Proceedings of Dynamic Systems and Applications 4 (2004) 1- xxx

APPROXIMATING LIMIT CYCLES OF A VAN DER POL EQUATION WITH DELAY

DAVID E. GILSINN

Mathematical and Computational Sciences Division
National Institute of Standards and Technology
100 Bureau Drive, Stop 8910
Gaithersburg, MD 20899-8910 [1]

ABSTRACT. In this paper a theorem of Stokes is used to establish the existence of a periodic solution of a Van der Pol equation with fixed delay in the neighborhood of an approximating solution that satisfies a certain noncriticality condition.

AMS (MOS) Subject Classification. 34K11, 34K13, 34K28.

1. Introduction

Given $x, X \epsilon \mathbb{R}^n$, how does one approximate a periodic solution of a delay differential equation (DDE) of the form

$$\dot{x} = X(x(t), x(t-h))? \tag{1}$$

Next, can one establish the existence of a true periodic solution in a neighborhood of the approximate solution and produce a numerical error bound? This latter question has been answered in the positive in the case of ordinary differential equations in Stokes, 1972, and Urabe, 1965, and for functional differential equations in Stokes, 1976. Although the results have been used to estimate approximation bounds in ordinary differential equations no application in functional or delay differential equations (DDEs) has been attempted. The object of this paper is to apply the functional differential equation result in Stokes, 1976, to a Van der Pol equation with delay terms.

2. The Main Approximation Theorem

Given $\overline{\omega} > 0$, let $C_0 = C[-\overline{\omega}, 0]$ be the space of continuous functions on $[-\overline{\omega}, 0]$. We introduce $t/\overline{\omega}$ for t in (1) to produce

$$\overline{\omega}\dot{x} = X(x(t), x(t-\overline{\omega}h)). \tag{2}$$

A solution of (2) will be designated by $x(t;\phi)$ where $\phi \epsilon C_0$ and $x(t;\phi) = \phi$ for $t \epsilon [-\overline{\omega}, 0]$. The following notation will be used. Let $x \epsilon \mathbb{R}^n$, and set

$$x_t = x(t;\phi), \quad |x_t| = |x(t;\phi)| = \sqrt{x_1(t;\phi)^2 + \cdots + x_n(t;\phi)^2} \tag{3}$$

Further let

$$\begin{aligned} x_{t,\overline{\omega}} &= x(t-\overline{\omega}h;\phi), \\ x_{\overline{\omega}} &= (x_t, x_{t,\overline{\omega}}), \\ |x_{\overline{\omega}}| &= \max\Big(\sup_{0\le t\le 2\pi}|x_t|, \sup_{0\le t\le 2\pi}|x_{t,\overline{\omega}}|\Big), \\ dX(x_{\overline{\omega}}; z) &= X_1(x_{\overline{\omega}})z_1 + X_2(x_{\overline{\omega}})z_2, \quad z = (z_1, z_2) \end{aligned} \tag{4}$$

[1]Contribution of the National Institute of Standards and Technology, a Federal agency, not subject to copyright.

where the subscript i of X represents the partial derivative with respect to the i^{th} component of $x_{\overline{\omega}}$ where $i = 1, 2$. Let $\overline{x}(t)$ be 2π periodic and $0 < \overline{\omega} < 2\pi$ satisfy

$$\overline{\omega}\dot{\overline{x}} = X(\overline{x}_{\overline{\omega}}) + k(t), \;\; k(t + 2\pi) = k(t). \tag{5}$$

The variational equation of (2) with respect to $\overline{x}_{\overline{\omega}}$ is given by

$$\overline{\omega}\dot{z}(t) = dX(\overline{x}_{\overline{\omega}}; z_{\overline{\omega}}), \tag{6}$$

and the adjoint to the variational equation is given by

$$\overline{\omega}\dot{y}(t) = -dX^T(\overline{x}_{\overline{\omega}}; y_{\overline{\omega}}), \tag{7}$$

Definition 2.1. *The pair $(\overline{\omega}, \overline{x})$ is said to be noncritical with respect to (2) if the variational equation (6) about $\overline{x}(t)$ has a characteristic multiplier, ρ_0, of multiplicity one with the remaining multipliers not equal to one. If $\nu_0(t, \psi)$, $\psi \epsilon C[2\pi, 2\pi + \overline{\omega}]$, with $|\nu_0|_2 = 1$, is the 2π periodic solution of the adjoint (7) corresponding to ρ_0 then*

$$\int_0^{2\pi} \nu_0^T(t;\psi) J(\dot{\overline{x}}, \overline{\omega})(t)\, dt \neq 0, \tag{8}$$

where

$$J(\dot{\overline{x}}, \overline{\omega})(t) = \dot{\overline{x}}_t + hX_2(x_{\overline{\omega}})\dot{\overline{x}}_{t,\overline{\omega}}. \tag{9}$$

This definition is given here for fixed delay differential equations but is given in a more general form for functional differential equations in Stokes, 1976, and is motivated by a result in Hale, 1971. We first state a lemma that is proven in Halanay, 1966.

Lemma 2.2. *If $\overline{x}(t)$ is noncritical and $f(t)$ is 2π periodic such that*

$$\int_0^{2\pi} \nu_0^T(t;\psi) f(t)\, dt = 0, \tag{10}$$

then there exists a unique 2π periodic solution of

$$\overline{\omega}\dot{z}(t) = dX(\overline{x}_{\overline{\omega}}; z_{\overline{\omega}}) + f(t), \tag{11}$$

which satisfies $|z| \leq M\ |f|_2$ for some $M > 0$, independent of f.

The main theorem is proven in Stokes, 1976 by a contraction argument.

Theorem 2.3. *Let $(\overline{\omega}, \overline{x})$ satisfy equation (6) and let $|k| \leq r$. Suppose there exist K_1 and K such that, for ψ_1, ψ_2,*

$$|dX(\overline{x}_{\overline{\omega}}; \phi_{\overline{\omega}})| \leq K_1|\phi|, \tag{12}$$

$$|dX(\overline{x}_{\overline{\omega}} + \psi_1; \phi_{\overline{\omega}}) - dX(\overline{x}_{\overline{\omega}} + \psi_2; \phi_{\overline{\omega}})| \leq K|\psi_1 - \psi_2||\phi|. \tag{13}$$

Assume $\overline{x}$ is noncritical in the sense of Definition 2.1 and let $\nu_0(t)$ be the appropriate solution of the adjoint to the variational equation such that $|\nu_0| = 1$. Let

$$\alpha = \left[\frac{1}{2\pi}\left|\int_0^{2\pi} \nu_0^T(t;\psi) J(\dot{\overline{x}}, \overline{\omega})(t)\, dt\right|\right]^{-1} \tag{14}$$

If M is the constant from Lemma 2.2, let

$$\lambda_1 = M(1 + \alpha|J(\dot{\overline{x}}, \overline{\omega})|), \tag{15}$$

$$\lambda_2 = (1 + MK_1)(\lambda_1/M). \tag{16}$$

Finally, if the following conditions hold

$$8\alpha r \leq \overline{\omega} \tag{17}$$

$$C(r, K, K_1, h, \alpha, \lambda_1, \lambda_2, \dot{\overline{x}}, \ddot{\overline{x}}, \overline{\omega}) < 1 \tag{18}$$

where the contraction constant C is a (lengthy) function of the prescribed parameters, then there exists an exact 2π periodic solution, x^*, and an exact frequency, ω^* so that

$$|x^* - \overline{x}| \leq 4\lambda_1 r, \tag{19}$$

$$|\omega^* - \overline{\omega}| \leq 2\alpha r. \tag{20}$$

In the particular case of a Van der Pol equation with delay we will show methods of estimating the parameters $\overline{x}, \overline{\omega}, r, K, K_1, M, \alpha, \lambda_1, \lambda_2, C$ in order to determine the bounds on the exact solution and frequency. But, in order to determine the bounds (19) and (20), we, first, need to compute the pair $(\overline{\omega}, \overline{x})$, second, to verify that the pair is noncritical in the sense of Definition 2.1 and, third, we need to estimate the particular value of M as given in Lemma 2.2. Once these steps have been accomplished all of the other parameters can be easily estimated. In the next three sections, we will temporarily assume that the pair $(\overline{\omega}, \overline{x})$ has been computed and we will develop computation tools necessary to verify their noncriticality and estimate the parameter M.

3. **Estimating Characteristic Multipliers**

In this section we assume that the variational equation with respect to the approximate solution, $\overline{x}(t)$, can be written in the form

$$\dot{z}(t;\phi) = A(t)z(t;\phi) + B(t)z(t-\overline{\omega};\phi) \tag{21}$$

where $A(t) = A(t+2\pi), B(t) = B(t+2\pi)$. Let $X(t,s)$ be the solution of (21) such that $X(s,s) = I_n, X(t,s) = 0$ for $t < s$ where I_n is the n^2 identity matrix on $\mathbb{R}^n$. The solution $X(t,s)$ is sometimes referred to as the "Fundamental Solution". The variation of constants formula for (21) is shown in Halanay, 1966, to be

$$z(t,\phi) = X(t,0)z(0) + \int_{-\overline{\omega}}^{0} X(t,\alpha+\overline{\omega})B(\alpha+\overline{\omega})z(\alpha,\phi)\,d\alpha \tag{22}$$

Define the operator

$$(U\phi)(s) = z(s+\overline{\omega},\phi) \tag{23}$$

where $\phi \epsilon C_0$, $s \epsilon \mathbb{R}$. If there is a non-trivial solution $z(t;\phi)$ of (21) such that $z(t+2\pi;\phi) = \rho z(t;\phi)$ then ρ is a characteristic multiplier of (21). If we combine (22) with (23) and note that $z(\alpha;\phi) = \phi$ for $\alpha \epsilon [-\overline{\omega}, 0]$, then characteristic multipliers are the eigenvalues of

$$(U\phi)(s) = X(s+2\pi,0)\phi(0) + \int_{-\overline{\omega}}^{0} X(s+2\pi,\alpha+\overline{\omega})B(\alpha+\overline{\omega})\phi(\alpha)\,d\alpha \tag{24}$$

where $\phi \epsilon C_0$. It is shown in Halanay, 1966, that we can restrict $s \epsilon [-\overline{\omega}, 0]$.

Although for simple equations it is possible to analytically compute $X(t,s)$ by the method of steps, in general it must be computed numerically (see, for example, the code described in Shampine & Thompson, 2001). In this paper we assume the availability of a DDE solver for at least fixed delay equations that maintains a memory of intermediate steps and the values of $X(t,s)$ taken between the initial value s and final value t.

Since $X(t,s)$ will be computed numerically, a natural approach to estimating the eigenvalues of (24) is to use the method of quadratures. To accomplish this we discretize the interval $[-\overline{\omega}, 0]$ into N equal intervals by $-\overline{\omega} = s_1 < s_2 < \cdots < s_{N+1} = 0$ and set $\Delta = \overline{\omega}/N$. The operator U can then be represented by a matrix,

$$\begin{pmatrix} (U\phi)(s_1) \\ \vdots \\ (U\phi)(s_i) \\ \vdots \\ (U\phi)(s_{N+1}) \end{pmatrix} = \begin{bmatrix} U_{1,1} & \cdots & U_{1,j} & \cdots & U_{1,N+1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{i,1} & \cdots & U_{i,j} & \cdots & U_{i,N+1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{N+1,1} & \cdots & U_{N+1,j} & \cdots & U_{N+1,N+1} \end{bmatrix} \tag{25}$$

where block elements for $i = 1, N+1, j = 1, N$ are $U_{i,j} = X(s_i+2\pi, s_j+\overline{\omega})B(s_j+\overline{\omega})\Delta$. The block elements in the last column of the matrix are given by $U_{i,N+1} = X(s_i+2\pi, s_{N+1}) + X(s_i+2\pi, s_{N+1}+\overline{\omega})B(s_{N+1}+\overline{\omega})\Delta$ for $i = 1, N+1$. We note that, since $0 < s_i + 2\pi \leq 2\pi$ and $S_{N+1} = 0$, all values of the X function in the block rows above the $i = N+1$ row can be obtained by interpolation from the stored numerical integration values. The eigenvalues of this matrix estimate the eigenvalues of U and thus the characteristic multipliers of (21).

4. **Determining Solutions of the Adjoint Equation Associated with Multipliers of the Variational Equation**

In order to estimate α in (14) let $t\epsilon[0,2\pi]$ and ψ be the initial function defined on $[2\pi,2\pi+\overline{\omega}]$. The adjoint equation is given by

$$\dot{y}(t;\psi) = -y(t;\psi)A(t) - y(t+\overline{\omega};\psi)B(t+\overline{\omega}), \tag{26}$$

where $y(t)$ is a row vector. Ordinarily solving the adjoint equation would require a "backward" integration. However, it was shown in Halanay, 1966, that the solution of the adjoint on $[0,2\pi]$ is given by

$$y(t;\psi) = \psi(2\pi)X(2\pi,t) + \int_{2\pi}^{2\pi+\overline{\omega}} \psi(\alpha)B(\alpha)X(\alpha-\overline{\omega},t)\,d\alpha. \tag{27}$$

The significance of this representation is that only a "forward" integration is required to solve for the X function. Let $\tilde{\phi}(s)$ be a continuous row vector function defined on $[-\overline{\omega},0]$. Then define the operator

$$\left(\tilde{U}\tilde{\phi}\right)(s) = \tilde{\phi}(-\overline{\omega})X(2\pi,s+\overline{\omega}) + \int_{-\overline{\omega}}^{0} \tilde{\phi}(\alpha)B(\alpha+\overline{\omega})X(2\pi+\alpha,s+\overline{\omega})\,d\alpha. \tag{28}$$

An associated operator $\tilde{V}$, defined on $[2\pi,2\pi+\overline{\omega}]$, is given in Halanay, 1966, as

$$\left(\tilde{V}\psi\right)(s) = y(s-2\pi,\psi) = \psi(2\pi)X(2\pi,s-2\pi) + \int_{2\pi}^{2\pi+\overline{\omega}} \psi(\alpha)B(\alpha)X(\alpha-\overline{\omega},s-2\pi)\,d\alpha. \tag{29}$$

It was also shown in Halanay, 1966, that an eigenvalue ρ of $\tilde{V}$ is associated with a $1/\rho$ multiplier of the adjoint equation, the eigenvalues of $U,\tilde{U},\tilde{V}$ are all the same, and the eigenvectors of $\tilde{U},\tilde{V}$ are related by $\tilde{\phi}(s) = \psi(s+2\pi+\overline{\omega})$, $s\epsilon[\overline{\omega},0]$. It turns out then, to solve the adjoint equation in row form on $[0,2\pi]$, we need only compute the significant eigenvalue and eigenvector of $\tilde{U}$. Using quadratures we discretize $\tilde{U}$ by setting $-\overline{\omega} = s_1 < \cdots < s_{N+1} = 0, \Delta = \overline{\omega}/N$. The j-th block column is given by

$$\left(\tilde{U}\tilde{\phi}\right)(s_j) = \left[\tilde{\phi}(s_1),\cdots,\tilde{\phi}(s_i),\cdots,\tilde{\phi}(s_{N+1})\right] \begin{bmatrix} X(2\pi,s_j+\overline{\omega}) + B(s_1+\overline{\omega})X(s_1+2\pi,s_j+\overline{\omega})\Delta \\ \vdots \\ B(s_i+\overline{\omega})X(s_i+2\pi,s_j+\overline{\omega})\Delta \\ \vdots \\ B(s_{N+1}+\overline{\omega})X(s_{N+1}+2\pi,s_j+\overline{\omega})\Delta \end{bmatrix}. \tag{30}$$

The eigenvector $\tilde{\phi}$ of the matrix on the right associated with the multiplier of the variational equation is computed and sustituted into the discretized form of equation (27) to give the value of $y(t)$ on the partition $0 = t_1 < \cdots < t_{M+1} = 2\pi, \Delta^* = 2\pi/M$ as

$$y(t_j) = \left[\tilde{\phi}(s_1),\cdots,\tilde{\phi}(s_i),\cdots,\tilde{\phi}(s_{N+1})\right] \begin{bmatrix} X(2\pi,t_j) + B(s_1+2\pi+\overline{\omega})X(s_1+2\pi,t_j)\Delta^* \\ \vdots \\ B(s_i+2\pi+\overline{\omega})X(s_i+2\pi,t_j)\Delta^* \\ \vdots \\ B(s_{N+1}+2\pi+\overline{\omega})X(s_{N+1}+2\pi,t_j)\Delta^* \end{bmatrix}. \tag{31}$$

5. **Estimating the M Parameter**

From Halanay, 1966, the variation of constants formula for

$$\dot{z}(t;\phi) = A(t)z(t;\phi) + B(t)z(t-\overline{\omega};\phi) + f(t), \tag{32}$$

where $t\epsilon[0,2\pi]$, is given by

$$z(t;\phi) = X(t,0)\phi(0) + \int_{-\overline{\omega}}^{0} X(t,\alpha+\overline{\omega})B(\alpha+\overline{\omega})z(\alpha)\,d\alpha + \int_{0}^{t} X(t,\alpha)f(\alpha)\,d\alpha. \tag{33}$$

The 2π periodic initial function condition with $s\epsilon[-\overline{\omega},0]$ is

$$\begin{aligned} \phi(s) &= X(s+2\pi,0)\phi(0) \\ &\quad + \int_{-\overline{\omega}}^{0} X(s+2\pi,\alpha+\overline{\omega})B(\alpha+\overline{\omega})\phi(\alpha)\,d\alpha + \int_{0}^{s+2\pi} X(s+2\pi,\alpha)f(\alpha)\,d\alpha. \end{aligned} \tag{34}$$

The first step in computing M involves relating ϕ to f. Let $|\phi| = \sup_{-\overline{\omega}\le s\le 0}|\phi(s)|$ and $|f|_2$ be the L_2 norm of f on $[0,2\pi]$. To eliminate $\phi(0)$ from (34), set $s=0$ in (34) and solve for $\phi(0)$ as

$$\begin{aligned} \phi(0) &= \int_{-\overline{\omega}}^{0}(I-X(2\pi,0))^{-1}X(2\pi,\alpha+\overline{\omega})B(\alpha+\overline{\omega})\phi(\alpha)\,d\alpha \\ &\quad + \int_{0}^{2\pi}(I-X(2\pi,0))^{-1}X(2\pi,\alpha)f(\alpha)\,d\alpha. \end{aligned} \tag{35}$$

Substitute (35) into (34) and combine terms as

$$\begin{aligned} \phi(s) &= \int_{-\overline{\omega}}^{0}\big[X(s+2\pi,0)(I-X(2\pi,0))^{-1}X(2\pi,\alpha+\overline{\omega}) \\ &\quad + X(s+2\pi,\alpha+\overline{\omega})\big]B(\alpha+\overline{\omega})\phi(\alpha)\,d\alpha \\ &\quad + \int_{0}^{2\pi}\big[X(s+2\pi,0)(I-X(2\pi,0))^{-1}X(2\pi,\alpha)+X(s+2\pi,\alpha)\big]f(\alpha)\,d\alpha. \end{aligned} \tag{36}$$

where $s\epsilon[-\overline{\omega},0]$. To relate the sup norm $|\phi|$ to the L_2 norm $|f|_2$ again use the method of quadratures. Partition $[-\overline{\omega},0]$ and $[0,2\pi]$ in equal steps as follows. Let $-\overline{\omega}=s_1<s_2<\cdots<s_{N+1}=0$, with $\Delta=\overline{\omega}/N$, and $0=t_1<t_2<\cdots<t_{M+1}=2\pi$, with $\Delta^*=2\pi/M$. The first integral on the right of (36) is discretized as a matrix with block element

$$H_1(i,j)=\big[X(s_i+2\pi,0)(I-X(2\pi,0))^{-1}X(2\pi,s_j+\overline{\omega})+X(s_i+2\pi,s_j+\overline{\omega})\big]B(s_j+\overline{\omega}), \tag{37}$$

for $i,j=1,\cdots,N+1$, times the vector $(\phi(s_1),\cdots,\phi(s_{N+1}))^T$. The second integral on the right of (36) is discretized as a matrix with block elements

$$H_2(i,j)=\big[X(s_i+2\pi,0)(I-X(2\pi,0))^{-1}X(2\pi,t_k)+X(s_i+2\pi,t_k)\big], \tag{38}$$

$i=1,\cdots,N+1$, $j=1,\cdots,M+1$, times the vector $(f(t_1),\cdots,f(t_{M+1})^T$. For each row $i=1,\cdots,N+1$

$$|\phi(s_i)| \le \left[\Delta\sum_{j=1}^{N+1}|H_1(i,j)|\right]|\phi| + \left[\Delta^*\sum_{k=1}^{M+1}|H_2(i,k)|^2\right]^{1/2}|f|_2. \tag{39}$$

Let $M_1=\max_{1\le i\le N+1}\left[\Delta\sum_{j=1}^{N+1}|H_1(i,j)|\right]$ and $M_2=\max_{1\le i\le N+1}\left[\Delta^*\sum_{k=1}^{M+1}|H_2(i,k)|^2\right]^{1/2}$. Then clearly $|\phi|\le M_1|\phi|+M_2|f|_2$ or $|\phi|\le M_2(1-M_1)^{-1}|f|_2$. In the second step the value of $\phi(0)$, given by equation (35), is substituted into equation (33) and terms combined to give

$$\begin{aligned} z(t;\phi) &= \int_{-\overline{\omega}}^{0}\big[X(t,0)(I-X(2\pi,0))^{-1}X(2\pi,\alpha+\overline{\omega}) \\ &\quad + X(t,\alpha+\overline{\omega})\big]B(\alpha+\overline{\omega})\phi(\alpha)\,d\alpha \\ &\quad + \int_{0}^{2\pi}\big[X(t,0)(I-X(2\pi,0))^{-1}X(2\pi,\alpha)+X(t,\alpha)\big]f(\alpha)\,d\alpha. \end{aligned} \tag{40}$$

The estimates proceed exactly as above. At block row k let

$$H_3(k,j)=\big[X(t_k,0)(I-X(2\pi,0))^{-1}X(2\pi,s_j+\overline{\omega})+X(t_k,s_j+\overline{\omega})\big]B(s_j+\overline{\omega}), \tag{41}$$

for $k=1,\cdots,M+1$ and

$$H_4(k,l)=\big[X(t_k,0)(I-X(2\pi,0))^{-1}X(2\pi,t_l)+X(t_k,t_l)\big]. \tag{42}$$

Then,

$$|z(t_k;\phi)| \le \left[\Delta \sum_{j=1}^{N+1} |H_3(k,j)|\right] |\phi| + \left[\Delta^* \sum_{l=1}^{M+1} |H_4(k,l)|^2\right]^{1/2} |f|_2. \tag{43}$$

Let $M_3 = \max_{1\le k\le M+1}\left[\Delta \sum_{j=1}^{N+1}|H_3(k,j)|\right]$ and $M_4 = \max_{1\le k\le M+1}\left[\Delta^* \sum_{l=1}^{M+1}|H_4(k,l)|^2\right]^{1/2}$. Then clearly $|z| \le M_3|\phi| + M_4|f|_2$. Finally, substituting the inequality for $|\phi|$ shows that $|z| \le M|f|_2$, where $M = M_2M_3(1-M_1)^{-1} + M_4$, which is independent of f. As in the previous section, each occurence of the X function in a matrix block element need not be separately computed. It is possible to solve for X in the last block row and interpolate the others above that block row.

6. **Application to a Van der Pol Equation with Delay**

In this section we will apply the main theorem to approximate the limit cycle of the Van der Pol equation with unit delay, given by

$$\ddot{x} + \lambda\left(x(t-1)^2 - 1\right)\dot{x}(t-1) + x = 0. \tag{44}$$

Since the period of the limit cycle is unknown we introduce an unknown frequency by substituting t/ω for t to obtain

$$\omega^2\ddot{x} + \omega\lambda\left(x(t-\omega)^2 - 1\right)\dot{x}(t-\omega) + x = 0, \tag{45}$$

for $t\epsilon[0,2\pi]$. To compare with an approximation result obtained in Stokes, 1972, we take $\lambda = 0.1$. By using Galerkin's method and a symbol manipulator the following approximate solution was obtained

$$\begin{aligned}
\overline{x}(t) &= 2.0185\cos(t)\\
&\quad + 2.5771e-3\cos(3t) + 2.5655e-2\sin(3t)\\
&\quad + 1.0667e-4\cos(5t) - 5.2531e-4\sin(5t)\\
&\quad - 7.1791e-6\cos(7t) - 2.2042e-6\sin(7t),\\
\overline{\omega} &= 1.0012.
\end{aligned} \tag{46}$$

The residual was estimated by substituting $(\overline{\omega},\overline{x})$ from equation (46) into equation (45) and finding the maximum of the absolute values of the residuals obtained at 1000 points in the interval $[0,2\pi]$. The result was $r = 6.0867e-6$.

The values of the constants K_1 and K were obtained in the following manner. The variational equation about the approximate frequency and solution takes the form

$$\dot{z}(t;\phi) = A(t)z(t;\phi) + B(t)Z(t-\overline{\omega};\phi), \tag{47}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -1/\overline{\omega}^2 & 0 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 0 \\ -2(\lambda/\overline{\omega})\overline{x}_1(t-\overline{\omega})\overline{x}_2(t-\overline{\omega}) & (\lambda/\overline{\omega})\left(1-\overline{x}_1(t-\overline{\omega})^2\right) \end{pmatrix}$$

We use the fact that the natural norm of a matrix, H, associated with a vector norm $|x| = \max_{1\le i\le n}|x_i|$ is $|H| = \max_{1\le i\le n}\sum_{j=1}^{n}|h_{ij}|$. With this definition it is not hard to show that

$$\begin{aligned}
|dX(\overline{x};\phi)| &\le \left|\begin{matrix} 0 & 1 \\ -1/\overline{\omega}^2 - 2(\lambda/\overline{\omega})\overline{x}_1(t-\overline{\omega})\overline{x}_2(t-\overline{\omega}) & (\lambda/\overline{\omega})\left(1-\overline{x}_1(t-\overline{\omega})^2\right) \end{matrix}\right| |\phi|\\
&\le 2.3776|\phi|.
\end{aligned} \tag{48}$$

Therefore, for $\lambda = 0.1$, $K_1 = 2.3776$. Working conservatively within the domain $D = \{x\epsilon C[0,2\pi] : |x-\overline{x}| \le 1\}$ it is not hard to show that

$$|dX(\overline{x}_{\overline{\omega}} + \psi_1;\phi_{\overline{\omega}}) - dX(\overline{x}_{\overline{\omega}} + \psi_2;\phi_{\overline{\omega}})| \le (6\lambda/\overline{\omega})\left(1+|\overline{x}|\right)|\psi_1-\psi_2|\,|\phi|. \tag{49}$$

For 1000 sample points in $[0, 2\pi]$ we can estimate $|\overline{x}| \leq 2.0225$, which implies that for $\lambda = 0.1$ that we can estimate $K = 1.8113$. Using the same sampled points we can estimate $|\dot{\overline{x}}| \leq 2.0258$, $|\ddot{\overline{x}}| \leq 2.1192$, and, from equation (9), we can estimate, using $h = 1$, $\left|J(\dot{\overline{x}}, \overline{\omega})\right| \leq 2.6082$. If we apply the methods of Section 5, using $A(t)$ and $B(t)$ defined in equation (47), we can estimate $M = 1.7411$. These results allowed us to estimate λ_1 and λ_2 in Theorem 2.3 as $\lambda_1 \leq 38.3722$ and $\lambda_2 \leq 113.2727$.

Next we estimate the characteristic multiplier of the variational equation relative to the function $\overline{x}(t)$ using the method of Section 3 as $\rho = 0.99806$ where N in the calculation was taken as 1000. All of the other eigenvalues have magnitudes less than one. Using the methods of Section 4 the solution of the adjoint to the variational equation was computed and the parameter α in Theorem 2.3 was estimated as

$$\alpha = \left[\frac{\Delta^*}{2\pi} \left| \sum_{i=1}^{M+1} y(t_i) J(\dot{\overline{x}}, \overline{\omega})(t_i) \right| \right]^{-1} \leq 8.0665 \tag{50}$$

where N and M for the calculations in Section 4 were taken as 500.

With the estimated parameters above it was possible to estimate the contraction conditions of the theorem as $8\alpha r \leq 2.9279e-4 \ll \overline{\omega}$ and $C(r, K, K_1, h, \alpha, \lambda_1, \lambda_2, \dot{\overline{x}}, \ddot{\overline{x}}, \overline{\omega}) \leq 0.3299 < 1$. This allowed us to conclude from Theorem 2.3 that there exists an exact solution x^* and an exact frequency ω^* of equation (45) such that $|x_* - \overline{x}| \leq 9.3424e-4$ and $|\omega^* - \overline{\omega}| \leq 9.8197e-5$.

7. Acknowledgements

The author wishes to thank Prof. Dianne O'Leary of the Department of Computer Science, University of Maryland, College Park, MD and Dr. Tim Burns of the Mathematical and Computational Sciences Division, National Institute of Standards and Technology, Gaithersburg, MD for many helpful discussions.

References

[1] S. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.
[2] J. Hale, *Functional Differential Equations*, Springer-Verlag, New York, 1971.
[3] L.F. Shampine and S. Thompson, Solving DDE's in MATLAB, Applied Numerical Mathematics, 37 (2001) 441–458.
[4] A. Stokes, On the Approximation of Nonlinear Oscillations, Journal of Differential Equations, 12 (1972) 535–558.
[5] A.P. Stokes, On the Existence of Periodic Solutions of Functional Differential Equations, Journal of Mathematical Analysis and Applications, 54 (1976) 634–652.
[6] M. Urabe, Galerkin's Procedures for Nonlinear Periodic Systems, Arch. Rational Mech., 20 (1965) 120–152.