjoint.

We can now give the definition of a noncritical approximate solution of Equation (5).

Definition 2. The pair $(\hat{\omega}, \hat{x})$, where $\hat{x}$ is at least twice continuously differentiable, is said to be noncritical with respect to Equation (5) if (a) the variational equation about the approximate solution $\hat{x}$, given by Equation (11), has a simple characteristic multiplier ρ_0, that need not be unity, with all of the other characteristic multipliers not equal to one. (b) Furthermore, if v_0, $|v_0|_2 = 1$, is the solution of Equation (12) corresponding to ρ_0, i.e., with multiplier $1/\rho_0$, then we must have that

$$\int_0^{2\pi} v_0^{\mathrm{T}} J(\dot{\hat{x}}, \hat{\omega}) dt \neq 0, \tag{15}$$

where

$$J(\dot{\hat{x}}, \hat{\omega}) = \dot{\hat{x}} + \hat{B}\dot{\hat{x}}_{\hat{\omega}.} \tag{16}$$

The fact that ρ_0 need not be unity for an approximate 2π-periodic solution will be demonstrated in the example in Section 11.

We will not give the proof of the next lemma, since it is also proven in Halanay [25]. The result, however, will be critical to the main approximation theorem.

Lemma 3. *The nonhomogeneous system*

$$\hat{\omega}\dot{x} = \hat{A}x + \hat{B}x_{\hat{\omega}} + f \tag{17}$$

has a unique 2π-periodic solution if and only if

$$\int_0^{2\pi} v_0^{\mathrm{T}} f dt = 0 \tag{18}$$

for all independent solutions v_0 of period 2π of Equation (12). Furthermore there exists an $M > 0$, independent of f, such that

$$|x| \le M|f|. \tag{19}$$

If $(\hat{\omega}, \hat{x})$ is noncritical according to Definition 2, then there will only be one 2π-periodic solution v_0 of Equation (12) that concerns us.

Lemmas 1 and 3 combine to form a part of a result called the Fredholm alternative. There are various forms of Fredholm type of results, but they all address conditions for the solvability of nonhomogeneous systems. In general, determining the number of independent 2π-periodic solutions of the formal adjoint in Lemma 1 is not a trivial problem, but for the case of simple characteristic multipliers this is not a problem. For a more extensive discussion of the Fredholm alternative see Kreyszig [26].

4 A perturbation problem

In this paper, we will look for an exact 2π-periodic solution, x, and an exact frequency, ω, for Equation (5) as a perturbation of the 2π-periodic approximate solution, $\hat{x}$, and approximate frequency, $\hat{\omega}$, of Equation (5). In particular, let

$$\begin{aligned} \omega &= \hat{\omega} + \beta \\ x &= \hat{x} + \frac{\hat{\omega}}{\omega} z. \end{aligned} \tag{20}$$

Note that we do not assume that $\beta > 0$.

Then, substituting Equation (20) into Equation (5), and using Equation (9), we can write the equation for z and β as

$$\hat{\omega}\dot{z} = dX(\hat{x}, \hat{x}_{\hat{\omega}}; z, z_{\hat{\omega}}) + R(z, \beta) - \beta J(\dot{\hat{x}}, \hat{\omega}) - k, \tag{21}$$

where

$$R(z,\beta) = \left[X\left(\hat{x} + \frac{\hat{\omega}}{\omega} z, \hat{x}_{\omega} + \frac{\hat{\omega}}{\omega} z_{\omega}\right) - X(\hat{x}, \hat{x}_{\hat{\omega}}) \right] - dX(\hat{x}, \hat{x}_{\hat{\omega}}; z, z_{\hat{\omega}}) + \beta \hat{B}\dot{\hat{x}}_{\hat{\omega}} \tag{22}$$

and $J(\dot{\hat{x}}, \hat{\omega})$ is given by Equation (16).

In the next lemma, we establish bounds and Lipschitz conditions for $R(z, \beta)$.

Lemma 4. *There exist functions $\mathcal{R}_0(z, \beta) > 0$, $\mathcal{R}_i(z, \beta, \tilde{z}, \tilde{\beta}) > 0$, $i = 1, 2$, such that $\mathcal{R}_0 \to 0$ as $(z, \beta) \to 0$ and $\mathcal{R}_i \to 0$ as $(z, \beta, \tilde{z}, \tilde{\beta}) \to 0$ and*

$$\begin{aligned} |R(z,\beta)| &\le \mathcal{R}_0(z,\beta), \\ |R(z,\beta) - R(\tilde{z},\tilde{\beta})| &\le \mathcal{R}_1(z,\beta,\tilde{z},\tilde{\beta})|z - \tilde{z}| \\ &\quad + \mathcal{R}_2(z,\beta,\tilde{z},\tilde{\beta})|\beta - \tilde{\beta}|. \end{aligned} \tag{23}$$

Proof: Appendix A.1. □

Since we will be considering $|\beta|$ small, we will begin by restricting β, which could be negative, so that

$$\hat{\omega} + \beta \geq \frac{\hat{\omega}}{2}. \tag{24}$$

We can select $|\beta| \leq \hat{\omega}/2$.

As a first step to establishing the existence of a 2π-periodic solution of Equation (21), we first study the existence of a 2π-periodic solution of

$$\hat{\omega}\dot{z} = dX(\hat{x}, \hat{x}_{\hat{\omega}}; z, z_{\hat{\omega}}) + g - \beta J(\dot{\hat{x}}, \hat{\omega}) - k, \tag{25}$$

where $g \in \mathcal{P}$. For this we have the following lemma

Lemma 5. *If $(\hat{\omega}, \hat{x})$ are noncritical with respect to Equation (5), then (a) there exists a unique β such that*

$$g - \beta J(\dot{\hat{x}}, \hat{\omega}) - k \perp v_0, \tag{26}$$

where v_0 is the solution of Equation (12) corresponding to the characteristic multiplier ρ_0 of Equation (11), and (b) there exists a unique 2π-periodic solution of Equation (25) that satisfies

$$|z| \leq M|g - \beta J(\dot{\hat{x}}, \hat{\omega}) - k|, \tag{27}$$

for some $M > 0$.

Proof: Take

$$\beta = \alpha \left[\int_0^{2\pi} v_0^{\mathrm{T}}(g - k)\, dt \right], \tag{28}$$

where

$$\alpha = \left[\int_0^{2\pi} v_0^{\mathrm{T}} J(\dot{\hat{x}}, \hat{\omega})\, dt \right]^{-1} \tag{29}$$

and apply Lemma 3. □

We can now establish bounds on β, z, and $\dot{z}$. For notation, designate the unique β and z in Lemma 5 by $\beta(g)$ and $z(g)$, respectively, and $\dot{z}$ by $\dot{z}(g)$.

Lemma 6. *There exist three constants, designated by λ_i, $i = 0, 1, 2$, such that*

$$\begin{aligned} |\beta(g)| &\leq \lambda_0(|g| + r), \\ |z(g)| &\leq \lambda_1(|g| + r), \\ |\dot{z}(g)| &\leq \lambda_2(|g| + r). \end{aligned} \tag{30}$$

Proof: From

$$\begin{aligned} |g|_2 &\leq \sqrt{2\pi}\ |g|, \\ |k|_2 &\leq \sqrt{2\pi}\ |k|, \end{aligned} \tag{31}$$

and the Cauchy–Schwarz inequality, applied to Equation (28),

$$|\beta(g)| \leq |\alpha||v_0^{\mathrm{T}}|_2|g - k|_2 \leq \sqrt{2\pi}|\alpha|(|g| + r) \tag{32}$$

from the bound $|k| \leq r$.

From Equations (27) and (32),

$$\begin{aligned} |z(g)| &\leq M[|g| + |k| + |\beta(g)||J(\dot{\hat{x}}, \hat{\omega})|], \\ &\leq M[1 + \sqrt{2\pi}|\alpha||J(\dot{\hat{x}}, \hat{\omega})|](|g| + r). \end{aligned} \tag{33}$$

From Equations (25), (32), and (33)

$$\begin{aligned} |\hat{\omega}||\dot{z}(g)| &\leq |dX(\hat{x},\ \hat{x}_{\hat{\omega}}; z(g),\ z(g)_{\hat{\omega}})| \\ &\quad + |g - \beta J(\dot{\hat{x}}, \hat{\omega}) - k| \\ &\leq [\mathcal{B}(|z(g)| + |z(g)_{\hat{\omega}}|)] \\ &\quad + [1 + \sqrt{2\pi}|\alpha||J(\dot{\hat{x}}, \hat{\omega})|](|g| + r) \\ &\leq (2M\mathcal{B})[1 + \sqrt{2\pi}|\alpha||J(\dot{\hat{x}}, \hat{\omega})|](|g| + r). \end{aligned} \tag{34}$$

Therefore, from Equations (29), (32), (33), (34),

$$\begin{aligned} \lambda_0 &= \sqrt{2\pi}|\alpha|, \\ \lambda_1 &= M[1 + \sqrt{2\pi}|\alpha||J(\dot{\hat{x}}, \hat{\omega})|], \\ \lambda_2 &= \frac{\lambda_1}{|\hat{\omega}|M}(1 + 2M\mathcal{B}). \end{aligned} \tag{35}$$

□

λ_1 and λ_2 can be significantly large due to the magnitude of M, as will be shown in the example in Section 11. M is a critical parameter that ultimately affects the size of the residual, r, needed to assure the bounds required in the main theorem in Section 6.

5 A map and its properties

In the main approximation theorem, we will show that the solution of the perturbation problem, represented by Equation (21), is the fixed point of a particular contraction map. In this section, we will define the map and establish some properties.

We begin by defining a subset of $\mathcal{P}$, designated by $\mathcal{N}_\delta$, as

$$\mathcal{N}_\delta = \{g \in \mathcal{P} : |g| \leq \delta\}, \tag{36}$$

where $\delta > 0$. Following Stokes [14] we will define a map $S\colon \mathcal{N}_\delta \to \mathcal{P}$ in terms of two mappings

$$\begin{aligned} &L\colon \mathcal{N}_\delta \to R \times \mathcal{P}_1, \\ &T\colon R \times \mathcal{P}_1 \to \mathcal{P}. \end{aligned} \tag{37}$$

To define L, let $g \in \mathcal{N}_\delta$, then Lemma 5 assures us of the existence of a unique $\beta(g)$ satisfying Equation (26) and a unique solution $z(g)$ satisfying Equation (25). Thus, define $L\colon \mathcal{N}_\delta \to R \times \mathcal{P}_1$ by

$$L(g) = (\beta(g), z(g)). \tag{38}$$

Now define $T\colon\ R \times \mathcal{P}_1 \to \mathcal{P}$ by

$$T(\beta,\ z) = R(z,\ \beta). \tag{39}$$

Finally, define $S\colon\ \mathcal{N}_\delta \to \mathcal{P}$ by

$$S(g) = T(L(g)) = R(z(g), \beta(g)). \tag{40}$$

Lemma 7. *For $g \in \mathcal{N}_\delta,\ \tilde{g} \in \mathcal{N}_\delta$ there exist two functions $E_1(\delta),\ E_2(\delta)$ and two positive constants $F_1,\ F_2$ so that*

$$|S(g)| \leq E_1(\delta), \tag{41}$$

$$|S(g) - S(\tilde{g})| \leq E_2(\delta)\, |g - \tilde{g}|\,,$$

where

$$\begin{aligned} &E_1(\delta) \leq F_1\delta^2, \\ &E_2(\delta) \leq F_2\delta. \end{aligned} \tag{42}$$

Proof: From Equations (40) and (23), we have

$$\begin{aligned} &|S(g)| \leq \mathcal{R}_0(z(g), \beta(g)), \\ &|S(g) - S(\tilde{g})| \\ &\quad \leq \mathcal{R}_1(z(g), \beta(g), z(\tilde{g}), \beta(\tilde{g}))|z(g) - z(\tilde{g})| \\ &\quad + \mathcal{R}_2(z(g), \beta(g), z(\tilde{g}), \beta(\tilde{g}))|\beta(g) - \beta(\tilde{g})|. \end{aligned} \tag{43}$$

By Cauchy–Schwarz, the fact that $|v_0^{\mathrm{T}}|_2 = 1$, and Equation (35)

$$\begin{aligned} |\beta(g) - \beta(\tilde{g})| &\leq |\alpha| \int_0^{2\pi} |v_0^{\mathrm{T}}(g - \tilde{g})|\, dt, \\ &\leq |\alpha| [\int_0^{2\pi} |g - \tilde{g}|^2\, dt]^{1/2}, \\ &\leq \lambda_0 |g - \tilde{g}|. \end{aligned} \tag{44}$$

From Lemma 5 and the definition of $\beta(g),\ \beta\,(\tilde{g})$ we have

$$\begin{aligned} &\int_0^{2\pi} v_0^{\mathrm{T}}(g - \beta(g)J(\dot{\hat{x}}, \hat{\omega}) - k)\, dt = 0, \\ &\int_0^{2\pi} v_0^{\mathrm{T}}(\tilde{g} - \beta(\tilde{g})J(\dot{\hat{x}}, \hat{\omega}) - k)\, dt = 0. \end{aligned} \tag{45}$$

Then, by subtracting,

$$\int_0^{2\pi} v_0^{\mathrm{T}}[(g - \tilde{g}) - (\beta(g) - \beta(\tilde{g}))J(\dot{\hat{x}}, \hat{\omega})]\, dt = 0. \tag{46}$$

Lemma 5 also shows that there exists a unique $\bar{z}$ such that

$$\begin{aligned} \hat{\omega}\dot{\bar{z}} = dX(\hat{x}, \hat{x}_{\hat{\omega}}; \bar{z}, \bar{z}_{\hat{\omega}}) + [(g - \tilde{g}) - (\beta(g) \\ - \beta(\tilde{g}))J(\dot{\hat{x}}, \hat{\omega})]. \end{aligned} \tag{47}$$

But from Equation (25), $z(g) - z\,(\tilde{g})$ also satisfies Equation (47), so that $\bar{z} = z(g) - z\,(\tilde{g})$ and from

Equation (27)

$$|z(g)-z(\tilde{g})| \leq M|(g-\tilde{g})-(\beta(g)-\beta(\tilde{g}))\, J(\dot{\hat{x}}, \hat{\omega})|, \\ \leq \lambda_1|g-\tilde{g}|. \quad (48)$$

Then Equation (41) follows from Equation (43) through Equation (48). □

The specific forms for $E_1(\delta)$ and $E_2(\delta)$ are given in Appendix A.2 as Equations (127) and (131), respectively, as well as the selection of F_1 and F_2 as Equations (129) and (133), respectively. As functions of the other parameters, $E_1(\delta)$ and $E_2(\delta)$ depend linearly on $\mathcal{K}$ and $\mathcal{B}$, but nonlinearly on λ_0, λ_1, and λ_2 and thus nonlinearly on M.

6 Main approximation theorem

In the main theorem, the constants F_1, F_2 are those from Lemma 7

Theorem 1. *If (a) $(\hat{\omega}, \hat{x})$ is noncritical with respect to Equation (5) in the sense of Definition 2, (b) δ is selected so that*

$$\delta \leq \min\{1/F_1, \quad 1/2F_2, \quad \hat{\omega}/4\lambda_0\}, \quad (49)$$

and (c) $r \leq \delta$, see (9), (10), then there exists an exact frequency, ω^*, and solution, x^*, of (5) such that

$$|x^*-\hat{x}| \leq 4\lambda_1\delta, \\ |\omega^*-\hat{\omega}| \leq 2\lambda_0\delta, \quad (50)$$

where λ_0, λ_1 are defined in Equation (35) and δ is defined in Equation (36).

Proof: Let β and z be defined as in Equation (20). By substituting Equation (20) into Equation (5) we have Equation (21). Associated with Equation (21) we consider Equation (25). We then define the set $\mathcal{N}_\delta$ in Equation (36) and consider the map $S\colon \mathcal{N}_\delta \to \mathcal{P}$ defined in Equation (40). From Equations (126), (127), and (129) we have $|S(g)| \leq F_1\delta^2$ for $g \in \mathcal{N}_\delta$. Furthermore, from Equations (130), (131), and (133) we have, for g, $\tilde{g} \in \mathcal{N}_\delta$, that $|S(g)-S(\tilde{g})| \leq F_2\delta\,|g-\tilde{g}|$. Now, if we select δ as in Equation (49) then $F_1\delta^2 \leq \delta$ and $F_2\delta \leq 1/2$, S maps $\mathcal{N}_\delta$ to itself and is a contraction. The last inequality that δ satisfies in Equation (49) assures that $\beta(g)$ satisfies Equation (24) by way of Lemma 6, provided r satisfies $r \leq \delta$. Therefore, S has a fixed point $g^* \in \mathcal{N}_\delta$. This implies that there exists a unique (β^*, z^*), z^* is 2π-periodic, satisfying Equation (21). Then, from Equation (20), there exists a unique (ω^*, x^*), x^* is 2π-periodic, satisfying Equation (5). From Equation (20), with $r \leq \delta$,

$$|\omega^*-\hat{\omega}| \leq |\beta^*| \leq \lambda_0(|g^*|+r) \leq 2\lambda_0\delta, \\ |x^*-\hat{x}| \leq \left|\frac{\hat{\omega}}{\hat{\omega}+\beta^*}\right| |z^*| \leq 2\lambda_1(|g^*|+r) \leq 4\lambda_1\delta. \quad (51)$$

□

We need to introduce a note here on the relationship between r and δ. In practice, the process of determining them is iterative. We start by determining an approximate solution and the residual r. We then compute all of the parameters that involve the approximate solution and compute δ from Equation (49). We then compare r against δ. If $r \leq \delta$ we are finished, otherwise we have to return and recompute another approximate solution with possibly smaller residual r and iterate the process. The author is not familiar with any result that guarantees that at some point $r \leq \delta$, although he suspects that this will eventually happen in most practical problems.

7 Approximating a solution and frequency

An approximate solution and frequency for Equation (5) can be developed by assuming a finite trigonometric polynomial of the form

$$\hat{x}_m = a_2 \cos t + \sum_{n=2}^{m} [a_{2n} \cos nt + a_{2n-1} \sin nt], \quad (52)$$

where the $\sin t$ term has been dropped so that we can estimate $a_1 = \hat{\omega}$, the frequency. Computational experience has suggested that dropping a low-order harmonic term provides a smaller residual estimate for Equation (5).

Note that we have centered the approximate solution about the origin, since we assumed $X(0, 0) = 0$. If we set $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_{2m})$ and

$$E_m(t, \bar{\mathbf{a}}) = a_1\dot{\hat{x}}_m(t) - X(\hat{x}_m(t), \hat{x}_m(t-a_1)), \quad (53)$$

then for a sufficiently fine mesh, specified by $\{t_i : i = 1, 2, \ldots, 2N\}$, in $[0, 2\pi]$, where

$$t_i = \frac{2i-1}{2N}\pi, \tag{54}$$

the determining equations for $\bar{\mathbf{a}}$ can be written as (see Urabe and Reiter [27])

$$\begin{aligned} F_1(\bar{\mathbf{a}}) &= \frac{1}{N}\sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \sin t_i = 0, \\ F_2(\bar{\mathbf{a}}) &= \frac{1}{N}\sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \cos t_i = 0, \\ F_{2n-1}(\bar{\mathbf{a}}) &= \frac{1}{N}\sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \sin nt_i = 0, \\ F_{2n}(\bar{\mathbf{a}}) &= \frac{1}{N}\sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \cos nt_i = 0, \end{aligned} \tag{55}$$

for $n = 2, \ldots, m$.

These equations give $2m$ equations in $2m$ unknowns. Standard numerical solvers, using, for example, Newton's method, for nonlinear equations can be used to solve for $\bar{\mathbf{a}}$. The number of harmonics, m, and the quadrature index, N, can be selected independently. Gilsinn [21] presents a vectorized algorithm for solving for $\bar{a}$.

8 Estimating characteristic multipliers

In this section, we assume that the variational equation, Equation (11), with respect to the approximate solution, $\hat{x}(t)$, can be written in the form

$$\dot{z}(t) = \hat{A}(t)z(t) + \hat{B}(t)z(t-\hat{\omega}), \tag{56}$$

where $\hat{A}(t) = \hat{A}(t+2\pi)$, $\hat{B}(t) = \hat{B}(t+2\pi)$ and we have reintroduced t to make the operator definitions more transparent. $\hat{A}$, $\hat{B}$ now include the scale factor $1/\hat{\omega}$. Let $Z(t, s)$ be the solution of Equation (56) such that $Z(s, s) = I_n$, $Z(t, s) = 0$ for $t < s$ where I_n is the n^2 identity matrix on $\mathbf{C}^n$. The solution $Z(t, s)$ is sometimes referred to as the "Fundamental Solution." Using the variation of constants formula for Equation (56), Halanay [25] shows that the solution of Equation (56) for the initial function $\phi \in C_{\hat{\omega}}$ is given by

$$z(t) = Z(t,0)\phi(0) + \int_{-\hat{\omega}}^{0} Z(t, \alpha+\hat{\omega})\hat{B}(\alpha+\hat{\omega})\phi(\alpha)\, d\alpha. \tag{57}$$

Define the operator

$$(U\phi)(s) = z(s+2\pi), \tag{58}$$

where $\phi \in C_{\hat{\omega}}$, $s \in [-\hat{\omega}, 0]$. The range of U is again $C_{\hat{\omega}}$. If there is a nontrivial solution $z(t)$ of Equation (56) such that $z(t+2\pi) = \rho z(t)$, then ρ is a characteristic multiplier of Equation (56). If we combine Equation (57) with Equation (58) and note that $z(\alpha) = \phi(\alpha)$ for $\alpha \in [-\hat{\omega}, 0]$, then the characteristic multipliers are the eigenvalues of

$$\begin{aligned} (U\phi)(s) = {} & Z(s+2\pi, 0)\phi(0) \\ & + \int_{-\hat{\omega}}^{0} Z(s+2\pi, \alpha+\hat{\omega})\hat{B}(\alpha+\hat{\omega})\phi(\alpha)\, d\alpha, \end{aligned} \tag{59}$$

where $\phi \in C_{\hat{\omega}}$. Halanay [25] shows that we can restrict $s \in [-\hat{\omega}, 0]$. This operator is sometimes referred to as the *Monodromy Operator*.

8.1 Estimating the fundamental solution

Since a detailed discussion of the algorithm for estimating the fundamental solution is given in Gilsinn and Potra [28], we will only outline the major steps here. A collocation method based on spectral methods will be used here. Although collocation methods for delay differential equations are well known (see Engelborghs et al. [12]), one of the difficulties in applying them arises in estimating the differentiation matrix on the left-hand side in a sufficiently rapid manner and in such a way that it maintains stability in the presence of roundoff. For a discussion of differentiation matrices see, for example, Baltensperger and Berut [29], Bayliss et al. [30], Solomonoff [31], Trefethen [32] and Welfert [33]. In this paper, we use a collocation method motivated by the pseudospectral methods of Gottlieb and Turkel [34], and Gottlieb et al. [35], and Trefethen [32]. Trefethen [32] points out that the reason spectral methods are so accurate for smooth functions is that their Fourier transforms decay rapidly. He also notes that this implies that aliasing errors introduced by discretization are small. In this method, the differentiation matrix is known exactly for Lagrange polynomials evaluated at Chebyshev extreme points. To compute the differentiation matrix in the example in Section 11 we used a function, called "cheb", given in Trefethen [32], which produced a matrix with stability in the presence of roundoff.

The method of steps is used here to solve the variational equation on $[0,\ 2\pi]$. To begin, we find a positive integer q so that $q\hat{\omega} \geq 2\pi$. Then, we consider the set of intervals $[0, \hat{\omega}],\ [\hat{\omega}, 2\hat{\omega}], \ldots, [(q-1)\hat{\omega}, q\hat{\omega}]$. On the first interval, we solve

$$\dot{z}_1(t) = \hat{A}(t)z_1(t) + \hat{B}z_1(t-\hat{\omega}), \tag{60}$$

where $z_1(t-\hat{\omega}) = \phi(s)$ for some initial function $\phi \in C_{\hat{\omega}},\ s = t - \hat{\omega}$. To compute the fundamental solution on the first interval, we solve n problems with $\phi(s) = 0$ and n initial conditions $z_1(0) = e_i,\ \ i = 1, 2, \ldots, n$ where $e_i = (0, 0, \ldots, 1, \ldots, 0)$ and 1 is in the i-th position. We then continue for $j = 2, \ldots, q$ and solve n problems at the j-th step

$$\dot{z}_j(t) = \hat{A}(t)z_j(t) + \hat{B}z_j(t-\hat{\omega}), \tag{61}$$

where $z_j((j-1)\hat{\omega}) = z_{j-1}((j-1)\hat{\omega})$. The final fundamental solution is the concatenation of all of the $z_j(t),\ j = 1, 2, \ldots, q$.

The solution at each step can be computed by a collocation procedure using Chebyshev points on $[-1, 1]$. To do this, each equation in the step sequence, $j = 1, 2, \ldots, q$ is transformed by

$$t = \frac{\hat{\omega}}{2}\eta + \frac{(2j-1)}{2}\hat{\omega} \tag{62}$$

to a sequence of ordinary differential equations

$$u_j'(\eta) = \frac{\hat{\omega}}{2}\hat{A}_j(\eta)u_j(\eta) + \frac{\hat{\omega}}{2}\hat{B}_j(\eta)u_{j-1}(\eta), \tag{63}$$

where, for $t \in [(j-1)\hat{\omega}, j\hat{\omega}]$,

$$\begin{aligned} &u_j(-1) = u_{j-1}(1), \quad u_j(\eta) = z_j(t), \\ &\hat{A}_j(\eta) = \hat{A}(t), \quad \hat{B}_j(\eta) = \hat{B}(t), \\ &u_{j-1}(\eta) = z_j(t-\hat{\omega}). \end{aligned} \tag{64}$$

Then, an N-degree Lagrange polynomial is interpolated through the $N+1$ Chebyshev extreme points

$$\eta_k = \cos\left(\frac{k\pi}{N}\right), \quad k = 0, 1, \ldots, N. \tag{65}$$

A sum of Lagrange polynomials with N unknown coefficients is substituted into the ordinary differential equation, represented by Equation (63), and the coefficients are solved for by matrix algebra. There are sets of coefficients for each of the time step intervals. The final solution on $[0, 2\pi]$ is the concatenation in the time domain of the q sets of Lagrange polynomials. In particular, $Z(t, \alpha)$ is estimated by the concatenation of q matrices $Z_i(t, \alpha),\ i = 1, 2, \ldots, q$, where each can be written in the form

$$Z_i(t, \alpha) = \sum_{k=0}^{N} W_k^{(i)} l_k\left(\frac{2}{\hat{\omega}}t - \frac{2\alpha + (2i-1)\hat{\omega}}{\hat{\omega}}\right). \tag{66}$$

For full details and proof of convergence, see Gilsinn and Potra [28].

8.2 Estimating the monodromy operator eigenvalues

To approximate the monodromy operator, represented by Equation (59), we will require a quadrature rule that satisfies

$$\sum_{k=1}^{P+1} w_k f(s_k) \to \int_{-\hat{\omega}}^{0} f(s)\,ds \tag{67}$$

as $P \to \infty$, for each continuous function $f \in C_{\hat{\omega}}$. The rule is satisfied if

$$\sum_{k=1}^{P+1} |w_k| \leq Q, \tag{68}$$

for some $Q > 0$ and $P = 1,\ 2, \ldots$. This is satisfied by, for example, Trapezoidal or Simpson rules.

Let $-\hat{\omega} = s_1 < s_2 < \cdots < s_{P+1} = 0$, and define

$$\begin{aligned} (U\phi)(s) = {} & Z(s+2\pi, 0)\phi(0) \\ & + \sum_{k=1}^{P+1} w_k Z(s+2\pi, s_k+\hat{\omega})B(s_k+\hat{\omega})\phi(s_k) \end{aligned} \tag{69}$$

for $\phi \in C_{\hat{\omega}}$.

Then, for each $s_i \in [-\hat{\omega}, 0]$,

$$(U\phi)(s_i) = Z(s_i + 2\pi, 0)\phi(0) + \sum_{j=1}^{P+1} w_j Z(s_i + 2\pi, s_j + \hat{\omega}) B(s_j + \hat{\omega})\phi(s_j). \tag{70}$$

Since $s_{P+1} = 0$, Equation (70) can be rewritten as

$$\begin{aligned}(U\phi)(s_i) = {} & \sum_{j=1}^{P} w_j Z(s_i + 2\pi, s_j + \hat{\omega}) B(s_j + \hat{\omega})\phi(s_j) \\ & + (Z(s_i + 2\pi, 0) + w_{P+1} Z(s_i \\ & + 2\pi, \hat{\omega}) B(\hat{\omega}))\phi(s_{P+1}),\end{aligned} \tag{71}$$

where $Z(s, \alpha)$ is the fundamental matrix of Equation (60). Equation (71) can be put in matrix form

$$\begin{pmatrix} (U\phi)(s_1) \\ \vdots \\ (U\phi)(s_i) \\ \vdots \\ (U\phi)(s_{P+1}) \end{pmatrix} = \begin{bmatrix} U_{1,1} & \cdots & U_{1,j} & \cdots & U_{1,P+1} \\ \vdots & \dots & \vdots & \dots & \vdots \\ U_{i,1} & \cdots & U_{i,j} & \cdots & U_{i,P+1} \\ \vdots & \dots & \vdots & \dots & \vdots \\ U_{P+1,1} & \cdots & U_{P+1,j} & \cdots & U_{P+1,P+1} \end{bmatrix}, \tag{72}$$

where the block elements for $i = 1, \dots, P+1$, $j = 1, \dots, P$ are $U_{i,j} = w_j Z(s_i + 2\pi, s_j + \hat{\omega}) B(s_j + \hat{\omega})$. The block elements in the last column of the matrix are given by $U_{i,P+1} = Z(s_i + 2\pi, 0) + w_{P+1} Z(s_i + 2\pi, \hat{\omega}) B(\hat{\omega})$ for $i = 1, \dots, P+1$. The relevant eigenvalue problem becomes

$$\begin{bmatrix} U_{1,1} & \cdots & U_{1,j} & \cdots & U_{1,P+1} \\ \vdots & \dots & \vdots & \dots & \vdots \\ U_{i,1} & \cdots & U_{i,j} & \cdots & U_{i,P+1} \\ \vdots & \dots & \vdots & \dots & \vdots \\ U_{P+1,1} & \cdots & U_{P+1,j} & \cdots & U_{P+1,P+1} \end{bmatrix} = \lambda \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix}. \tag{73}$$

A convergence proof of the eigenvalues is also given in Gilsinn and Potra [28].

9 Determining solutions of the adjoint equation associated with multipliers of the variational equation

In order to estimate α in Equation (29), let $t \in [0, 2\pi]$ and ψ be the initial function defined on $[2\pi, 2\pi + \hat{\omega}]$. The adjoint equation is given by

$$\dot{y}(t) = -y(t)\hat{A}(t) - y(t + \hat{\omega})\hat{B}(t + \hat{\omega}), \tag{74}$$

where $y(t)$ is a row vector. Ordinarily, solving the adjoint equation would require a "backward" integration. However, Halanay [25] showed that the solution of the adjoint on $[0, 2\pi]$ is given in row vector form by

$$y(t) = \psi(2\pi) Z(2\pi, t) + \int_{2\pi}^{2\pi+\hat{\omega}} \psi(\alpha)\hat{B}(\alpha) Z(\alpha - \hat{\omega}, t)\, d\alpha. \tag{75}$$

The significance of this representation is that only a "forward" integration is required to solve for the fundamental solution, Z, of Equation (56). This allows us to directly use the collocation algorithm developed in Section 8.1.

Let $\tilde{\phi}(s)$ be a continuous row vector function defined on $[-\hat{\omega}, 0]$. Then define the operator

$$\left(\tilde{U}\tilde{\phi}\right)(s) = \tilde{\phi}(-\hat{\omega})Z(2\pi, s+\hat{\omega}) + \int_{-\hat{\omega}}^{0} \tilde{\phi}(\alpha)\hat{B}(\alpha+\hat{\omega})Z(2\pi+\alpha, s+\hat{\omega})\,d\alpha \tag{76}$$

$s \in [-\hat{\omega}, 0]$. An associated operator $\tilde{V}$, defined on $[2\pi, 2\pi+\hat{\omega}]$, is given in Halanay [25] as

$$(\tilde{V}\psi)(s) = y(s-2\pi, \psi) = \psi(2\pi)Z(2\pi, s-2\pi) + \int_{2\pi}^{2\pi+\hat{\omega}} \psi(\alpha)\hat{B}(\alpha)Z(\alpha-\hat{\omega}, s-2\pi)\,d\alpha. \tag{77}$$

Halanay [25] also showed that an eigenvalue ρ of $\tilde{V}$ is associated with a $1/\rho$ multiplier of the adjoint equation, the eigenvalues of $U,\ \tilde{U},\ \tilde{V}$ are all the same, and the eigenvectors of $\tilde{U},\ \tilde{V}$ are related by $\tilde{\phi}(s) = \psi(s+2\pi+\hat{\omega}),\ s \in [\hat{\omega}, 0]$. It turns out then, to solve the adjoint equation in row form on $[0, 2\pi]$, we need only compute the significant eigenvalue and eigenvector of $\tilde{U}$. Therefore, using quadratures, we discretize $\tilde{U}$ by setting $-\hat{\omega} = s_1 < \cdots < s_{P+1} = 0,\ s_{i+1} - s_i = \hat{\omega}/P,\ i = 1, \ldots, P$. The *j*-th block column is given by

$$\left(\tilde{U}\tilde{\phi}\right)(s_j) = \left[\tilde{\phi}(s_1), \cdots, \tilde{\phi}(s_i), \cdots, \tilde{\phi}(s_{P+1})\right] \tag{78}$$
$$\begin{bmatrix} Z(2\pi, s_j+\hat{\omega}) + \hat{B}(s_1+\hat{\omega})Z(s_1+2\pi, s_j+\hat{\omega})w_j \\ \vdots \\ \hat{B}(s_i+\hat{\omega})Z(s_i+2\pi, s_j+\hat{\omega})w_j \\ \vdots \\ \hat{B}(s_{P+1}+\hat{\omega})Z(s_{P+1}+2\pi, s_j+\hat{\omega})w_j \end{bmatrix}.$$

The eigenvector $\tilde{\phi}$ of the matrix on the right, associated with the multiplier of the variational equation, is computed and substituted into the discretized form of Equation (75) to give the value of $y(t)$ on the partition $0 = t_1 < \cdots < t_{O+1} = 2\pi,\ \ t_{i+1} - t_i = 2\pi/O$, $i = 1, \ldots, O$, as

$$y(t_j) = \left[\tilde{\phi}(s_1), \ldots, \tilde{\phi}(s_i), \ldots, \tilde{\phi}(s_{P+1})\right] \tag{79}$$
$$\begin{bmatrix} Z(2\pi, t_j) + \hat{B}(s_1+2\pi+\hat{\omega})Z(s_1+2\pi, t_j)w_j \\ \vdots \\ \hat{B}(s_i+2\pi+\hat{\omega})Z(s_i+2\pi, t_j)w_j \\ \vdots \\ \hat{B}(s_{P+1}+2\pi+\hat{\omega})Z(s_{P+1}+2\pi, t_j)w_j \end{bmatrix},$$

using $\tilde{\phi}(s) = \psi\,(s+2\pi+\hat{\omega})\,,\ s \in [-\hat{\omega}, 0]$.

We then can estimate α by

$$\alpha = \left[\sum_{j=1}^{O+1} w_j y(t_j) J\left(\dot{\hat{x}}, \hat{\omega}\right)\left(t_j\right)\right]^{-1}. \tag{80}$$

Note that α may be complex but in the final error estimates we only use $|\alpha|$. See Appendix A.2, Section 12 and Gilsinn [20] for further details.

10 Estimating the M parameter

From Halanay [25], the variation of constants formula for

$$\dot{z}(t) = \hat{A}(t)z(t) + \hat{B}(t)z(t-\hat{\omega}) + f(t), \tag{81}$$

where $t \in [0, 2\pi]$, is given by

$$z(t) = Z(t, 0)\phi(0) + \int_{-\hat{\omega}}^{0} Z(t, \alpha+\hat{\omega})\hat{B}(\alpha+\hat{\omega})z(\alpha)\,d\alpha + \int_{0}^{t} Z(t, \alpha)f(\alpha)\,d\alpha. \tag{82}$$

The 2π-periodic initial function condition with $s \in [-\hat{\omega}, 0]$ is

$$\phi(s) = Z(s+2\pi, 0)\phi(0) + \int_{-\hat{\omega}}^{0} Z(s+2\pi, \alpha+\hat{\omega})\hat{B}(\alpha+\hat{\omega})\phi(\alpha)\,d\alpha + \int_{0}^{s+2\pi} Z(s+2\pi, \alpha)f(\alpha)\,d\alpha. \tag{83}$$

The first step in computing M involves relating ϕ to f. Let $|\phi| = \sup_{-\hat{\omega} \le s \le 0} |\phi(s)|$ and similarly for $|f|$ on

$[0, 2\pi]$. To eliminate $\phi(0)$ from Equation (83), set $s = 0$ in Equation (83) and solve for $\phi(0)$ as

$$\phi(0) = \int_{-\hat{\omega}}^{0} (I - Z(2\pi, 0))^{-1} Z(2\pi, \alpha + \hat{\omega}) \times \hat{B}(\alpha + \hat{\omega})\phi(\alpha)\, d\alpha + \int_{0}^{2\pi} (I - Z(2\pi, 0))^{-1} Z(2\pi, \alpha) f(\alpha)\, d\alpha. \tag{84}$$

Substitute Equation (84) into Equation (83) and combine terms as

$$\phi(s) = \int_{-\hat{\omega}}^{0} \big[Z(s+2\pi, 0)(I-Z(2\pi, 0))^{-1} Z(2\pi, \alpha+\hat{\omega}) + Z(s + 2\pi, \alpha + \hat{\omega})]\, \hat{B}(\alpha + \hat{\omega})\phi(\alpha)\, d\alpha + \int_{0}^{2\pi} \big[Z(s + 2\pi, 0)(I - Z(2\pi, 0))^{-1} Z(2\pi, \alpha) + Z(s + 2\pi, \alpha)\big] f(\alpha)\, d\alpha, \tag{85}$$

where $s \in [-\hat{\omega}, 0]$.

Let $-\hat{\omega} = s_1 < s_2 < \cdots < s_{P+1} = 0$, $s_{i+1} - s_i = \frac{\hat{\omega}}{P}$, $i = 1, \ldots, P$, and $0 = t_1 < t_2 < \cdots < t_{O+1} = 2\pi$, $t_{j+1} - t_j = \frac{2\pi}{O}$, $j = 1, \ldots, O$. We can discretize Equation (85) by setting

$$\phi(s_i) = \sum_{j=1}^{P+1} H_1(i, j)\phi(s_j) + \sum_{k=1}^{O+1} H_2(i, j) f(t_k), \tag{86}$$

where v_j, $j = 1, \ldots, P + 1$, and u_k, $k = 1, \ldots, O + 1$ are appropriate quadrature coefficients, and

$$H_1(i, j) = v_j[Z(s_i + 2\pi, 0)(I - Z(2\pi, 0))^{-1} Z(2\pi, s_j + \hat{\omega}) + Z(s_i + 2\pi, s_j + \hat{\omega})]\hat{B}(s_j + \hat{\omega}),$$

$$H_2(i, j) = u_k\big[Z(s_i + 2\pi, 0)(I - Z(2\pi, 0))^{-1} Z \times (2\pi, t_k) + Z(s_i + 2\pi, t_k)\big]. \tag{87}$$

In vector matrix form Equation (86) can be written

$$\begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix} = H_1 \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix} + H_2 \begin{pmatrix} f(t_1) \\ \vdots \\ f(t_{O+1}) \end{pmatrix}. \tag{88}$$

Using a generalized inverse we can solve for the ϕ vector with minimum norm by

$$\begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix} = (I - H_1)^{+} H_2 \begin{pmatrix} f(t_1) \\ \vdots \\ f(t_{O+1}) \end{pmatrix}. \tag{89}$$

In the second step, the value of $\phi(0)$, given by Equation (84), is substituted into Equation (82) and terms combined to give

$$z(t) = \int_{-\hat{\omega}}^{0} \big[Z(t, 0)(I - Z(2\pi, 0))^{-1} Z(2\pi, \alpha + \hat{\omega}) + Z(t, \alpha + \hat{\omega})]\, \hat{B}(\alpha + \hat{\omega})\phi(\alpha)\, d\alpha + \int_{0}^{2\pi} \big[Z(t, 0)(I - Z(2\pi, 0))^{-1} Z(2\pi, \alpha) + Z(t, \alpha)]\, f(\alpha)\, d\alpha. \tag{90}$$

This can be discretized by setting

$$z(t_k) = \sum_{i=1}^{P+1} H_3(k, i)\phi(s_i) + \sum_{j=1}^{O+1} H_4(k, j) f(t_k), \tag{91}$$

where

$$H_3(k, i) = v_i[Z(t_k, 0)(I-Z(2\pi, 0))^{-1} Z(2\pi, s_i+\hat{\omega}) + Z(t_k, s_i + \hat{\omega})]\hat{B}(s_i + \hat{\omega}), \tag{92}$$

$$H_4(k, j) = u_j[Z(t_k, 0)(I - Z(2\pi, 0))^{-1} Z\left(2\pi, t_j\right) + Z(t_k, t_j)].$$

In vector matrix form, Equation (91) can be written

$$\begin{pmatrix} z(t_1) \\ \vdots \\ z(t_{O+1}) \end{pmatrix} = H_3 \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix} + H_4 \begin{pmatrix} f(t_1) \\ \vdots \\ f(t_{O+1}) \end{pmatrix}. \tag{93}$$

By substituting Equation (89) into Equation (93) we have

$$\begin{pmatrix} z(t_1) \\ \vdots \\ z(t_{O+1}) \end{pmatrix} = \left[H_3\,(I-H_1)^+\,H_2+H_4\right] \begin{pmatrix} f(t_1) \\ \vdots \\ f(t_{O+1}) \end{pmatrix}. \tag{94}$$

Therefore

$$|z| \le M|f|, \tag{95}$$

where $M = \|H_3(I-H_1)^+H_2 + H_4\|_\infty$.

11 Application to a Van der Pol equation with delay

In this section, we will apply the main theorem to approximate the limit cycle of the Van der Pol equation with unit delay, given by

$$\ddot{x} + \lambda(x(t-1)^2 - 1)\dot{x}(t-1) + x = 0. \tag{96}$$

Since the period of the limit cycle is unknown, we introduce an unknown frequency by substituting t/ω for t to obtain

$$\omega^2\ddot{x} + \omega\lambda(x(t-\omega)^2 - 1)\dot{x}(t-\omega) + x = 0, \tag{97}$$

for $t \in [0, 2\pi]$. To compare with an approximation result obtained in Stokes [15], we take $\lambda = 0.1$.

The first step was to estimate an approximate 2π-periodic solution, frequency, and residual to Equation (97). By using Galerkin's method described in Section 7, the following approximate solution was obtained

$$\begin{aligned} \hat{x}(t) &= 2.0185\cos(t) \\ &\quad + 2.5771\times10^{-3}\sin(2t) + 2.5655\times10^{-2}\cos(2t) \\ &\quad + 1.0667\times10^{-4}\sin(3t) - 5.2531\times10^{-4}\cos(3t) \\ &\quad - 7.1780\times10^{-6}\sin(4t) - 2.2043\times10^{-6}\cos(4t), \\ \hat{\omega} &= 1.0012. \end{aligned} \tag{98}$$

where we have displayed only the first few harmonics. This solution was estimated based on 11 harmonics, 40,000 sampled points over $[0, 2\pi]$, and 100 Chebyshev extreme points, represented by Equation (65). The residual was estimated by substituting $(\hat{\omega}, \hat{x})$ from Equation (98) into Equation (97) and finding the maximum of the absolute values of the residuals obtained in the interval $[0, 2\pi]$. The result was $r = 3.1086 \times 10^{-15}$. This residual is significantly better than the one given in Stokes [15]. The distribution of the residuals for the current case is shown in Fig. 1. The phase plot of the approximate solution is shown in Fig. 2. For $t \in [0, 2\pi]$, we can then immediately estimate $|\hat{x}| \le 2.0436$, $|\dot{\hat{x}}| \le 2.0279$, $|\ddot{\hat{x}}| \le 2.1165$.

In the second step, the values of the constants $\mathcal{B}$ and $\mathcal{K}$ were obtained in a straightforward manner from the

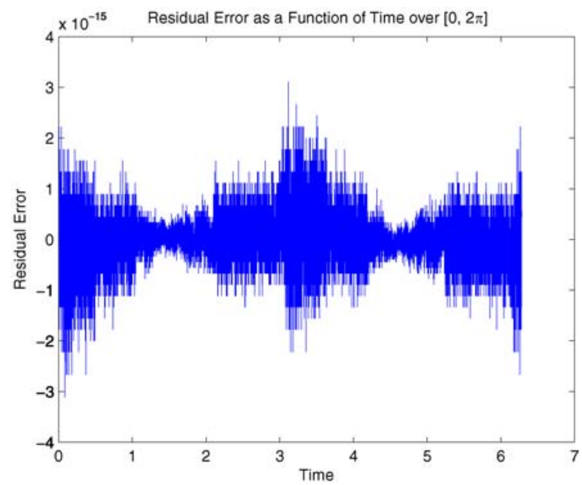

Fig. 1 Residual error of approximate solution for the Van der Pol equation

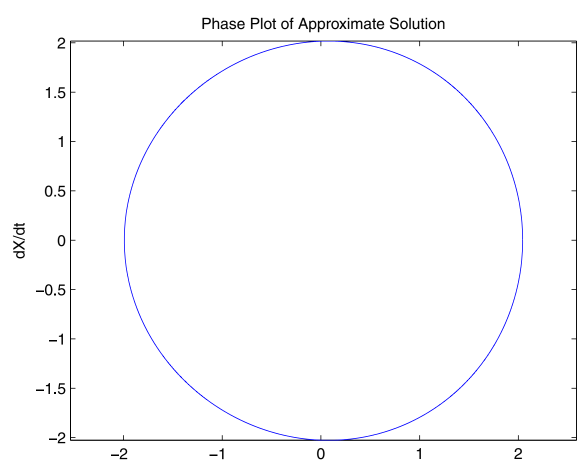

Fig. 2 Phase plot of approximate solution for the Van der Pol equation

variational equation about the approximate frequency and solution given by

$$\dot{z}(t) = \hat{A}(t)z(t) + \hat{B}(t)Z(t-\hat{\omega}), \tag{99}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \hat{A}(t) = \begin{pmatrix} 0 & 1 \\ -1/\hat{\omega}^2 & 0 \end{pmatrix},$$

$$\hat{B}(t) = \begin{pmatrix} 0 & 0 \\ -2(\lambda/\hat{\omega})\hat{x}_1(t-\hat{\omega})\hat{x}_2(t-\hat{\omega}) & (\lambda/\hat{\omega})(1-\hat{x}_1(t-\hat{\omega})^2) \end{pmatrix}.$$

We use the fact that the natural norm of a matrix, H, associated with a vector norm $|x| = \max_{1\le i\le n} |x_i|$ is $|H| = \max_{1\le i\le n} \sum_{j=1}^{n} |h_{ij}|$. With this definition it is not hard to show that

$$|dX(\hat{x};\phi)| \le \begin{vmatrix} 0 \\ -1/\hat{\omega}^2 - 2(\lambda/\hat{\omega})\hat{x}_1(t-\hat{\omega}) \\ 1 \\ \hat{x}_2(t-\hat{\omega})(\lambda/\hat{\omega})(1-\hat{x}_1(t-\hat{\omega})^2) \end{vmatrix} |\phi|,$$
$$\le 2.3776|\phi|. \tag{100}$$

Therefore, for $\lambda = 0.1$, $\mathcal{B} = 2.3776$. Working conservatively within the domain $D = \{x \in C[0, 2\pi]: |x - \hat{x}| \le 1\}$ it is not hard to show that

$$|dX(\hat{x}_{\hat{\omega}} + \psi_1; \phi_{\hat{\omega}}) - dX(\hat{x}_{\hat{\omega}} + \psi_2; \phi_{\hat{\omega}})|$$
$$\le (6\lambda/\hat{\omega})(1 + |\hat{x}|)|\psi_1 - \psi_2||\phi|. \tag{101}$$

Then from Equations (98) and (101) we can estimate $\mathcal{K} = 1.8157$ and, from Equation (16), we can estimate $|J(\hat{\dot{x}}, \hat{\omega})| \le 2.7546$.

Next, we can estimate the characteristic multipliers of the variational equation relative to the function $\hat{x}(t)$. For the quadrature steps in Sections 8 and 9, P and O were taken as 200 and 1200, respectively. These gave mesh widths of about 1/200 on both $[-\hat{\omega}, 0]$ and $[0, 2\pi]$. Using the method of Section 8, we computed two simple conjugate eigenvalues with magnitude 1.0430. All of the other eigenvalues have magnitudes near zero. These are, of course, the eigenvalues of the monodromy operator U. The fundamental matrix Z in Equation (59) is computed using the collocation method of Section 8.1 (See Fig. 3). The monodromy operator is formulated as in Section 8.2. The eigenvalues of the monodromy operator U are plotted in Fig. 4. Note that the significant complex conjugate eigenvalues are near the unit circle but are not exactly on it. This is due to the fact that Equation (98) is only an approximate solution. The eigenvalues are complex conjugates because the left-hand matrix in Equation (73) is real and nonsymmetric since the fundamental solution Z is nonsymmetric (See Fig. 3). We can confirm that the eigenvalues of the operator $\tilde{U}$ are the same as those of U. Graphically, this is shown in Fig. 5.

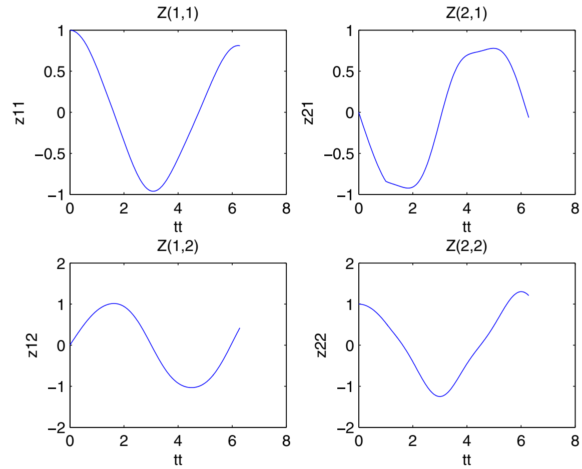

Fig. 3 Fundamental matrix for the variational equational relative to the approximate solution for the Van der Pol equation

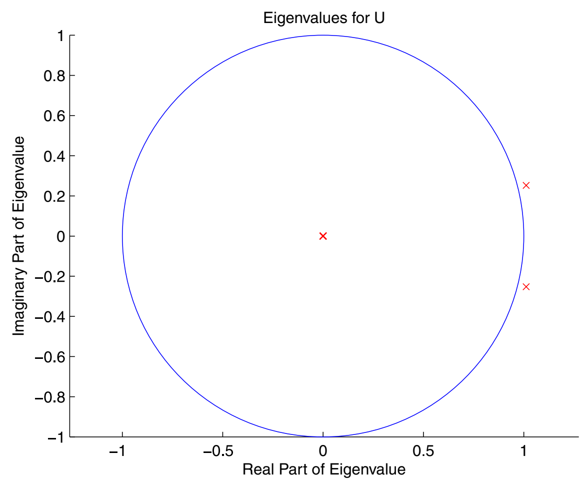

Fig. 4 Eigenvalues for the Monodromy operator

In the next step, we estimate the parameter α using the methods of Section 9. The solution of the adjoint to the variational equation was computed using Equation (79) and the parameter α in Equation (29) was

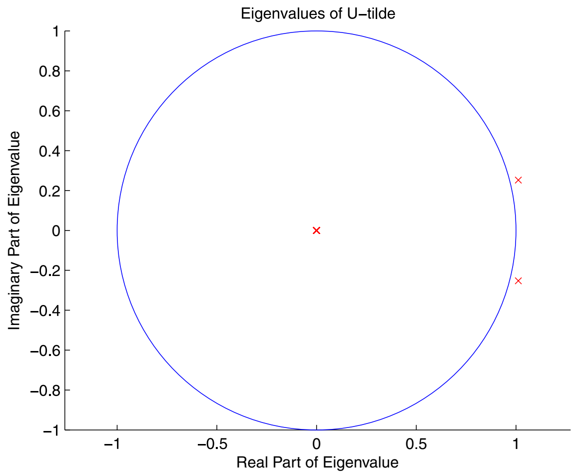

Fig. 5 Eigenvalues for $\tilde{U}$

estimated by simple quadrature, with $\Delta = 2\pi/O$ for a sufficiently large mesh, $0 = t_1 < t_2 < \cdots < t_{O+1} = 2\pi$, as

$$\alpha = \left[\Delta \left| \sum_{i=1}^{O+1} y(t_i) J(\dot{\hat{x}}, \hat{\omega})(t_i) \right| \right]^{-1}. \tag{102}$$

The absolute value of α is estimated as 3.3547.

If we now apply the methods of Section 10, using $\hat{A}(t)$ and $\hat{B}(t)$ defined in Equation (99), we can estimate $M = 2.7618 \times 10^2$. These results allow us to estimate λ_0, λ_1 and λ_2 in Lemma 6 as $\lambda_0 = 8.4091$, $\lambda_1 = 6.6736 \times 10^3$, and $\lambda_2 = 3.1720 \times 10^4$. Note the magnitude of the parameters.

With the estimates above, we can compute $F_1 = 2.5941 \times 10^9$, $F_2 = 1.0798 \times 10^{10}$ from Equations (129) and (133), respectively. Then, we compute $\delta = 4.6305 \times 10^{-11}$ from Equation (49). Then, $F_1\delta^2 = 5.5623 \times 10^{-12}$ is less than δ and $F_2\delta = 0.5$. Furthermore, $r < \delta$. Therefore, the conditions of the main theorem are satisfied and we can conclude from Theorem 1 that there exists an exact solution x^* and an exact frequency ω^* of Equation (97) such that $|x^* - \hat{x}| \leq 1.2361 \times 10^{-6}$ and $|\omega^* - \hat{\omega}| \leq 7.7877 \times 10^{-10}$.

12 Conclusions

Although there seem to be a large number of parameters to be computed and inequalities to be tested in order to produce the final error estimates, the process is feasible. All of the steps can be completed within a single code. A preliminary code, centered around the example in Section 11, will be published as a report of the National Institute of Standards and Technology (NIST) in Gilsinn [36].

From the computational point of view, the longest compute times involve the construction of the block matrices Equations (72) and (78). Computing the approximate solution and the fundamental solution of the variational equation are relatively fast compared to these matrix constructions. It behooves anyone wishing to apply the methods of this paper to spend some effort vectorizing the matrix construction algorithms in Sections 8.2 and 9 as much as possible or linking to compiled portions of code in C++ or FORTRAN.

The parameter M in the Fredholm Lemma 3 is a significant parameter. From the example above, it is clear that it would be desirable to obtain as small a value for that M as possible, since its magnitude affects the λ_i, $i = 1, 2$ parameters and λ_1 appears in the final error estimates. In particular, in the example above, the effect of M causes a very fine residual r for the approximate solution, represented by Equation (98), to produce a pessimistic error estimate between the approximate solution and the exact solution in the end. From Equation (35), the critical parameter λ_1 is linearly dependent on M.

As long as a delay differential equation with a single constant delay can be put into the form of Equation (5), the methods developed in this paper should be directly applicable. Although Stokes' result in [14] assures us that the main approximation result is true in the case of general functional equations, developing the specific bounds and Lipschitz conditions for $R(z, \beta)$ in Appendix A.1 is a very detailed and nontrivial construction for systems with multiple or functional delays. Furthermore, the computational algorithms require some significant modifications. Some of the difficulties can be seen in a simple case of two constant delays. Introducing an unknown frequency does not produce a simple equation similar to Equation (5). Extra terms arise in the arguments of the x function that involve the difference between the maximum of the two delays and each individual delay. One has to follow the details of the steps in the paper to determine the impact on the various parameters and inequalities. Also, developing the computational algorithms requires the proper handling of these details. It is not to say that the modifications are not feasible, but just to note that they will take some effort.

Appendix A.1: Bounds and Lipschitz condition for $R(z,\beta)$

In this section, we give a proof of Lemma 4. To begin, a lengthy, but direct, calculation shows

$$\begin{aligned}
&R(z,\beta)\\
&\quad=\int_0^1\left[X_1\left(\hat{x}+s\frac{\hat{\omega}}{\omega}z,\hat{x}_{\hat{\omega}}+s(\hat{x}_\omega-\hat{x}_{\hat{\omega}})+s\frac{\hat{\omega}}{\omega}z_\omega\right).\right.\\
&\qquad\left.-X_1(\hat{x},\hat{x}_{\hat{\omega}})\right]\frac{\hat{\omega}}{\omega}z\,ds\\
&\qquad+\int_0^1\left[X_2\left(\hat{x}+s\frac{\hat{\omega}}{\omega}z,\hat{x}_{\hat{\omega}}+s\left(\hat{x}_\omega-\hat{x}_{\hat{\omega}}\right)+s\frac{\hat{\omega}}{\omega}z_\omega\right)\right.\\
&\qquad\left.-X_2(\hat{x},\hat{x}_{\hat{\omega}})\right]\frac{\hat{\omega}}{\omega}z_\omega\,ds\\
&\qquad+\int_0^1\left[X_2\left(\hat{x}+s\frac{\hat{\omega}}{\omega}z,\hat{x}_{\hat{\omega}}+s(\hat{x}_\omega-\hat{x}_{\hat{\omega}})+s\frac{\hat{\omega}}{\omega}z_\omega\right)\right.\\
&\qquad\left.-X_2(\hat{x},\hat{x}_{\hat{\omega}})\right]\left(\hat{x}_\omega-\hat{x}_{\hat{\omega}}\right)ds\\
&\qquad+\left(\frac{\hat{\omega}}{\omega}-1\right)X_1(\hat{x},\hat{x}_{\hat{\omega}})z+\left(\frac{\hat{\omega}}{\omega}-1\right)X_2(\hat{x},\hat{x}_{\hat{\omega}})z_{\hat{\omega}}\\
&\qquad+\left[X_2\left(\hat{x},\hat{x}_{\hat{\omega}}\right)\left(\hat{x}_\omega-\hat{x}_{\hat{\omega}}\right)+\beta X_2\left(\hat{x},\hat{x}_{\hat{\omega}}\right)\dot{\hat{x}}_{\hat{\omega}}\right]\\
&\qquad+\frac{\hat{\omega}}{\omega}X_2\left(\hat{x},\hat{x}_{\hat{\omega}}\right)\left(z_\omega-z_{\hat{\omega}}\right).
\end{aligned}\tag{103}$$

From

$$\hat{x}_\omega-\hat{x}_{\hat{\omega}}=\int_0^1\dot{\hat{x}}\left(t-\hat{\omega}-s\beta\right)(-\beta)\,ds \tag{104}$$

we have

$$|\hat{x}_\omega-\hat{x}_{\hat{\omega}}|\leq|\beta||\dot{\hat{x}}|. \tag{105}$$

Similarly

$$|z_\omega-z_{\hat{\omega}}|\leq|\beta|\,|\dot{z}|\,. \tag{106}$$

Also, from

$$\begin{aligned}
&(\hat{x}_\omega-\hat{x}_{\hat{\omega}})+\beta\dot{\hat{x}}_{\hat{\omega}}\\
&\quad=-\beta\int_0^1[\dot{\hat{x}}_{\hat{\omega}}(t-\hat{\omega}-s\beta)-\dot{\hat{x}}_{\hat{\omega}}(t-\hat{\omega})]ds\\
&\quad=\beta^2\int_0^1\int_0^1\ddot{\hat{x}}(t-\hat{\omega}-us\beta)s\,du\,ds
\end{aligned}\tag{107}$$

we have

$$|(\hat{x}_\omega-\hat{x}_{\hat{\omega}})+\beta\dot{\hat{x}}_{\hat{\omega}}|\leq\frac{\beta^2}{2}|\ddot{\hat{x}}| \tag{108}$$

Using Equations (103) through (108), along with Equations (3) and (4), we have

$$|R(z,\beta)|\leq\mathcal{R}(z,\beta), \tag{109}$$

where

$$\begin{aligned}
\mathcal{R}(z,\beta)&=2\mathcal{K}\left|\frac{\hat{\omega}}{\omega}\right|^2|z|^2+2\frac{|\beta||z|}{|\omega|}\left(|\hat{\omega}|+|\beta|\right)\\
&\qquad+\frac{\beta^2}{2}(\mathcal{K}|\dot{\hat{x}}|^2+\mathcal{B}|\ddot{\hat{x}}|)+\mathcal{B}\left|\frac{\hat{\omega}}{\omega}\right||\beta|\,|\dot{z}|
\end{aligned}\tag{110}$$

To establish the Lipschitz condition, we start with the inequality

$$\begin{aligned}
&|dX\left(\hat{x}+a_{11},\hat{x}_{\hat{\omega}}+a_{12};b_{11},b_{12}\right)\\
&\qquad-dX\left(\hat{x}+a_{21},\hat{x}_{\hat{\omega}}+a_{22};b_{21},b_{22}\right)|\\
&\quad\leq\mathcal{K}\left(|b_{11}|+|b_{12}|\right)\left(|a_{11}-a_{21}|+|a_{12}-a_{22}|\right)\\
&\qquad+\mathcal{K}\left(|a_{12}|+|a_{22}|\right)\left(|b_{11}-b_{21}|+|b_{12}-b_{22}|\right)\\
&\qquad+\mathcal{B}\left(|b_{11}-b_{21}|+|b_{12}-b_{22}|\right).
\end{aligned}\tag{111}$$

We need to define some functions that will help simplify the relations somewhat. Let

$$\begin{aligned}
\gamma&=s\beta+(1-s)\tilde{\beta},\\
q&=sz+(1-s)\tilde{z},\\
\dot{q}&=s\dot{z}+(1-s)\dot{\tilde{z}},
\end{aligned}\tag{112}$$

for $0\leq s\leq 1$, and define

$$\begin{aligned}
\psi_1(q,\gamma)&=\hat{x}+\frac{\hat{\omega}}{\hat{\omega}+\gamma}q,\\
\psi_2(q,\gamma)&=\hat{x}_{\hat{\omega}+\gamma}+\frac{\hat{\omega}}{\hat{\omega}+\gamma}q_{\hat{\omega}+\gamma},\\
\phi_1(q,\gamma)&=-\frac{\hat{\omega}}{(\hat{\omega}+\gamma)^2}q,\\
\phi_2(q,\gamma)&=-\left[\dot{\hat{x}}_{\hat{\omega}+\gamma}+\frac{\hat{\omega}}{\hat{\omega}+\gamma}q_{\hat{\omega}+\gamma}\right]\\
&\qquad-\frac{\hat{\omega}}{(\hat{\omega}+\gamma)^2}\dot{q}_{\hat{\omega}+\gamma}.
\end{aligned}\tag{113}$$

Since we have earlier chosen β, $\tilde{\beta}$ so that

$$\begin{aligned}\hat{\omega}+\beta &\geq \frac{\hat{\omega}}{2},\\ \hat{\omega}+\tilde{\beta} &\geq \frac{\hat{\omega}}{2},\end{aligned} \tag{114}$$

it is easy to see that

$$\left|\frac{\hat{\omega}}{\hat{\omega}+\gamma}\right| \leq 2. \tag{115}$$

From Equation (112) we have the following integrals

$$\begin{aligned}
\int_0^1 |q|^2 ds &\leq \frac{1}{3}(|z|+|\tilde{z}|)^2,\\
\int_0^1 |\gamma| ds &\leq \frac{1}{2}(|\beta|+|\tilde{\beta}|),\\
\int_0^1 |q| ds &\leq \frac{1}{2}(|z|+|\tilde{z}|),\\
\int_0^1 |\gamma||q| ds &\leq \frac{1}{3}(|\beta|+|\tilde{\beta}|)(|z|+|\tilde{z}|),\\
\int_0^1 |\dot{q}||q| ds &\leq \frac{1}{3}(|\dot{z}|+|\dot{\tilde{z}}|)(|z|+|\tilde{z}|),\\
\int_0^1 |\dot{q}||\gamma| ds &\leq \frac{1}{3}(|\dot{z}|+|\dot{\tilde{z}}|)(|\beta|+|\tilde{\beta}|),\\
\int_0^1 |\dot{q}| ds &\leq \frac{1}{2}(|\dot{z}|+|\dot{\tilde{z}}|).
\end{aligned} \tag{116}$$

Define the function

$$F(z,\beta) = X\left(\hat{x}+\frac{\hat{\omega}}{\hat{\omega}+\beta}z, \hat{x}_{\hat{\omega}+\beta}+\frac{\hat{\omega}}{\hat{\omega}+\beta}z_{\hat{\omega}+\beta}\right). \tag{117}$$

Taking partial derivatives of Equation (117),

$$\begin{aligned}
&d_1F(z,\beta;y)\\
&\quad= \frac{\hat{\omega}}{\hat{\omega}+\beta}dX\\
&\qquad\times\left(\hat{x}+\frac{\hat{\omega}}{\hat{\omega}+\beta}z, \hat{x}_{\hat{\omega}+\beta}+\frac{\hat{\omega}}{\hat{\omega}+\beta}z_{\hat{\omega}+\beta}; y, y_{\hat{\omega}+\beta}\right),\\
&d_2F(z,\beta;\eta)\\
&\quad= \eta dX\\
&\qquad\times\left(\hat{x}+\frac{\hat{\omega}}{\hat{\omega}+\beta}z, \hat{x}_{\hat{\omega}+\beta}+\frac{\hat{\omega}}{\hat{\omega}+\beta}z_{\hat{\omega}+\beta};\right.\\
&\qquad-\frac{\hat{\omega}}{(\hat{\omega}+\beta)^2}z, -\left[\dot{\hat{x}}_{\hat{\omega}+\beta}+\frac{\hat{\omega}}{\hat{\omega}+\beta}\dot{z}_{\hat{\omega}+\beta}\right]\\
&\qquad\left.-\frac{\hat{\omega}}{(\hat{\omega}+\beta)^2}z_{\hat{\omega}+\beta}\right),\\
&d_1F(0,0;y) = dX\left(\hat{x},\hat{x}_{\hat{\omega}}; y, y_{\hat{\omega}}\right),\\
&d_2F(0,0;\eta) = \eta dX(\hat{x},\hat{x}_{\hat{\omega}}; 0, -\dot{\hat{x}}_{\hat{\omega}}).
\end{aligned} \tag{118}$$

From the definition of $R(z,\beta)$ and Equation (117) we have

$$\begin{aligned}R(x,\beta)-R(\tilde{z},\tilde{\beta}) = F(z,\beta)&-F(\tilde{z},\tilde{\beta})\\ &-d_2F(0,0;\beta-\tilde{\beta})\\ &-d_1F(0,0;z-\tilde{z}).\end{aligned} \tag{119}$$

From the definition of γ and q in Equation (112) we define the derivative with respect to s as

$$\begin{aligned}&d_sF(q,\gamma;ds)\\ &\quad= [d_1F(q,\gamma;z-\tilde{z})\\ &\qquad+d_2F(q,\gamma;\beta-\tilde{\beta})]\,ds.\end{aligned} \tag{120}$$

By the Fundamental Theorem of Calculus

$$\int_0^1 d_sF(q,\gamma;ds) = F(z,\beta)-F(\tilde{z},\tilde{\beta}). \tag{121}$$

We can then write, using Equations (113) and (118),

$$\begin{aligned}&R(z,\beta)-R(\tilde{z},\tilde{\beta})\\ &\quad= \int_0^1 [d_1F(q,\gamma;z-\tilde{z})-d_1F(0,0;z-\tilde{z})]ds\\ &\qquad+\int_0^1 [d_2F(q,\gamma;\beta-\tilde{\beta})-d_2F(0,0;\beta-\tilde{\beta})]ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 [dX(\psi_1(q,\gamma), \psi_2(q,\gamma); \psi_1(z,\gamma) \\
&\quad - \psi_1(\tilde{z},\gamma), \psi_2(z,\gamma) - \psi_2(\tilde{z},\gamma)) \\
&\quad - dX(\psi_1(0,0), \psi_2(0,0); \psi_1(z,0) \\
&\quad - \psi_1(\tilde{z},0), \psi_2(z,0) - \psi_2(\tilde{z},0))]ds \\
&\quad + \int_0^1 [\, dX(\psi_1(q,\gamma), \psi_2(q,\gamma); \\
&\quad \times (\beta - \tilde{\beta})\phi_1(q,\gamma), (\beta - \tilde{\beta})\phi_2(q,\gamma) \\
&\quad - dX(\psi_1(0,0), \psi_2(0,0); (\beta - \tilde{\beta})\phi_1(0,0), \\
&\quad \times (\beta - \tilde{\beta})\phi_2(0,0))]ds. \qquad (122)
\end{aligned}
$$

From Equation (113) we note that $\psi_1(0,0) = \hat{x}$ and $\psi_2(0,0) = \hat{x}_{\hat{\omega}}$.

Then, using Equations (111) through (122) it is possible to show with some effort that

$$
\begin{aligned}
\mathcal{R}_1(z,\beta,\tilde{z},\tilde{\beta}) &= 8\mathcal{K}(|z| + |\tilde{z}|) \\
&\quad + \left(2\mathcal{K}|\dot{\hat{x}}| + \frac{\mathcal{B}}{|\hat{\omega}|}\right)(|\beta| + |\tilde{\beta}|), \\
\mathcal{R}_2(z,\beta,\tilde{z},\tilde{\beta}) &= \frac{\mathcal{K}}{3}\left(\frac{16}{|\tilde{\omega}|} + 4\right)(|z| + |\tilde{z}|)^2 \\
&\quad + \left(2\mathcal{K}|\dot{\hat{x}}| + \mathcal{B}\left(1 + \frac{2}{|\hat{\omega}|}\right)\right) \\
&\quad \times (|z| + |\tilde{z}|) + \frac{2\mathcal{B}}{|\hat{\omega}|}(|\dot{z}| + |\dot{\tilde{z}}|) \\
&\quad + \frac{16\mathcal{K}}{3|\hat{\omega}|}(|z| + |\tilde{z}|)(|\dot{z}| + |\dot{\tilde{z}}|) \\
&\quad + \frac{\mathcal{B}|\ddot{\hat{x}}|}{2}(|\beta| + |\tilde{\beta}|). \qquad (123)
\end{aligned}
$$

□

Appendix 2: Bounds and Lipschitz conditions for *S*(*g*)

Let $g \in \mathcal{N}_\delta$, then, from Lemma 7 and the selection of β so that $\hat{\omega} + \beta(g) \geq \frac{\hat{\omega}}{2}$, we have

$$
\left|\frac{\hat{\omega}}{\hat{\omega} + \beta(g)}\right| \leq 2, \qquad (124)
$$

and

$$
\begin{aligned}
|S(g)| &= |\mathcal{R}_0(z(g), \beta(g))| \\
&\leq 2\mathcal{K}\left|\frac{\hat{\omega}}{\hat{\omega} + \beta(g)}\right|^2 |z(g)|^2 \\
&\quad + 2\frac{|\beta(g)||z(g)|}{|\hat{\omega} + \beta(g)|}(|\hat{\omega}| + |\beta(g)|) \\
&\quad + \frac{\beta(g)^2}{2}(\mathcal{K}|\dot{\hat{x}}|^2 + \mathcal{B}|\ddot{\hat{x}}|) \\
&\quad + \mathcal{B}\left|\frac{\hat{\omega}}{\hat{\omega} + \beta(g)}\right||\beta(g)||\dot{z}(g)|. \qquad (125)
\end{aligned}
$$

If we combine Equations (35), (124), and (125) we have

$$
\begin{aligned}
|S(g)| \leq \Bigg\{&32\mathcal{K}\lambda_1^2 + \frac{16\lambda_0\lambda_1}{|\hat{\omega}|}(|\hat{\omega}| + 2\lambda_0\delta) \\
&+ 2\lambda_0^2(\mathcal{K}|\dot{\hat{x}}|^2 + \mathcal{B}|\ddot{\hat{x}}|) + 8\mathcal{B}\lambda_0\lambda_2\Bigg\}\delta^2. \qquad (126)
\end{aligned}
$$

Set

$$
\begin{aligned}
E_1(\delta) = \Bigg\{&32\mathcal{K}\lambda_1^2 + \frac{16\lambda_0\lambda_1}{|\hat{\omega}|}(|\hat{\omega}| + 2\lambda_0\delta) \\
&+ 2\lambda_0^2(\mathcal{K}|\dot{\hat{x}}|^2 + \mathcal{B}|\ddot{\hat{x}}|) + 8\mathcal{B}\lambda_0\lambda_2\Bigg\}\delta^2. \qquad (127)
\end{aligned}
$$

and let F_1 be any constant such that

$$
\begin{aligned}
F_1 \geq\ &32\mathcal{K}\lambda_1^2 + \frac{16\lambda_0\lambda_1}{|\hat{\omega}|}(|\hat{\omega}| + 2\lambda_0\delta) \\
&+ 2\lambda_0^2(\mathcal{K}|\dot{\hat{x}}|^2 + \mathcal{B}|\ddot{\hat{x}}|) + 8\mathcal{B}\lambda_0\lambda_2. \qquad (128)
\end{aligned}
$$

Since the right-hand side of Equation (128) is not independent of δ we can use the fact that we need to select $\delta \leq \hat{\omega}/4\lambda_0$ in Equation (49) to set, since $\hat{\omega}$ is taken as positive,

$$
\begin{aligned}
F_1 &= 32\mathcal{K}{\lambda_1}^2 + 24\lambda_0\lambda_1 \\
&\quad + 2{\lambda_0}^2(\mathcal{K}|\dot{\hat{x}}|^2 + \mathcal{B}|\ddot{\hat{x}}|) + 8\mathcal{B}\lambda_0\lambda_2. \qquad (129)
\end{aligned}
$$

Now let $g,\ \tilde{g} \in \mathcal{N}_\delta$ and again set $r = \delta$. Then, from Equations (35), (41), and (123) and, choosing $|g| \leq \delta$, we have, with some algebra,

$$
\begin{aligned}
&|S(g)-S(\tilde{g})| \\
&\le \Bigg[\lambda_1\Bigg\{8\mathcal{K}(|z(g)|+|z(\tilde{g})|) \\
&\quad+\left(2\mathcal{K}|\dot{x}|+\frac{\mathcal{B}}{|\hat{\omega}|}\right)(|\beta(g)|+|\beta(\tilde{g})|)\Bigg\} \\
&\quad+\lambda_0\Bigg\{\frac{\mathcal{K}}{3}\left(\frac{16}{|\hat{\omega}|}+4\right)(|z(g)|+|z(\tilde{g})|)^2 \\
&\quad+\left(2\mathcal{K}\left|\dot{x}\right|+\mathcal{B}\left(1+\frac{2}{|\hat{\omega}|}\right)\right)(|z(g)|+|z(\tilde{g})|) \\
&\quad+\frac{2\mathcal{B}}{|\hat{\omega}|}(|\dot{z}(g)|+|\dot{z}(\tilde{g})|) \\
&\quad+\frac{16\mathcal{K}}{3|\hat{\omega}|}(|z(g)|+|z(\tilde{g})|)(|\dot{z}(g)|+|\dot{z}(\tilde{g})|) \\
&\quad+\frac{\mathcal{B}\left|\ddot{x}\right|}{2}(|\beta(g)|+|\beta(\tilde{g})|)\Bigg\}\Bigg]|g-\tilde{g}| \\
&\le \Bigg[\lambda_1\Bigg\{32\mathcal{K}\lambda_1+4\lambda_0\left(2\mathcal{K}\left|\dot{x}\right|+\frac{\mathcal{B}}{|\hat{\omega}|}\right)\Bigg\} \\
&\quad+\lambda_0\Bigg\{\frac{16{\lambda_0}^2}{3}\left(\frac{16}{|\hat{\omega}|}+4\right)\delta \\
&\quad+4\lambda_1\left(2\mathcal{K}|\dot{x}|+\mathcal{B}\left(1+\frac{2}{|\hat{\omega}|}\right)\right) \\
&\quad+\frac{8\mathcal{B}\lambda_2}{|\hat{\omega}|}+\frac{256\mathcal{K}\lambda_1\lambda_2}{3|\hat{\omega}|}\delta+2\mathcal{B}|\ddot{x}|\lambda_0\Bigg\}\Bigg]\delta\,|g-\tilde{g}|\,.
\end{aligned}
\tag{130}
$$

Finally, we set

$$
\begin{aligned}
E_2(\delta)=&\Bigg[\lambda_1\Bigg\{32\mathcal{K}\lambda_1+4\lambda_0\left(2\mathcal{K}\left|\dot{x}\right|+\frac{\mathcal{B}}{|\hat{\omega}|}\right)\Bigg\} \\
&+\lambda_0\Bigg\{\frac{16\lambda_0^2}{3}\left(\frac{16}{|\hat{\omega}|}+4\right)\delta \\
&+4\lambda_1\left(2\mathcal{K}\left|\dot{x}\right|+\mathcal{B}\left(1+\frac{2}{|\hat{\omega}|}\right)\right) \\
&+\frac{8\mathcal{B}\lambda_2}{|\hat{\omega}|}+\frac{256\mathcal{K}\lambda_1\lambda_2}{3|\hat{\omega}|}\delta+2\mathcal{B}|\ddot{x}|\lambda_0\Bigg\}\Bigg]\delta,
\end{aligned}
\tag{131}
$$

and let F_2 be any constant such that

$$
\begin{aligned}
F_2\ge\;&\lambda_1\Bigg\{32\mathcal{K}\lambda_1+4\lambda_0\left(2\mathcal{K}\left|\dot{x}\right|+\frac{\mathcal{B}}{|\hat{\omega}|}\right)\Bigg\} \\
&+\lambda_0\Bigg\{\frac{16{\lambda_0}^2}{3}\left(\frac{16}{|\hat{\omega}|}+4\right)\delta \\
&+4\lambda_1\left(2\mathcal{K}\left|\dot{x}\right|+\mathcal{B}\left(1+\frac{2}{|\hat{\omega}|}\right)\right) \\
&+\frac{8\mathcal{B}\lambda_2}{|\hat{\omega}|}+\frac{256\mathcal{K}\lambda_1\lambda_2}{3|\hat{\omega}|}\delta+2\mathcal{B}|\ddot{x}|\lambda_0\Bigg\}.
\end{aligned}
\tag{132}
$$

As in the selection of F_1 we again use the fact that we need to set $\delta\le\hat{\omega}/4\lambda_0$ to write

$$
\begin{aligned}
F_2=\;&\lambda_1\Bigg\{32\mathcal{K}\lambda_1+4\lambda_0\left(2\mathcal{K}\left|\dot{x}\right|+\frac{\mathcal{B}}{|\hat{\omega}|}\right)\Bigg\} \\
&+\lambda_0\Bigg\{\frac{16{\lambda_0}^2}{3}\left(\frac{16}{|\hat{\omega}|}+4\right)\frac{\hat{\omega}}{4\lambda_0} \\
&+4\lambda_1\left(2\mathcal{K}\left|\dot{x}\right|+\mathcal{B}\left(1+\frac{2}{|\hat{\omega}|}\right)\right) \\
&+\frac{8\mathcal{B}\lambda_2}{|\hat{\omega}|}+\frac{256\mathcal{K}\lambda_1\lambda_2}{3|\hat{\omega}|}\frac{\hat{\omega}}{4\lambda_0}+2\mathcal{B}\left|\ddot{x}\right|\lambda_0\Bigg\}.
\end{aligned}
\tag{133}
$$

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