Metrologia **44** (2007) 117–127 doi:10.1088/0026-1394/44/2/003

# Trapezoidal and triangular distributions for Type B evaluation of standard uncertainty

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Received 6 September 2006
Published 26 February 2007
Online at stacks.iop.org/Met/44/117

**Abstract**
The *Guide to the Expression of Uncertainty in Measurement* (GUM), published by the International Organization for Standardization (ISO), recognizes Type B state-of-knowledge probability distributions specified by scientific judgment as valid means to quantify uncertainty. The ISO-GUM discusses symmetric probability distributions only. Sometimes an asymmetric distribution is needed. We describe a trapezoidal distribution which may be asymmetric depending on the settings of its parameters. We describe the probability density function, cumulative distribution function (cdf), inverse function of the cdf, moment generating function, moments about origin (zero), expected value and variance of a trapezoidal distribution. We show that triangular and rectangular distributions are special cases of the trapezoidal distribution. Then we derive the moment generating functions, moments, expected values and variances of various special cases of the trapezoidal distribution. Finally, we illustrate through a real life example how a Type B asymmetric trapezoidal distribution may be useful in quantifying a correction for bias (systematic error) in a result of measurement and in quantifying the standard uncertainty associated with the correction.

## 1. Introduction

The International Organization for Standardization (ISO) *Guide to the Expression of Uncertainty in Measurement* (GUM) recognizes Type B (non-statistical) evaluations. A Type B estimate and standard uncertainty for an input quantity are the expected value and standard deviation of a state-of-knowledge probability distribution specified by scientific judgment based on all available information, [1, section 4.3]. The ISO-GUM [1] discusses the following probability distributions for a Type B input variable of the measurement equation: normal (Gaussian) distribution, rectangular (uniform) distribution, isosceles triangular distribution and isosceles trapezoidal distribution. These distributions are symmetric about their expected values and they are useful in many applications. However, in some applications an asymmetric distribution is needed [1, sections 4.3.8, F.2.4.4 and G.5.3]. In section 2, we describe a trapezoidal distribution, which may be asymmetric depending on the settings of its parameters. We give explicit and simple expressions for the moment generating function (mgf) and the moments of a trapezoidal distribution. In particular, we give expressions for the expected value and variance of a trapezoidal distribution. These expressions are new contributions to the literature. We also discuss how random numbers from a trapezoidal distribution may be numerically generated. In section 3, we discuss various special cases of the trapezoidal distribution, which include triangular and rectangular distributions. Then we derive the mgfs, moments, expected values and variances of various special cases of the trapezoidal distribution.

A particular use of a Type B probability distribution is to quantify a correction for bias (systematic error) in a result of measurement. The exact bias is unknowable; therefore, a correction for bias carries uncertainty. According

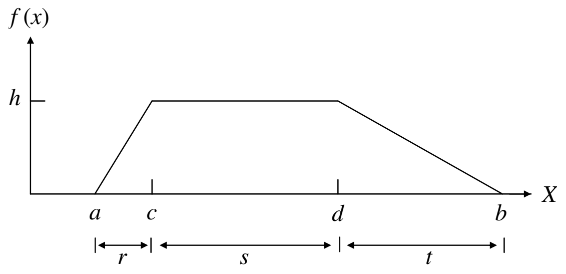


**Figure 1.** Probability density function of Trapezoid $(a, c, d, b)$.

to the ISO-GUM, a result of measurement that is subject to non-negligible bias should be corrected[3] and the uncertainty associated with the correction should be quantified and included in the combined standard uncertainty associated with the corrected result [1, section 3.2]. This is a signal contribution of the ISO-GUM because it establishes, for the first time, an approach to account for the uncertainty from unknown bias. A correction for bias and its associated standard uncertainty are generally the expected value and standard deviation of a Type B state-of-knowledge probability distribution[4] for an input variable which represents a correction (or a correction factor) in the measurement equation. In section 4, we review the ISO-GUM's approach to quantify and incorporate a correction for bias and its associated standard uncertainty. In section 5, we illustrate through a real life example how a Type B asymmetric trapezoidal distribution may be useful in quantifying a correction for bias and its associated standard uncertainty. A summary appears in section 6.

## 2. Trapezoidal distribution

A random variable $X$ has a trapezoidal distribution if its probability density function (pdf) $f(x)$ has the shape of a trapezoid shown in figure 1. A trapezoidal distribution may be defined by four parameters, $a$, $c$, $d$ and $b$, where $a$ and $b$ are the end points and the points $c$ and $d$ identify the rectangular part of the trapezoid. We refer to such a distribution as Trapezoid $(a, c, d, b)$, where $a \leqslant c$, $c \leqslant d$ and $d \leqslant b$. We use the symbols $r$, $s$ and $t$ for the widths of the line segments between $a$, $c$, $d$ and $b$, where $r = (c - a)$, $s = (d - c)$ and $t = (b - d)$. Clearly $r + s + t = (b - a)$ is the width of the trapezoid.

The area of the trapezoid shown in figure 1 is $h(c-a)/2 + h(d - c) + h(b - d)/2$. For a trapezoid to represent a pdf its area should be one. By equating the area $h(c - a)/2 + h(d - c) + h(b - d)/2$ to one and solving for $h$ we get the height $h = 2/[(b - a) + (d - c)] = 2/(r + 2s + t)$. Then algebraic equations of the straight lines in figure 1 give the pdf. Thus, the pdf $f(x)$ of the trapezoidal probability distribution, Trapezoid $(a, c, d, b)$, is

$$f(x) = 0 \qquad \text{if } x \leqslant a, \tag{2.1}$$

$$f(x) = \frac{(x-a)}{(c-a)}h \qquad \text{if } a \leqslant x \leqslant c, \tag{2.2}$$

$$f(x) = h \qquad \text{if } c \leqslant x \leqslant d, \tag{2.3}$$

$$f(x) = \frac{(b-x)}{(b-d)}h \qquad \text{if } d \leqslant x \leqslant b, \tag{2.4}$$

$$f(x) = 0 \qquad \text{if } b \leqslant x, \tag{2.5}$$

where

$$h = \frac{2}{[(b-a)+(d-c)]} = \frac{2}{(r+2s+t)}. \tag{2.6}$$

The cumulative distribution function (cdf) $F(x)$, defined as $F(x) = \Pr(X \leqslant x)$, is obtained by integrating the pdf of $X$ defined above in (2.1)–(2.6), within the limits from $-\infty$ to $x$. Thus, the cdf $F(x)$ of the trapezoidal distribution, Trapezoid $(a, c, d, b)$, is

$$F(x) = 0 \qquad \text{if } x \leqslant a, \tag{2.7}$$

$$F(x) = \frac{h(x-a)^2}{2(c-a)} \qquad \text{if } a \leqslant x \leqslant c, \tag{2.8}$$

$$F(x) = \frac{h}{2}(c-a) + h(x-c) \qquad \text{if } c \leqslant x \leqslant d, \tag{2.9}$$

$$F(x) = 1 - \frac{h(b-x)^2}{2(b-d)} \qquad \text{if } d \leqslant x \leqslant b. \tag{2.10}$$

and

$$F(x) = 1 \qquad \text{if } b \leqslant x. \tag{2.11}$$

In particular,

$$F(a) = 0, \tag{2.12}$$

$$F(c) = \frac{h}{2}(c-a), \tag{2.13}$$

$$F(d) = 1 - \frac{h}{2}(b-d) = \frac{[(d-a)+(d-c)]}{[(b-a)+(d-c)]} \tag{2.14}$$

and

$$F(b) = 1. \tag{2.15}$$

The inverse function $x = F^{-1}(y)$ of the cdf $F(x)$ is determined by setting $y = F(x)$, for $0 \leqslant y \leqslant 1$, and solving equations (2.8)–(2.10) for $x$. Thus the inverse function $x = F^{-1}(y)$ of the cdf $F(x)$ is

$$F^{-1}(y) = a + \sqrt{\frac{2(c-a)}{h}} \times \sqrt{y} \qquad \text{if } 0 \leqslant y \leqslant \frac{h}{2}(c-a), \tag{2.16}$$

$$F^{-1}(y) = \frac{(a+c)}{2} + \frac{y}{h} \qquad \text{if } \frac{h}{2}(c-a) \leqslant y \leqslant 1 - \frac{h}{2}(b-d), \tag{2.17}$$

$$F^{-1}(y) = b - \sqrt{\frac{2(b-d)}{h}} \times \sqrt{1-y} \qquad \text{if } 1 - \frac{h}{2}(b-d) \leqslant y \leqslant 1. \tag{2.18}$$

[3] The exception is when one is highly certain (very sure) that the bias is negligible. This corresponds to the situation where both the expected value and the standard deviation of a state-of-knowledge probability distribution for the correction variable are close to zero.

[4] There is no direct correspondence between the classification of uncertainties as Type A or Type B evaluations and the classification of the uncertainties as arising from random effects or those associated with the corrections for systematic effects [2, section D2].

In particular,

$$F^{-1}(0) = a, \tag{2.19}$$

$$F^{-1}\left(\frac{h}{2}(c-a)\right) = c, \tag{2.20}$$

$$F^{-1}\left(1-\frac{h}{2}(b-d)\right) = d \tag{2.21}$$

and

$$F^{-1}(1) = b, \tag{2.22}$$

as one would expect in view of the equations from (2.12) to (2.15).

### 2.1. *Moment generating function and moments of Trapezoid $(a, c, d, b)$*

The mgf of the pdf of a random variable $X$, denoted by $M(t)$, is defined as $M(t) = E(\mathrm{e}^{tX})$, provided that this expected value exists in some neighbourhood of zero[5]. As the name suggests, an mgf can be used to generate the moments about origin (zero), $E(X^k)$, for $k = 1, 2, \ldots$. As we shall see, in equation (2.24), the mgf of a trapezoidal distribution exists for all values of $t$. Therefore, we may express it as

$$M(t) = \int_a^b \mathrm{e}^{tX} f(x)\,\mathrm{d}x = \int_a^b \left(\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\right) f(x)\,\mathrm{d}x$$

$$= 1 + \sum_{k=1}^{\infty} E(X^k)\frac{t^k}{k!}, \tag{2.23}$$

where $f(x)$ is the pdf of the random variable $X$. The mgf of the trapezoidal distribution Trapezoid $(a, c, d, b)$ is obtained by successively integrating $\mathrm{e}^{tX}$ with respect to the three parts of the pdf $f(x)$ defined in equations (2.1)–(2.6). Thus, we have

$$M(t) = \int_a^b \mathrm{e}^{tX} f(x)\,\mathrm{d}x = \frac{h}{t^2}\left(\frac{\mathrm{e}^{bt}-\mathrm{e}^{dt}}{b-d} - \frac{\mathrm{e}^{ct}-\mathrm{e}^{at}}{c-a}\right). \tag{2.24}$$

We can expand (2.24) to get the following expression for the mgf of Trapezoid $(a, c, d, b)$:

$$M(t) = 1 + \sum_{k=1}^{\infty} \frac{h}{(k+2)(k+1)} \times \left(\frac{b^{k+2}-d^{k+2}}{b-d} - \frac{c^{k+2}-a^{k+2}}{c-a}\right)\frac{t^k}{k!}. \tag{2.25}$$

By comparing (2.23) and (2.25), the $k$th moment, $E(X^k)$, of the trapezoidal distribution Trapezoid $(a, c, d, b)$ is

$$E(X^k) = \frac{h}{(k+2)(k+1)}\left(\frac{b^{k+2}-d^{k+2}}{b-d} - \frac{c^{k+2}-a^{k+2}}{c-a}\right) \tag{2.26}$$

for $k = 1, 2, \ldots$, where $h = 2/(r+2s+t) = 2/[(b-a)+(d-c)]$.

The most important moments of a trapezoidal distribution, from the viewpoint of its use to quantify uncertainty in measurement, are its expected value, $E(X)$, and variance, $V(X) = E(X-E(X))^2$, which is the second central moment (moment about mean). The expected value $E(X)$ is obtained by substituting $k = 1$ in (2.26). Thus we have

$$E(X) = \frac{h}{6}\left(\frac{b^3-d^3}{b-d} - \frac{c^3-a^3}{c-a}\right). \tag{2.27}$$

The expected value (2.27) can alternatively be expressed as

$$E(X) = \frac{(b^2-a^2)+(d^2-c^2)-ac+bd}{3[(b-a)+(d-c)]}. \tag{2.28}$$

The second moment, $E(X^2)$, is obtained by substituting $k = 2$ in (2.26). Thus we have

$$E(X^2) = \frac{h}{12}\left(\frac{b^4-d^4}{b-d} - \frac{c^4-a^4}{c-a}\right). \tag{2.29}$$

The second moment (2.29) can alternatively be expressed as

$$E(X^2) = \frac{(b^3-a^3)+(d^3-c^3)-(ac^2+a^2c)+(bd^2+b^2d)}{6[(b-a)+(d-c)]}. \tag{2.30}$$

The variance $V(X)$ can be obtained from the formula $V(X) = E(X-E(X))^2 = E(X^2)-(E(X))^2$. It is useful, however, to obtain a concise expression for the variance $V(X)$. The variance $V(X)$ of Trapezoid $(a, c, d, b)$ depends only on the widths $r$, $s$ and $t$ of the left-side triangle, the middle rectangle and the right-side triangle of the trapezoid. So an expression for $V(X)$ can be obtained in terms of $r$, $s$ and $t$. We have succeeded in obtaining the following expression for the variance $V(X)$ of Trapezoid $(a, c, d, b)$:

$$V(X) = \frac{3(r+2s+t)^4 + 6(r^2+t^2)(r+2s+t)^2 - (r^2-t^2)^2}{[12(r+2s+t)]^2}, \tag{2.31}$$

where $r = (c-a)$, $s = (d-c)$, $t = (b-d)$ and $(r+2s+t) = [(b-a)+(d-c)] = 2/h$. The numerator of (2.31) can alternatively be expressed as a fourth degree polynomial in $s$ with coefficients made of $r$ and $t$. Thus, another expression for the variance $V(X)$ of Trapezoid $(a, c, d, b)$ is

$$V(X) = \frac{6s^4 + 12(r+t)s^3 + [12(r+t)^2 - 6rt]s^2}{18(r+2s+t)^2} + \frac{6(r+t)(r^2+rt+t^2)s + (r+t)^2(r^2+rt+t^2)}{18(r+2s+t)^2}. \tag{2.32}$$

Both expressions (2.31) and (2.32) for the variance $V(X)$ are equivalent and useful.

*Comment 1.* A 'generalized trapezoidal distribution' having seven parameters, $a$, $b$, $c$, $d$, $n_1$, $n_3$ and $\alpha$, is described in [3]. They report closed form expressions for the first two moments $E(X)$ and $E(X^2)$ of a four-parameter trapezoidal distribution. However, our expressions for $E(X)$ and $E(X^2)$, given in (2.27) and (2.29), are much simpler than in [3]. Reference [3] does not report a closed form expression for the variance $V(X)$.

[5] In keeping with a statistical convention, we use the symbol $t$ for the argument of a moment generating function $M(t)$. In this paper, we have also used the symbol $t$ for the width of the line segment $t = (b-d)$.

### 2.2. *Generation of random numbers from Trapezoid $(a, c, d, b)$*

The ISO-GUM's approach to determine the expected value and variance for an output variable (measurand) is to propagate the expected values and variances of probability distributions for the input variables through a linear approximation of the measurement equation. An alternative approach proposed in [4] is to propagate probability distributions by numerical simulation of the measurement equation. The latter approach requires generation of random numbers from the probability distributions specified for the input variables. Random numbers from a trapezoidal distribution can be easily generated from the inverse cdf function $x = F^{-1}(y)$ defined in (2.16)–(2.18). Suppose $\{y_1, \ldots, y_n\}$ is a set of random numbers from a rectangular distribution on the interval [0, 1] obtained by a random number generator. Then the set of numbers $\{x_1, \ldots, x_n\}$, where $x_i = F^{-1}(y_i)$, for $i = 1, \ldots, n$, is a set of random numbers from the trapezoidal distribution Trapezoid $(a, c, d, b)$.

## 3. Special cases of the trapezoidal distribution Trapezoid $(a, c, d, b)$

The trapezoidal probability distribution Trapezoid $(a, c, d, b)$, sketched in figure 1, is very versatile. By varying the intermediate points $c$ and $d$ with respect to the end points $a$ and $b$ and with respect to each other, we can obtain various useful special cases. In this section, we describe the mgfs, the moments (about zero), the expected values and the variances of the following special cases of the trapezoidal distribution Trapezoid $(a, c, d, b)$:

(i) only left-side sloping trapezoidal distribution,
(ii) only right-side sloping trapezoidal distribution,
(iii) isosceles trapezoidal distribution,
(iv) triangular distribution,
(v) left-side sloping right triangular distribution,
(vi) right-side sloping right triangular distribution,
(vii) isosceles triangular distribution and
(viii) rectangular distribution.

Random numbers from these distributions can be generated from the corresponding inverse cdfs $x = F^{-1}(y)$. The inverse cdfs for these special case distributions can be obtained from the inverse cdf $x = F^{-1}(y)$ defined in (2.16)–(2.18) for a trapezoidal distribution by proceeding along the same lines as used in the following subsections to determine the mgfs and the moments.

### 3.1. *Only left-side sloping Trapezoid $(a, c, b)$*

The trapezoidal distribution of figure 1 approaches an only left-side sloping trapezoidal distribution shown in figure 2 as the point $d$ approaches the point $b$. When the point $d$ approaches the point $b$, the width $t$ of the right-side triangle approaches zero. The pdf $f(x)$, the cdf $F(x)$, the inverse cdf $F^{-1}(y)$ and the mgf $M(t)$ of an only left-side sloping trapezoidal distribution are obtained from equations (2.1) through (2.18) and (2.24) by taking the limit as $d$ approaches $b$.

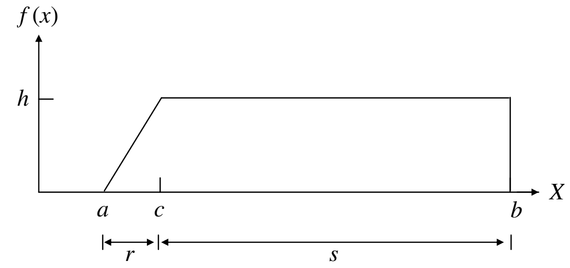


**Figure 2.** Probability density function of only left-side sloping Trapezoid $(a, c, b)$.

*L'Hospital's Rule.* When determining the limits we sometimes need to use L'Hospital's Rule for indeterminate forms which states the following. If the limits of each of the two functions $\phi_1(z)$ and $\phi_2(z)$ are zero but the limit of $[\phi_1'(z)/\phi_2'(z)]$ is finite, where $\phi_1'(z)$ and $\phi_2'(z)$ are the derivatives of $\phi_1(z)$ and $\phi_2(z)$, respectively, then $\lim[\phi_1(z)/\phi_2(z)] = \lim[\phi_1'(z)/\phi_2'(z)]$. The rule extends to second and higher order derivatives.

The mgf and the moments of an only left-side sloping trapezoidal distribution with parameters $a$, $c$ and $b$, where $a \leqslant c$ and $c \leqslant b$, are determined by taking the limits of (2.24) and (2.26) as $d$ approaches $b$. Thus we have

$$M(t) = \frac{h}{t}(\mathrm{e}^{bt}) - \frac{h}{t^2}\left(\frac{\mathrm{e}^{ct} - \mathrm{e}^{at}}{c - a}\right) \tag{3.1}$$

and

$$E(X^k) = \frac{h}{(k+2)(k+1)}\left((k+2)b^{k+1} - \frac{c^{k+2} - a^{k+2}}{c - a}\right), \tag{3.2}$$

for $k = 1, 2, \ldots$, where $h = 2/(r+2s) = 2/[(b-a)+(b-c)]$.

The expected value of an only left-side sloping trapezoidal distribution can be obtained by substituting $b$ for $d$ in (2.28). Thus we have

$$E(X) = \frac{3b^2 - a^2 - c^2 - ac}{3(2b - a - c)}. \tag{3.3}$$

The expected value (3.3) agrees with (3.2) evaluated for $k = 1$.

The variance of a left-sided sloping trapezoidal distribution is obtained by setting $t = 0$ in (2.32). Thus we have

$$V(X) = \frac{6s^4 + 12rs^3 + 12r^2s^2 + 6r^3s + r^4}{18(r + 2s)^2}, \tag{3.4}$$

where $r = (c - a)$ and $s = (b - c)$.

### 3.2. *Only right-side sloping Trapezoid $(a, d, b)$*

The trapezoidal distribution of figure 1 approaches an only right-side sloping trapezoidal distribution shown in figure 3 as the point $c$ approaches the point $a$. When the point $c$ approaches the point $a$, the width $r$ of the left-side triangle approaches zero. The pdf $f(x)$, the cdf $F(x)$, the inverse cdf $F^{-1}(y)$ and the mgf $M(t)$ of an only right-side sloping trapezoidal distribution are obtained from equations (2.1) through (2.18) and (2.24) by taking the limit as $c$ approaches $a$.

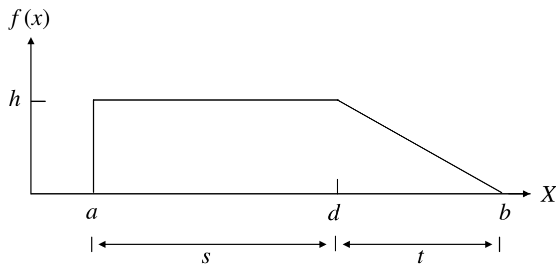


**Figure 3.** Probability density function of only right-side sloping Trapezoid $(a, d, b)$.

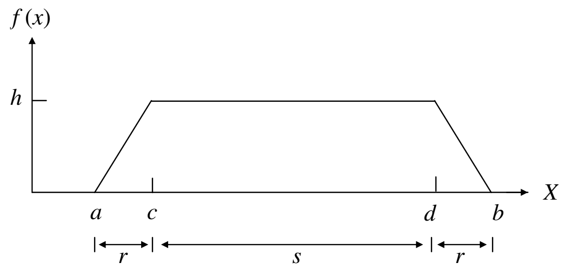


**Figure 4.** Probability density function of isosceles Trapezoid $(a, c, d, b)$.

The mgf and the moments of an only right-side sloping trapezoidal distribution with parameters $a$, $d$ and $b$, where $a \leqslant d$ and $d \leqslant b$, are determined by taking the limits of (2.24) and (2.26) as $c$ approaches $a$. Thus we have

$$M(t) = \frac{h}{t^2}\left(\frac{e^{bt} - e^{dt}}{c - a}\right) - \frac{h}{t}(e^{at}) \tag{3.5}$$

and

$$E(X^k) = \frac{h}{(k+2)(k+1)}\left(\frac{b^{k+2} - d^{k+2}}{b - d} - (k+2)a^{k+1}\right), \tag{3.6}$$

for $k = 1, 2, \ldots$, where $h = 2/(2s+t) = 2/[(b-a)+(d-a)]$.

The expected value of an only right-side sloping trapezoidal distribution can be obtained by substituting $a$ for $c$ in (2.28). Thus we have

$$E(X) = \frac{b^2 + d^2 + bd - 3a^2}{3(b + d - 2a)}. \tag{3.7}$$

The expected value (3.7) agrees with (3.6) evaluated for $k = 1$.

The variance of a right-sided sloping trapezoidal distribution is obtained by setting $r = 0$ in (2.32). Thus we have

$$V(X) = \frac{6s^4 + 12ts^3 + 12t^2s^2 + 6t^3s + t^4}{18(t + 2s)^2}, \tag{3.8}$$

where $s = (d - a)$ and $t = (b - d)$.

We note that (3.8) is the same as (3.4) where $r$ is replaced with $t$. This is what one would expect because the variance is unchanged if the right and left sides of a trapezoid are flipped vertically.

### 3.3. *Isosceles Trapezoid $(a, c, d, b)$*

The trapezoidal distribution Trapezoid $(a, c, d, b)$ of figure 1 becomes an isosceles trapezoidal distribution shown in figure 4 when $r = t$ or equivalently $r = (c-a) = (b-d)$ and $c = (a+r)$ and $d = (b - r)$. Thus, an isosceles trapezoidal distribution is defined by only three parameters: $a$, $b$ and $r$.

The mgf and the moments of an isosceles trapezoidal distribution, determined by substituting $c = (a + r)$ and $d = (b - r)$ in (2.24) and (2.26), are

$$M(t) = \frac{h}{t^2}\left(\frac{e^{bt} - e^{(b-r)t}}{r} - \frac{e^{(a+r)t} - e^{at}}{r}\right) \tag{3.9}$$

and

$$E(X^k) = \frac{h}{(k+2)(k+1)} \times \left(\frac{b^{k+2} - (b-r)^{k+2}}{r} - \frac{(a+r)^{k+2} - a^{k+2}}{r}\right), \tag{3.10}$$

where $k = 1, 2, \ldots$, where $h = 2/[(b - a) + (d - c)] = 1/(r + s) = 1/(b - a - r)$.

The expected value of an isosceles trapezoidal distribution can be obtained by substituting $c = (a + r)$ and $d = (b - r)$ in (2.28). Thus we have

$$E(X) = \frac{a + b}{2}. \tag{3.11}$$

The expected value (3.11) agrees with (3.10) evaluated for $k = 1$.

The variance of an isosceles trapezoidal distribution is obtained by replacing $t$ with $r$ in (2.32). Thus we have

$$V(X) = \frac{2r^2 + 2rs + s^2}{12}. \tag{3.12}$$

By substituting $2r^2 = r^2 + r^2 = (c - a)^2 + (b - d)^2$ and $2rs = (c - a)(d - c) + (d - c)(b - d)$, in (3.12), the variance of an isosceles trapezoidal distribution, expressed in terms of the widths $(c - a)$, $(d - c)$ and $(b - d)$, is

$$V(X) = \big\{(c - a)^2 + (d - c)^2 + (b - d)^2 + (c - a)(d - c) + (d - c)(b - d)\big\}/12, \tag{3.13}$$

The expression (3.13) simplifies to

$$V(X) = \frac{a^2 + b^2 + c^2 + d^2 - ac - ad - bc - bd}{12}, \tag{3.14}$$

where $c = (a + r)$ and $d = (b - r)$.

*Comment 2.* The ISO-GUM [1, section 4.3.9] discusses an isosceles trapezoidal distribution defined on a base of width $2a$ and a top of width $2a\beta$, where $0 \leqslant \beta \leqslant 1$. The Trapezoid $(-a, -a\beta, a\beta, a)$ is such a distribution. This distribution approaches a rectangular distribution on the interval $(-a, a)$ as $\beta$ approaches one, and it approaches an isosceles triangular distribution on the interval $(-a, a)$ as $\beta$ approaches zero. If we substitute $-a$ for $a$, $-a\beta$ for $c$, $a\beta$ for $d$ and $a$ for $b$ in (3.14), we get $V(X) = a^2(1 + \beta)^2/6$. This is the expression given in the ISO-GUM [1, section 4.3.9].

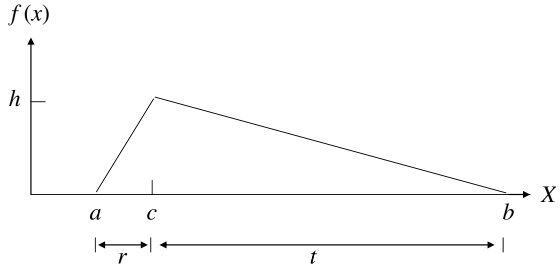


**Figure 5.** Probability density function of Triangle (*a, c, b*).

*Comment 3.* Suppose a random variable $X_1$ has a rectangular distribution on the interval $(-k, +k)$ and suppose another random variable $X_2$ has a rectangular distribution on the interval $(-\delta, +\delta)$ where $k \geqslant \delta$. Then the probability distribution of the sum $X = X_1 + X_2$ is the isosceles trapezoidal distribution Trapezoid $(-(k+\delta), -(k-\delta), (k-\delta), (k+\delta))$ [5, sections 4.07–4.13]. If we substitute $a = -(k+\delta)$, $c = -(k-\delta)$, $d = (k-\delta)$ and $b = (k+\delta)$) in (3.14), we get $V(X) = (k^2+\delta^2)/3$. This is the expression given in [5]. When $k = \delta$, the distribution of $X = X_1 + X_2$ is an isosceles triangular distribution on the interval $(-2k, +2k)$ with variance $V(X) = (2/3)k^2$.

### 3.4. Triangle $(a, c, b)$

The trapezoidal distribution of figure 1 approaches a triangular distribution shown in figure 5 as the point $d$ approaches the point $c$. When $d$ approaches $c$, the width $s$ of the middle rectangle approaches zero. The pdf $f(x)$, the cdf $F(x)$, the inverse cdf $F^{-1}(y)$ and the mgf $M(t)$ of a triangular distribution are obtained from equations (2.1) through (2.18) and (2.24) by taking the limit as $d$ approaches $c$.

The mgf and the moments of a triangular distribution with parameters $a$, $c$ and $b$, where $a \leqslant c$ and $c \leqslant b$, are determined by taking the limits of (2.24) and (2.26) as $d$ approaches $c$. Thus we have

$$M(t) = \frac{h}{t^2}\left(\frac{\mathrm{e}^{bt} - \mathrm{e}^{ct}}{b-c} - \frac{\mathrm{e}^{ct} - \mathrm{e}^{at}}{c-a}\right) \tag{3.15}$$

and

$$E(X^k) = \frac{h}{(k+2)(k+1)}\left(\frac{b^{k+2} - c^{k+2}}{b-c} - \frac{c^{k+2} - a^{k+2}}{c-a}\right), \tag{3.16}$$

for $k = 1, 2, \ldots$, where $h = 2/(r+t) = 2/(b-a)$.

The expected value of a triangular distribution can be obtained by replacing $d$ with $c$ in (2.28). We get

$$E(X) = \frac{a+b+c}{3}. \tag{3.17}$$

The expected value (3.17) agrees with (3.16) evaluated for $k = 1$.

The variance of a triangular distribution is obtained by setting $s = 0$ in (2.32). We get

$$V(X) = \frac{r^2 + rt + t^2}{18}. \tag{3.18}$$

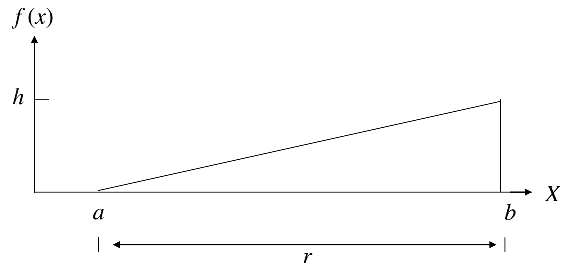


**Figure 6.** Probability density function of left-side sloping right Triangle $(a, b)$.

An expression for the variance of a triangular distribution in terms of the parameters $a$, $c$ and $b$ is obtained by substituting $(c-a)$ for $r$ and $(b-c)$ for $t$ in (3.18). Thus we have

$$V(X) = \frac{a^2 + b^2 + c^2 - ab - ac - bc}{18}. \tag{3.19}$$

*Comment 4.* Expressions (3.17) and (3.19) agree with [6]. The expected value and variance of a triangular distribution with end points $-\alpha_1$ and $\alpha_2$ and with peak at zero were reported in [7] as $E(X) = (\alpha_2 - \alpha_1)/3$ and $V(X) = (\alpha_2 - \alpha_1)^2/18 + \alpha_1\alpha_2/6$. If we substitute $a = -\alpha_1$, $b = 0$ and $c = \alpha_2$ in (3.17) and (3.19), we get these results.

### 3.5. Left-side sloping right Triangle (a,b)

The trapezoidal distribution of figure 1 approaches a left-side sloping right triangular distribution shown in figure 6 as both the points $d$ and $c$ approach the point $b$. Then both $s$ and $t$ approach zero. The pdf $f(x)$, the cdf $F(x)$, the inverse cdf $F^{-1}(y)$ and the mgf $M(t)$ of a left-side sloping right triangular distribution are obtained from equations (2.1) through (2.18) and (2.24) by taking the limits as both $d$ and $c$ approach $b$.

The mgf and the moments of a left-side sloping right triangular distribution with parameters $a$ and $b$ are determined by taking the limits of (2.24) and (2.26) as both $d$ and $c$ approach $b$. Thus, we have

$$M(t) = \frac{h}{t}(\mathrm{e}^{bt}) - \frac{h}{t^2}\left(\frac{\mathrm{e}^{bt} - \mathrm{e}^{at}}{b-a}\right) \tag{3.20}$$

and

$$E(X^k) = \frac{h}{(k+2)(k+1)}\left((k+2)b^{k+1} - \frac{b^{k+2} - a^{k+2}}{b-a}\right), \tag{3.21}$$

for $k = 1, 2, \ldots$, where $h = 2/r = 2/(b-a)$.

The expected value of a left-side sloping right triangular distribution can be obtained by substituting $b$ for both $d$ and $c$ in (2.28). We get

$$E(X) = \frac{a+2b}{3}. \tag{3.22}$$

The expected value (3.22) agrees with (3.21) evaluated for $k = 1$.

The variance of a left-side sloping right triangular distribution is obtained by setting $s = 0$ and $t = 0$ in (2.32). We get

$$V(X) = \frac{r^2}{18} = \frac{(b-a)^2}{18}. \tag{3.23}$$

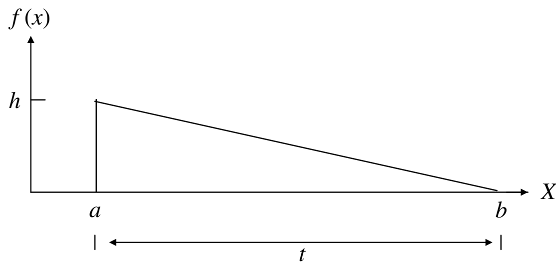


**Figure 7.** Probability density function of right-side sloping right Triangle $(a, b)$.

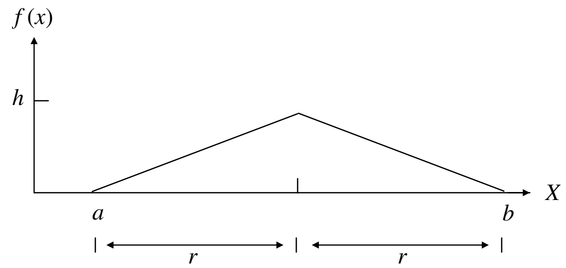


**Figure 8.** Probability density function of isosceles Triangle $(a, b)$.

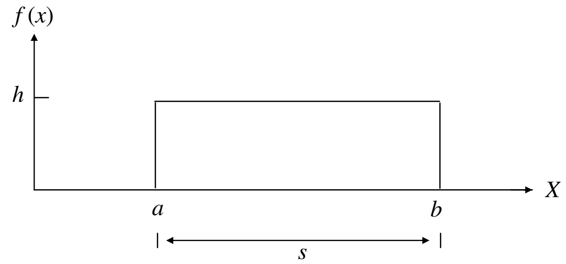


**Figure 9.** Probability density function of Rectangle $(a, b)$.

### *3.6. Right-side sloping right Triangle (a,b)*

The trapezoidal distribution of figure 1 approaches a right-side sloping right triangular distribution shown in figure 7 as both the points $c$ and $d$ approach the point $a$. Then both $r$ and $s$ approach zero. The pdf $f(x)$, the cdf $F(x)$, the inverse cdf $F^{-1}(y)$ and the mgf $M(t)$ of a right-side sloping right triangular distribution are obtained from equations (2.1) through (2.18) and (2.24) by taking the limits as both $c$ and $d$ approach $a$.

The mgf and moments of a right-side sloping right triangular distribution with parameters $a$ and $b$ are determined by taking the limits of (2.24) and (2.26) as both $c$ and $d$ approach $a$. Thus, we have

$$M(t) = \frac{h}{t^2}\left(\frac{e^{bt} - e^{at}}{b - a}\right) - \frac{h}{t}(e^{at}) \tag{3.24}$$

and

$$E(X^k) = \frac{h}{(k+2)(k+1)}\left(\frac{b^{k+2} - a^{k+2}}{b - a} - (k+2)a^{k+1}\right), \tag{3.25}$$

for $k = 1, 2, \ldots$, where $h = 2/t = 2/(b - a)$.

The expected value of a right-side sloping right triangular distribution can be obtained by substituting $a$ for both $c$ and $d$ in (2.28). We get

$$E(X) = \frac{2a + b}{3}. \tag{3.26}$$

The expected value (3.26) agrees with (3.25) evaluated for $k = 1$.

The variance of a left-side sloping right triangular distribution is obtained by setting $r = 0$ and $s = 0$ in (2.32). We get

$$V(X) = \frac{t^2}{18} = \frac{(b - a)^2}{18}. \tag{3.27}$$

### *3.7. Isosceles Triangle (a,b)*

The triangular distribution of figure 5 becomes an isosceles triangular distribution shown in figure 8 when $c = (a + b)/2$ or equivalently $r = t$. Thus, an isosceles triangular distribution is defined by only two parameters $a$ and $b$.

The mgf and moments of an isosceles triangular distribution with parameters $a$ and $b$ are determined by substituting $c = (a + b)/2$ in (3.15) and (3.16). Thus we have

$$M(t) = \frac{h^2}{t^2}(e^{bt} - 2e^{(a+b)t/2} + e^{at}) \tag{3.28}$$

and

$$E(X^k) = \frac{h^2}{(k+2)(k+1)}(b^{k+2} - 2[(a + b)/2]^{k+2} + a^{k+2}), \tag{3.29}$$

for $k = 1, 2, \ldots$, where $h = 1/r = 2/(b - a)$.

The expected value and variance of an isosceles triangular distribution are obtained from (3.17)–(3.19) by replacing $c$ with $(a + b)/2$ and $t$ with $r$. Thus we have

$$E(X) = \frac{a + b}{2}. \tag{3.30}$$

and

$$V(X) = \frac{r^2}{6} = \frac{(b - a)^2}{24}. \tag{3.31}$$

The expected value (3.30) agrees with (3.29) evaluated for $k = 1$.

### *3.8. Rectangle (a,b)*

The trapezoidal distribution of figure 1 approaches a rectangular distribution shown in figure 9 as the point $c$ approaches the point $a$ and the point $d$ approaches the point $b$. Then both $r$ and $t$ approach zero. The pdf $f(x)$, the cdf $F(x)$, the inverse cdf $F^{-1}(y)$ and the mgf $M(t)$ of a rectangular distribution are obtained from equations (2.1) through (2.18) and (2.24) by taking the limit as $c$ approaches $a$ and $d$ approaches $b$.

The mgf and moments of a triangular distribution with parameters $a$ and $b$ are determined by taking the limits of (2.24) and (2.26) as $c$ approaches $a$ and $d$ approaches $b$. Thus we have

$$M(t) = \frac{h}{t}(e^{bt} - e^{at}) \tag{3.32}$$

and

$$E(X^k) = \frac{h}{(k+1)}(b^{k+1} - a^{k+1}), \tag{3.33}$$

for $k = 1, 2, \ldots$, where $h = 1/s = 1/(b - a)$.

**Table 1.** Expected values and variances of trapezoidal, triangular, and rectangular probability distributions.

| | Expected value $E(X)$ | Variance $V(X)$ |
|---|---|---|
| Trapezoid $(a, c, d, b)$, figure 1 | $\frac{h}{6}\left(\frac{b^3-d^3}{b-d}-\frac{c^3-a^3}{c-a}\right)$ where $h = 2\div[(b-a)+(d-c)]$ | $\frac{3(r+2s+t)^4+6(r^2+t^2)(r+2s+t)^2-(r^2-t^2)^2}{[12(r+2s+t)]^2}$ where $r=(c-a)$, $s=(d-c)$, $t=(b-d)$, and $(r+2s+t)=[(b-a)+(d-c)]$ |
| Only left-side sloping Trapezoid $(a, c, b)$, figure 2 | $\frac{3b^2-a^2-c^2-ac}{3(2b-a-c)}$ | $\frac{6s^4+12rs^3+12r^2s^2+6r^3s+r^4}{18(r+2s)^2}$ where $r=(c-a)$ and $s=(b-c)$ |
| Only right-side sloping Trapezoid $(a, d, b)$, figure 3 | $\frac{b^2+d^2+bd-3a^2}{3(b+d-2a)}$ | $\frac{6s^4+12ts^3+12t^2s^2+6t^3s+t^4}{18(t+2s)^2}$ where $s=(d-a)$ and $t=(b-d)$ |
| Isosceles Trapezoid $(a, c, d, b)$, figure 4 | $\frac{a+b}{2}$ | $\frac{a^2+b^2+c^2+d^2-ac-ad-bc-bd}{12}$ |
| Triangle $(a, c, b)$, figure 5 | $\frac{a+b+c}{3}$ | $\frac{a^2+b^2+c^2-ab-ac-bc}{18}$ |
| Left-side sloping right Triangle $(a, b)$, figure 6 | $\frac{a+2b}{3}$ | $\frac{(b-a)^2}{18}$ |
| Right-side sloping right Triangle $(a, b)$, figure 7 | $\frac{2a+b}{3}$ | $\frac{(b-a)^2}{18}$ |
| Isosceles Triangle $(a, b)$, figure 8 | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{24}$ |
| Rectangle $(a, b)$, figure 9 | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |

The expected value of a rectangular distribution, obtained by substituting $a$ for $c$ and $b$ for $d$ in (2.28), is

$$E(X) = \frac{a+b}{2}. \tag{3.34}$$

The expected value (3.34) agrees with (3.33) evaluated for $k = 1$.

The variance of a rectangular distribution is obtained by substituting $r = 0$ and $t = 0$ in (2.32) and then replacing $s$ with $(b-a)$. Thus we have

$$V(X) = \frac{s^2}{12} = \frac{(b-a)^2}{12}. \tag{3.35}$$

For quick reference, the expected values and variances of the trapezoidal distribution Trapezoid $(a, c, d, b)$ and its special cases are summarized in table 1.

## 4. The ISO-GUM's approach to incorporate correction for bias and its associated uncertainty

An important use of a Type B probability distribution is to quantify the correction for bias in a result of measurement and to quantify the uncertainty associated with the correction. A bias may be additive or multiplicative.

### *4.1. Additive bias*

Suppose $x$ is a result of measurement for the value $Y$ of a measurand and its associated standard uncertainty is $u(x)$. Suppose the expected value of the sampling distribution of $x$ is $X$, then the difference $B = X - Y$ is the additive bias[6] in $x$. A metrological term for the additive bias is offset. A measurement equation is required to incorporate a correction for bias in the result $x$. All inputs and output of a measurement equation are variables having state-of-knowledge probability distributions. The measurement equation corresponding to the additive bias $X - Y$ is

$$Y = X + C, \tag{4.1}$$

where $C$ is a variable with a state-of-knowledge probability distribution for the negative $Y - X$ of bias [8, section 2.2]. In the measurement equation (4.1), $X$ is a variable representing the state of knowledge about the expected value of $x$ and $Y$ is a variable representing the state of knowledge about the value of the measurand. The ISO-GUM [1, section 4.1.6] interprets the result $x$ and the uncertainty $u(x)$ as the expected value, $E(X)$, and the standard deviation, $S(X)$, of a state-of-knowledge probability distribution for $X$. (This interpretation is justified when Bayesian statistics is used for Type A evaluations [8]. However, when sampling theory is used, this interpretation is a declaration.) Suppose that the expected value, $E(C)$, and the standard deviation, $S(C)$, of a state-of-knowledge probability distribution for $C$ are $c$ and $u(c)$, respectively. Then a corrected result $y$ for $Y$ is determined by substituting the result $x$ for the variable $X$ and the correction $c$ for the variable $C$ in the

[6] As in the ISO-GUM, we use the same symbol $X$ for the expected value of the sampling distribution of $x$ as well as for a variable having a state-of-knowledge probability distribution about the expected value. Likewise, we use the same symbol $Y$ for the value of the measurand as well as for a variable having a state-of-knowledge probability distribution about the value of the measurand.

measurement equation (4.1). Thus

$$y = x + c. \tag{4.2}$$

If the state-of-knowledge probability distributions for $X$ and $C$ are uncorrelated, then the combined standard uncertainty $u(y)$, determined by propagating the uncertainties using the measurement equation (4.1), is

$$u(y) = [u^2(x) + u^2(c)]^{1/2}. \tag{4.3}$$

The ISO-GUM [1] regards the result $y$ and uncertainty $u(y)$ as the expected value, $E(Y)$, and the standard deviation, $S(Y)$, of a state-of-knowledge distribution for $Y$. In the example discussed in section 5, the biases are additive; therefore, expressions (4.2) and (4.3) are used to determine the corrected result and its associated uncertainty.

### 4.2. Multiplicative bias

In some applications, multiplicative bias, defined as the ratio $B = X/Y$, is more appropriate than the additive bias $X - Y$. A metrological term for correcting the multiplicative bias is scaling. The measurement equation corresponding to the multiplicative bias $X/Y$ is

$$Y = X \times C, \tag{4.4}$$

where $C$ is a variable with a state-of-knowledge probability distribution for the reciprocal $Y/X$ of multiplicative bias. The corrected result $y$ based on (4.4) is

$$y = x \times c, \tag{4.5}$$

where the correction factor $c$ is the expected value of $C$. The uncertainty $u(y)$ is determined from a linear approximation of the measurement equation (4.4) which is

$$\frac{Y - y}{y} \approx \frac{X - x}{x} + \frac{C - c}{c}. \tag{4.6}$$

The expression for propagating uncertainties based on the linear approximation (4.6) is

$$u_r(y) = [u_r^2(x) + u_r^2(c)]^{1/2}, \tag{4.7}$$

where $u_r(y) = u(y)/|y|$, $u_r(x) = u(x)/|x|$ and $u_r(c) = u(c)/|c|$ are the relative standard uncertainties associated with $y$, $x$ and $c$, respectively [8]. Then

$$u(y) = |y| \times u_r(y) = |y| \times [u_r^2(x) + u_r^2(c)]^{1/2}. \tag{4.8}$$

In a physical chemistry paper [9], the bias is multiplicative; therefore, expressions (4.5) and (4.8) are used there to determine the corrected result and its associated uncertainty.

### 4.3. Trapezoidal versus rectangular distribution for the correction variable

The expected value $E(C) = c$ and the standard deviation $S(C) = u(c)$ are usually quantified by specifying a Type B state-of-knowledge distribution for the correction variable $C$ [1, section 4.3]. Often, the distribution for $C$ is specified to be a rectangular distribution on some interval $(-a, a)$ with the expected value zero and the standard deviation $a/\sqrt{3}$. A rectangular distribution represents uniform (equal) probability in the interval $(-a, a)$ and zero probability outside. A trapezoidal distribution, for example, Trapezoid $(-a, -a\beta, a\beta, a)$ where $0 < \beta < 1$, represents uniform (equal) probability in the sub-interval $(-a\beta, a\beta)$ and the probability decreases to zero linearly as we move out to the end points of the interval $(-a, a)$. Often, the end points of the rectangular distribution are not known exactly. Also, it is more realistic to expect that the values near the end points are less probable than the values near the middle of the interval. Therefore, a trapezoidal probability distribution with (one or both) sides sloping is often a more realistic state-of-knowledge distribution for $C$ than a rectangular distribution. We quote the ISO-GUM [1, section 4.3.9]

> 'In 4.3.7, because there was no specific knowledge about the possible values of $X_i$ within its estimated bounds $a_-$ to $a_+$, one could only assume that it was equally probable for $X_i$ to take any value within those bounds, with zero probability of being outside them. Such step function discontinuities in a probability distribution are often unphysical. In many cases it is more realistic to expect that values near the bounds are less likely than those near the midpoint. It is then reasonable to replace the symmetric rectangular distribution with a symmetric trapezoidal distribution having equal sloping sides (an isosceles trapezoid), . . . .'

In some cases an asymmetric distribution for the correction variable $C$ is either required or is a more realistic representation of the state of knowledge. The expressions for the expected values and variances tabulated in section 3 make the trapezoidal distribution, Trapezoid $(a, c, d, b)$, and its special cases easy to use.

**Table 2.** The amounts of Hg in mg kg$^{-1}$ as determined by the two methods with uncertainties as reported in [10].

| | Method 1 | Method 2 |
|---|---|---|
| $x_i$ | 0.368 mg kg$^{-1}$ | 0.310 mg kg$^{-1}$ |
| $s_i$ | 0.0110 mg kg$^{-1}$ | 0.0086 mg kg$^{-1}$ |
| $n_i$ | 4 | 20 |
| $s(x_i)$ | 0.0055 mg kg$^{-1}$ | 0.0019 mg kg$^{-1}$ |
| $c_i$ | 0 | |
| $u(c_i)$ | 0.0060 mg kg$^{-1}$ | |
| $u(x_i)$ | 0.0081 mg kg$^{-1}$ | 0.0019 mg kg$^{-1}$ |

## 5. Asymmetric correction for bias and uncertainty

Our interest in trapezoidal distribution arose from a 'two-method problem' addressed in [10, section 4]. The amount of mercury (Hg) in a reference material was measured by two methods. Method 1: cold vapour atomic absorption spectrometry (CVAAS) in NIST[7]. Method 2: another method in another US laboratory. The results, as reported in [10, section 4], are displayed in table 2. The two-method problem is how to determine a combined result for the amount of Hg (the measurand) based on the results displayed in table 2 and more importantly how to determine the uncertainty associated with the combined result.

[7] National Institute of Standards and Technology, Gaithersburg, MD, USA.

In table 2, $x_i = \sum_j x_{ij}/n_i$ is the arithmetic mean and $s_i = \sqrt{[\sum_j (x_{ij} - x_i)^2/(n_i - 1)]}$ is the experimental standard deviation of $n_i$ independent measurements, and $s(x_i) = s_i/\sqrt{n_i}$ is an estimate of the standard deviation of the mean $x_i$, for $i = 1$ and 2. A correction is applied to the result $x_i$ when it is believed to have a recognized systematic effect. The (additive) correction applied to $x_i$ is $c_i$ with uncertainty $u(c_i)$. The correction and uncertainty applied to the result $x_1$ are indicated in table 2. The result $x_2$ is not corrected (supposedly, because no systematic effect was recognized). The uncertainties, $u(x_i)$, as reported in [10] are shown in table 2. These uncertainties are obtained from the expression (4.3) by substituting $s(x_i)$ for $u(x)$ and $u(c_i)$ for $u(c)$ [10].

In [10], the arithmetic mean of the two results $x_1$ and $x_2$ is used as the initial combined result $x_A = (x_1 + x_2)/2 = 0.339\,\text{mg kg}^{-1}$. The corresponding standard uncertainty is $u(x_A) = \sqrt{[(1/2)^2u^2(x_1) + (1/2)^2u^2(x_2)]} = 0.0042\,\text{mg kg}^{-1}$. The resulting two-standard uncertainty interval is $(x_A \pm 2 \times u(x_A)) = (0.339 \pm 2 \times 0.0042)\,\text{mg kg}^{-1} = (0.3306,\ 0.3474)\,\text{mg kg}^{-1}$. This interval excludes both results $x_1 = 0.368\,\text{mg kg}^{-1}$ and $x_2 = 0.310\,\text{mg kg}^{-1}$. The results $x_1$ and $x_2$ are highly plausible values of the common measurand; therefore, $u(x_A) = 0.0042\,\text{mg kg}^{-1}$ is deemed to be an undervaluation of the uncertainty associated with $x_A$ [10]. The undervaluation of $u(x_A)$ is attributed to an unknown bias[8] in $x_A$ [7, 11]. Following the ISO-GUM's approach (discussed in our section 4), the authors of [10] apply the correction $c = 0$ for bias in $x_A$ with uncertainty $u(c) = 0.0167\,\text{mg kg}^{-1}$ to obtain the corrected result $y = x_A + c = x_A = 0.339\,\text{mg kg}^{-1}$ with uncertainty $u(y) = \sqrt{[u^2(x_A) + u^2(c)]} = 0.0173\,\text{mg kg}^{-1}$. The corresponding two-standard uncertainty interval is $(y \pm 2 \times u(y)) = (0.339 \pm 2 \times 0.0173)\,\text{mg kg}^{-1} = (0.3045,\ 0.3735)\,\text{mg kg}^{-1}$. This interval includes both results $x_1 = 0.368\,\text{mg kg}^{-1}$ and $x_2 = 0.310\,\text{mg kg}^{-1}$. Thus, the uncertainty $u(y) = 0.0173\,\text{mg kg}^{-1}$ is regarded as a satisfactory expression of the uncertainty associated with the corrected combined result $y = x_A = 0.339\,\text{mg kg}^{-1}$.

### *5.1. Results from rectangular and trapezoidal distributions*

In [10] the correction $c = 0$ and uncertainty $u(c) = 0.0167\,\text{mg kg}^{-1}$ were determined from a rectangular distribution on the interval $(x_2 - x_A,\ x_1 - x_A) = (-0.029, 0.029)$, which has the expected value zero and the standard deviation $0.029/\sqrt{3} = 0.0167$. (Since $x_2$ is less than $x_1$, the lower limit of the rectangular distribution is $x_2 - x_A$ and the upper limit is $x_1 - x_A$.) As an alternative probability distribution to determine the correction $c$ and uncertainty $u(c)$, we consider the trapezoidal distribution, Trapezoid $(x_2 - u(x_2) - x_A,\ x_2 + u(x_2) - x_A,\ x_1 - u(x_1) - x_A,\ x_1 + u(x_1) - x_A)$. Both distributions are sketched in figure 10. The expected value and standard deviation of the trapezoidal distribution, determined from the expressions (2.28) and (2.32), are $E(C) = 0.0002$ and $S(C) = 0.0171$; thus, $c = 0.0002\,\text{mg kg}^{-1}$ and $u(c) = 0.0171\,\text{mg kg}^{-1}$. This trapezoidal distribution is slightly asymmetric about zero; therefore, it represents a small non-zero correction. The result $x_A$, correction $c$, corrected result $y = x + c$ and the corresponding uncertainties based on the two distributions are shown in table 3. We note that both rectangular and trapezoidal probability distributions yield two-standard uncertainty intervals $(y \pm 2 \times u(y))$ that include the results $x_1$ and $x_2$.

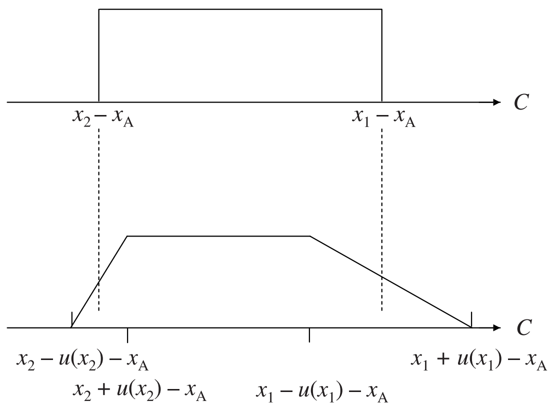


**Figure 10.** Sketches of the rectangular and trapezoidal distributions.

**Table 3.** Result $x_A$, correction $c$, corrected result $y$ and corresponding uncertainties in mg kg$^{-1}$ determined from rectangular and trapezoidal distributions.

| | Rectangular distribution | Trapezoidal distribution |
|---|---|---|
| $x_A$ | 0.3390 | 0.3390 |
| $u(x_A)$ | 0.0042 | 0.0042 |
| $c$ | 0.0000 | 0.0002 |
| $u(c)$ | 0.0167 | 0.0171 |
| $y$ | 0.3390 | 0.3392 |
| $u(y)$ | 0.0173 | 0.0176 |

### *5.2. Motivation underlying trapezoidal distribution*

Suppose $X_A$ is the expected value[9] of the sampling distribution of the mean $x_A$, then the additive bias in $x_A$ is $X_A - Y$. The measurement equation required to incorporate correction for bias in $x_A$ is $Y = X_A + C$. A Type B distribution for the correction variable $C$ is a state-of-knowledge probability distribution for the negative $Y - X_A$ of bias where both $Y$ and $X_A$ are regarded as random variables with state-of-knowledge distributions. Following the ISO-GUM [1], the result $x_A$ is interpreted as the expected value of a state-of-knowledge probability distribution for $X_A$. (This interpretation is justified when Bayesian statistics is used for Type A evaluations.) Thus, we may heuristically think that a probability distribution for $C$ represents subjective belief probabilities about the possible values of $Y - x_A$, where $x_A$ is known [7, 8, 12]. In [10], the limits of the rectangular distribution for $C$ are set as $x_2 - x_A$ and $x_1 - x_A$. This corresponds to uniform (equal) belief probability for values of $Y$ in the interval $(x_2, x_1)$ and zero probability for values outside this interval. Since the possible values of $Y$ are in the interval $(x_1, x_2)$ as well as to the left and to the right of this interval, one cannot be definite about the limits

[8] When only one method is used, the bias in the result may remain hidden, unknown and unaccounted for in the reported uncertainty. When multiple methods are used the possibility that the results may be biased is exposed. The bias in the result $x_i$ is $E(x_i) - Y$, for $i = 1$ and 2, and the bias in $x_A$ is $E(x_A) - Y$, where $E(.)$ is the expected value of the sampling distribution.

[9] We use the same symbol $X_A$ for the expected value as well as for a random variable having a state-of-knowledge distribution about the expected value.

of a rectangular distribution specified for $C$. Therefore, we considered a trapezoidal distribution for $C$. In setting the parameters of the trapezoidal distribution as $x_2 - u(x_2) - x_A$, $x_2 + u(x_2) - x_A$, $x_1 - u(x_1) - x_A$ and $x_1 + u(x_1) - x_A$ we thought as follows. The result of method 1 is not just the single value $x_1$ but a range of values, for example $(x_1 \pm 2 \times u(x_1))$, that could reasonably be attributed to $Y$. In particular, based on method 1 alone, the range of values $(x_1 - u(x_1), x_1 + u(x_1))$ may be regarded as highly plausible values for $Y$. Similarly, based on method 2 alone, the range of values $(x_2 - u(x_2), x_2 + u(x_2))$ may be regarded as highly plausible values for $Y$. Therefore, a trapezoidal distribution with the left-side sloping on the interval $(x_2 - u(x_2) - x_A, x_2 + u(x_2) - x_A)$ and the right-side sloping on the interval $(x_1 - u(x_1) - x_A, x_1 + u(x_1) - x_A)$ may be a better state-of-knowledge distribution for $C$ than a rectangular distribution with sharp cutoffs at $x_2 - x_A$ and $x_1 - x_A$.

## 6. Summary

A Type B estimate and standard uncertainty are determined from a state-of-knowledge probability distribution specified by scientific judgment. All probability distributions discussed in the ISO-GUM are symmetric. However, in some applications an asymmetric distribution is needed. We described a trapezoidal distribution which may be asymmetric depending on the settings of its parameters. A trapezoidal distribution is versatile. In particular, asymmetric and isosceles triangular distributions and rectangular distribution are special cases. The ISO-GUM's procedure to determine a result and standard uncertainty for an output quantity (measurand) is to propagate the expected values and the standard deviations of the probability distributions specified for the input quantities through a linear approximation of the measurement equation. We presented explicit and simple expressions for the expected values and the variances of a trapezoidal distribution and of its special cases. (We also presented explicit and simple expressions for the moment generating functions and moments of all orders.) A working group of the BIPM/JCGM[10] has drafted Supplement 1 to the ISO-GUM (draft GUMS1). The draft GUMS1 [4] propagates probability distributions specified for the input variables through a numerical simulation of the measurement equation to determine a probability distribution for the value of the output variable. Numerical simulation requires generation of random numbers from the probability distributions specified for the input variables. We showed how random numbers from a trapezoidal distribution and from its special cases may be easily generated. Thus, we have proposed many useful alternatives to the use of a rectangular distribution for the Type B input variables. A particular application, in which a Type B probability distribution is often used, is to determine the correction for bias and the standard uncertainty associated with the correction in a result of measurement. We illustrated how a trapezoidal distribution may be used to determine an asymmetric correction for bias and its associated standard uncertainty.

## Acknowledgments

The authors would like to thank the referee, Burt Rust and Blaza Toman for their comments on previous drafts.

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