# SOFT-DECISION METRICS FOR CODED ORTHOGONAL SIGNALING IN SYMMETRIC ALPHA-STABLE NOISE

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## ABSTRACT

This paper derives new soft decision metrics for coded orthogonal signaling in symmetric $\alpha$-stable noise, which has been used to model impulsive noise. In addition to the optimum metrics for Gaussian ($\alpha = 2$) noise and Cauchy ($\alpha = 1$) noise, a class of generalized likelihood ratio (GLR) metrics with lower side information requirements is derived. Through numerical results for a turbo code example, the Cauchy decoder is found to be robust for a wide range of $\alpha$, and GLR metrics are found which provide performance gains relative to the Gaussian metric, but with lower complexity and less *a priori* information.

## 1. INTRODUCTION

While noise in communications systems is often modeled as a Gaussian process, some systems experience noise or interference that is better characterized by the more general class of $\alpha$-stable distributions ($0 < \alpha \leq 2$), of which the Gaussian distribution is a special case ($\alpha = 2$). Consequently, systems designed specifically for Gaussian noise can perform poorly in impulsive noise environments. One example of such an environment (and an application of this work) is a wireless link in an ad hoc network modeled as a Poisson field of interferers, for which the index of stability, $\alpha$, of the interference was shown to be inversely related to the path loss exponent [1]. The smaller the value of $\alpha$, the more impulsive the noise becomes.

The purpose of this paper is to derive soft decision metrics for receivers experiencing $\alpha$-stable noise that vary in complexity and required side information. We study these metrics in the context of non-coherently detected orthogonal signals. While previous work on non-coherent receivers in $\alpha$-stable noise addressed uncoded systems [2, 3], this work applies to coded systems using soft decision decoding, and numerical results are given for a turbo code example.

## 2. SYSTEM MODEL

Encoded bits are mapped $\log_2 M$ bits at a time to one of $M$ orthogonal signals, such as $M$-ary FSK signals, and transmitted over a channel that injects additive noise modeled as a sequence of independent and identically distributed (i.i.d.) symmetric (about the origin) $\alpha$-stable (S$\alpha$S) random variables. The received signal is correlated, in-phase and quadrature, with each of the $M$ signals. The output of the $i$th correlator, $0 \leq i \leq M-1$, is modeled as

$$\mathbf{Z}_i = a_i \mathbf{S}_i + \mathbf{Y}_i$$

where all vectors are two-dimensional, representing the in-phase and quadrature components, $\mathbf{Y}_i$ is the additive S$\alpha$S noise at the output of the demodulator, and $a_i$ is the amplitude of the received signal. The desired signal $\mathbf{S}_i$ may be expressed as

$$\mathbf{S}_i = \begin{cases} \begin{bmatrix}\cos\theta_i & \sin\theta_i\end{bmatrix} & ; \quad i = i' \\ 0 & ; \quad i \neq i'. \end{cases}$$

where $\theta_i$ is the relative phase of the signal, and the $i'$th signal is transmitted. This model allows for the amplitude and phase to differ for different signals. We shall assume that the random noise vectors $\mathbf{Y}_i$, $0 \leq i \leq M-1$, are i.i.d., such as when the noise results from independent Poisson field processes [1, 3].

Soft decisions of coded symbols are generated from the demodulator outputs, plus any available side information, and passed to the decoder in the form of log-likelihood ratios (LLRs). The LLR of the $j$th coded bit, $c_j$, is defined as

$$L_j(\mathbf{z}, \mathbf{a}, \boldsymbol{\theta}) \triangleq \log \frac{\Pr[c_j = 1 | \mathbf{z}, \mathbf{a}, \boldsymbol{\theta}]}{\Pr[c_j = 0 | \mathbf{z}, \mathbf{a}, \boldsymbol{\theta}]} \quad (1)$$

where $\mathbf{z} = \begin{bmatrix}\mathbf{z}_0 & \mathbf{z}_1 & \ldots & \mathbf{z}_{M-1}\end{bmatrix}$ is a vector of the outputs of all $M$ in-phase and quadrature correlators, and likewise, $\mathbf{a} = \begin{bmatrix}a_0 & a_1 & \ldots & a_{M-1}\end{bmatrix}$ and $\boldsymbol{\theta} = \begin{bmatrix}\theta_0 & \theta_1 & \ldots & \theta_{M-1}\end{bmatrix}$.

Since each transmitted signal represents a length-$\log_2 M$ sequence of coded bits, $\mathbf{c}$, (1) can be expressed as

$$L_j(\mathbf{z}, \mathbf{a}, \boldsymbol{\theta}) = \log \frac{\sum_{\mathbf{c}:c_j=1} p(\mathbf{c}|\mathbf{z}, \mathbf{a}, \boldsymbol{\theta})}{\sum_{\mathbf{c}:c_j=0} p(\mathbf{c}|\mathbf{z}, \mathbf{a}, \boldsymbol{\theta})} \quad (2)$$

where $p(\mathbf{c})$ is the probability of sequence $\mathbf{c}$. Using Bayes' rule and assuming all coded sequences are equiprobable,[1] (2) can be written in terms of the conditional probability density function of the correlator outputs:

$$L_j(\mathbf{z}, \mathbf{a}, \boldsymbol{\theta}) = \log \frac{\sum_{\mathbf{c}:c_j=1} f(\mathbf{z}|\mathbf{c}, \mathbf{a}, \boldsymbol{\theta})}{\sum_{\mathbf{c}:c_j=0} f(\mathbf{z}|\mathbf{c}, \mathbf{a}, \boldsymbol{\theta})}. \quad (3)$$

Writing the conditional density function in (3) in terms of the density of the noise, we have

$$f(\mathbf{z}|\mathbf{c}, \mathbf{a}, \boldsymbol{\theta}) = f_{\mathbf{Y}}(\mathbf{z}_i - a_i \mathbf{s}_i(\theta_i)) \prod_{k \neq i} f_{\mathbf{Y}}(\mathbf{z}_k) \quad ; \quad \mathbf{c} \rightarrow i \quad (4)$$

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[1] This assumption can be relaxed by inserting $p(\mathbf{c})$ in (3).

where the notation $\mathbf{c} \rightarrow i$ means that the coded bit sequence $\mathbf{c}$ results in transmission of the $i$th signal.

Typically, orthogonal signals are detected non-coherently, and in a Bayesian approach the conditional density (4) would be averaged over the unknown random vector $\boldsymbol{\theta}$. We shall assume that the components of $\boldsymbol{\theta}$ are i.i.d., uniform on $(0, 2\pi)$. Similarly, if the amplitudes are unknown, the conditional density would be averaged over the appropriate distribution of $\mathbf{a}$.

## 3. DECISION METRICS

The LLR input to the decoder is derived in this section under various assumptions for the S$\alpha$S noise and available side information. A S$\alpha$S random vector has characteristic function

$$\Phi(\boldsymbol{\omega}) = \exp\left(-\gamma \|\boldsymbol{\omega}\|^{\alpha}\right)$$

where the index of stability $\alpha$ is limited to the interval $0 < \alpha \leq 2$ and the dispersion $\gamma > 0$ [4]. Closed forms for the density of a S$\alpha$S random vector exist only for the cases of $\alpha = 1$ and $\alpha = 2$, which correspond to the Cauchy and Gaussian distributions, respectively. In the Gaussian case, the variance $\sigma^2$ of each component is related to the dispersion through $\sigma^2 = 2\gamma$.

The LLR is derived below first under the assumption of Gaussian noise ($\alpha = 2$), then under Cauchy noise ($\alpha = 1$). Both of these metrics assume knowledge of the dispersion, $\gamma$, and signal amplitude, $\mathbf{a}$. Three additional metrics that rely on less side information are derived using the generalized likelihood ratio approach. The performance of these metrics is compared under different noise environments in the subsequent section.

### 3.1. Gaussian Metric

Under the assumption of S2S noise, the random vectors $\mathbf{Y}_i$, $0 \leq i \leq M-1$, are i.i.d. bivariate Gaussian with density

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{4\pi\gamma} \exp\left(-\frac{\|\mathbf{y}\|^2}{4\gamma}\right). \tag{5}$$

Using (5) in (4), the conditional density of $\mathbf{z}$ is

$$\begin{aligned} &f(\mathbf{z}|\mathbf{c}, \mathbf{a}, \boldsymbol{\theta}) \\ &= \frac{1}{(4\pi\gamma)^M} \exp\left[-\frac{\|\mathbf{z}_i - a_i \mathbf{s}_i(\theta_i)\|^2 + \sum_{k\neq i} \|\mathbf{z}_k\|^2}{4\gamma}\right] \quad (6) \\ &= \frac{1}{(4\pi\gamma)^M} \exp\left[-\frac{a_i^2 - 2a_i w_i \cos(\theta_i - \phi) + \|\mathbf{z}\|^2}{4\gamma}\right] \quad (7) \end{aligned}$$

where the last line uses $\mathbf{z}_i = w_i \begin{bmatrix} \cos\phi & \sin\phi \end{bmatrix}$.

For non-coherent detection, (7) is averaged over $\theta_i$, giving

$$\begin{aligned} &f(\mathbf{z}|\mathbf{c}, \mathbf{a}) \\ &= \frac{\exp\left(-\frac{a_i^2 + \|\mathbf{z}\|^2}{4\gamma}\right)}{(4\pi\gamma)^M} \frac{1}{2\pi} \int_0^{2\pi} \exp\left[\frac{a_i w_i \cos(\theta - \phi)}{2\gamma}\right] \mathrm{d}\theta \\ &= \frac{\exp\left(-\frac{a_i^2 + \|\mathbf{z}\|^2}{4\gamma}\right)}{(4\pi\gamma)^M} \mathrm{I}_0\left(\frac{a_i w_i}{2\gamma}\right) \quad (8) \end{aligned}$$

where $\mathrm{I}_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,\mathrm{d}\theta$ is the zeroth-order modified Bessel function of the first kind.

Using (8), and after canceling terms, the Gaussian LLR for non-coherent detection with known amplitudes and dispersion is

$$\begin{aligned} L_j(\mathbf{z}, \mathbf{a})\,|_{\alpha=2, \gamma\text{ known}} &= \log \frac{\sum_{\mathbf{c}:c_j=1} f(\mathbf{z}|\mathbf{c}, \mathbf{a})}{\sum_{\mathbf{c}:c_j=0} f(\mathbf{z}|\mathbf{c}, \mathbf{a})} \\ &= \log \frac{\sum_{i:c_j=1} e^{-a_i^2/4\gamma}\, \mathrm{I}_0\left(\frac{a_i w_i}{2\gamma}\right)}{\sum_{i:c_j=0} e^{-a_i^2/4\gamma}\, \mathrm{I}_0\left(\frac{a_i w_i}{2\gamma}\right)} \quad (9) \end{aligned}$$

where in the second line the summations are over all $M/2$ signals to which are mapped coded sequences for which $c_j = 1$ and $c_j = 0$, respectively.

### 3.2. Cauchy Metric

Under the assumption of Cauchy (i.e., S1S) noise, the density of the noise vector is

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{\gamma/2\pi}{\left(\gamma^2 + \|\mathbf{y}\|^2\right)^{3/2}}. \tag{10}$$

The conditional density of $\mathbf{z}$ averaged over $\theta_i$ is

$$f(\mathbf{z}|\mathbf{c}, \mathbf{a}) = \mathrm{E}_{\theta_i}\left[f_{\mathbf{Y}}(\mathbf{z}_i - a_i \mathbf{s}_i(\theta_i))\right] \prod_{k\neq i} f_{\mathbf{Y}}(\mathbf{z}_k). \tag{11}$$

The expectation above evaluates to

$$\begin{aligned} &\mathrm{E}_{\theta_i}\left[f_{\mathbf{Y}}(\mathbf{z}_i - a_i \mathbf{s}_i(\theta_i))\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma/2\pi\,\mathrm{d}\theta}{\left[\gamma^2 + w_i^2 + a_i^2 - 2a_i w_i \cos(\theta - \phi)\right]^{3/2}} \\ &= \frac{\gamma/\pi^2}{(\beta_i - \delta_i)\sqrt{\beta_i + \delta_i}} \mathrm{E}\left(\sqrt{\frac{2\delta_i}{\beta_i + \delta_i}}\right) \quad (12) \end{aligned}$$

where $\mathrm{E}(k) \triangleq \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\phi}\,\mathrm{d}\phi$ is the complete elliptic integral of the second kind, $\beta_i = \gamma^2 + w_i^2 + a_i^2$, $\delta_i = 2a_i w_i$, and where [5, (2.575.4)] was used to obtain (12).[2]

Using (11) together with (10) and (12), and after simplifying, the Cauchy LLR for non-coherent detection with known amplitudes and dispersion is

$$L_j(\mathbf{z}, \mathbf{a})\,|_{\alpha=1, \gamma\text{ known}} = \log \frac{\sum_{i:c_j=1} g(w_i, a_i)}{\sum_{i:c_j=0} g(w_i, a_i)} \tag{13}$$

where

$$g(w_i, a_i) = \mathrm{E}\left(\sqrt{\frac{2\delta_i}{\beta_i + \delta_i}}\right) \frac{\left(\gamma^2 + w_i^2\right)^{3/2}}{(\beta_i - \delta_i)\sqrt{\beta_i + \delta_i}}.$$

### 3.3. Generalized Likelihood Ratio Metrics

The metrics derived in the previous section require knowledge of the amplitude vector $\mathbf{a}$ and the noise dispersion parameter $\gamma$. This knowledge may not be available in practice. A Bayesian approach towards elimination of $\mathbf{a}$ and $\gamma$ does not seem feasible: first, averaging the likelihood functions in closed form does not appear to be possible. Second, the choice of priors $p(\mathbf{a})$ and $p(\gamma)$ requires *a priori* information that may be unavailable. An alternative approach

[2] A previously published evaluation of this integral in [2], and later used again in [3], contained an error.

is to derive metrics using the generalized likelihood ratio (GLR) paradigm. The philosophy behind GLR is to maximize the likelihood function with respect to the unknown parameters (rather than to average it over the parameter vector, as done in the Bayesian approach) [6]. The GLR approach is less commonly seen in the communication theory literature, but it has been used before with success [7, 8]. In this section, we derive three different decision metrics (that need different amounts of *a priori* knowledge) based on the GLR paradigm.

#### 3.3.1. *GLR for Gaussian distribution ($\alpha = 2$)*

Using (6), we have that

$$f(\mathbf{z};\mathbf{a},\boldsymbol{\theta},\gamma,\mathbf{c}) = \frac{1}{(4\pi\gamma)^M}\exp\left(-\frac{\|\mathbf{z}_i - a_i\mathbf{s}_i(\theta_i)\|^2}{4\gamma}\right) \cdot \prod_{k=0,k\neq i}^{M-1}\exp\left(-\frac{\|\mathbf{z}_k\|^2}{4\gamma}\right). \tag{14}$$

Jointly maximizing (14) w.r.t. $\mathbf{a}$ and $\boldsymbol{\theta}$ gives[3]

$$\hat{f}(\mathbf{z};\gamma,\mathbf{c}) \triangleq \max_{\mathbf{a},\boldsymbol{\theta}} f(\mathbf{z};\mathbf{a},\boldsymbol{\theta},\gamma,\mathbf{c}) = \frac{1}{(4\pi\gamma)^M}\exp\left(-\sum_{k=0,k\neq i}^{M-1}\frac{\|\mathbf{z}_k\|^2}{4\gamma}\right).$$

The resulting metric after simplifying is

$$L_j(\mathbf{z})|_{\text{GLR},\alpha=2,\gamma\text{ known}} = \log\frac{\sum_{i:c_j=1} e^{w_i^2/4\gamma}}{\sum_{i:c_j=0} e^{w_i^2/4\gamma}}. \tag{15}$$

(For $M = 2$, (15) reduces to $L_j(\mathbf{z}) = (w_1^2 - w_0^2)/(4\gamma)$.)

If $\gamma$ is unknown, we can proceed one step further and eliminate it by maximizing $\hat{f}(\mathbf{z};\gamma,\mathbf{c})$ w.r.t. $\gamma$. This leads to

$$\hat{f}(\mathbf{z};\mathbf{c}) \triangleq \max_{\gamma}\hat{f}(\mathbf{z};\gamma,\mathbf{c}) = \frac{e^{-M}}{\pi^M}\frac{1}{\left(\frac{1}{M}\sum_{k=0,k\neq i}^{M-1}\|\mathbf{z}_k\|^2\right)^M}$$

and the associated bit metric

$$L_j(\mathbf{z})|_{\text{GLR},\alpha=2,\gamma\text{ unknown}} = \log\frac{\sum_{i:c_j=1}\left(\sum_{k=0,k\neq i}^{M-1} w_k^2\right)^{-M}}{\sum_{i:c_j=0}\left(\sum_{k=0,k\neq i}^{M-1} w_k^2\right)^{-M}}. \tag{16}$$

(For $M = 2$, (16) simplifies to $L_j(\mathbf{z}) = 2\log(w_1^2/w_0^2)$.)

#### 3.3.2. *GLR for Cauchy distribution ($\alpha = 1$)*

For $\alpha = 1$, we have from (10) that

$$f(\mathbf{z};\mathbf{a},\boldsymbol{\theta},\gamma,\mathbf{c}) \tag{17}$$

$$= \frac{\gamma/2\pi}{(\gamma^2 + \|\mathbf{z}_i - a_i\mathbf{s}_i(\theta_i)\|^2)^{3/2}}\prod_{k=0,k\neq i}^{M-1}\frac{\gamma/2\pi}{(\gamma^2 + \|\mathbf{z}_k\|^2)^{3/2}}.$$

Maximizing (17) w.r.t. $\mathbf{a}$ and $\boldsymbol{\theta}$ yields

$$\hat{f}(\mathbf{z};\gamma,\mathbf{c}) = \frac{1}{2\pi\gamma^2}\prod_{k=0,k\neq i}^{M-1}\frac{\gamma/2\pi}{(\gamma^2 + \|\mathbf{z}_k\|^2)^{3/2}} \tag{18}$$

[3] We use $\hat{(\cdot)}$ to denote profile (concentrated) likelihoods.

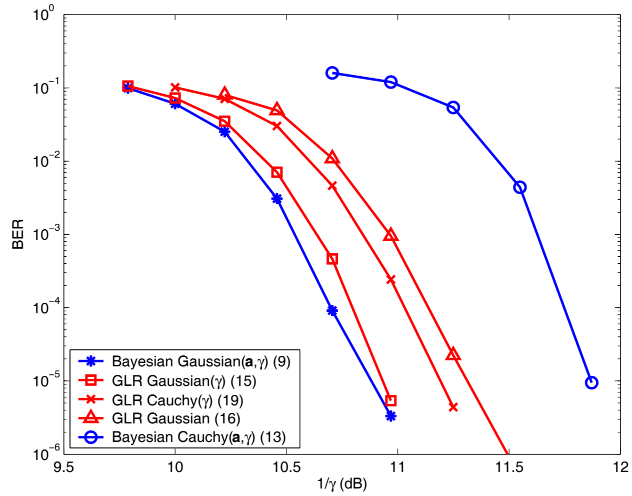


**Fig. 1**. Performance in S($\alpha = 2$)S (Gaussian) noise (side info. required by each metric, and equations, indicated in ()'s in legend)

and the resulting metric after simplification is

$$L_j(\mathbf{z})|_{\text{GLR},\alpha=1,\gamma\text{ known}} = \log\frac{\sum_{i:c_j=1}\left(\gamma^2 + w_i^2\right)^{3/2}}{\sum_{i:c_j=0}\left(\gamma^2 + w_i^2\right)^{3/2}}. \tag{19}$$

(For $M = 2$, (19) reduces to $L_j(\mathbf{z}) = \frac{3}{2}\log\left(\frac{\gamma^2+w_1^2}{\gamma^2+w_0^2}\right)$.)

Unlike the Gaussian case, (18) is monotonic with $\gamma$, and therefore the GLR metric for unknown $\gamma$ does not exist under the Cauchy assumption.

## 4. QUANTITATIVE RESULTS

Quantitative results for the performance of the aforementioned metrics are obtained through Monte Carlo simulation of a coded binary FSK system in S$\alpha$S noise. Information bits are encoded with a rate $1/2$ binary parallel concatenated convolutional (turbo) code with interleaver size of 1024 bits and constituent encoder constraint length of four. Coded bits are mapped to binary FSK channel symbols with unit amplitude. After non-coherent detection and LLR computation, decoding is performed iteratively by a pair of soft-input/soft-output MAP decoders and is terminated after eight iterations. Generation of the bivariate S$\alpha$S noise is straightforward for $\alpha = 1$ and 2; for other values of $\alpha$ we approximate the noise with the first 100 terms in a bivariate version of the series representation of a S$\alpha$S random variable given in [4, Theorem 1.4.2].

Fig. 1 compares the performance of the metrics developed in Section 3 on an additive white Gaussian noise (AWGN) channel in terms of bit error rate (BER) versus the inverse of the dispersion, $1/\gamma$, in dB. Recall that $1/\gamma$ is proportional to the conventional signal-to-noise ratio in AWGN. As expected, the Bayesian-derived Gaussian metric performs best, as it is matched to the noise distribution and assumes knowledge of the amplitudes and dispersion. Nevertheless, we observe that the Bayesian Cauchy and the GLR metrics perform within 1 dB and 0.5 dB of the Gaussian metric, respectively. It is notable that the GLR metrics achieve this performance without knowledge of the amplitudes (and, in one case, without the dispersion) and with lower computational complexity than the Bayesian-derived Gaussian and Cauchy metrics.

Fig. 2 compares performance on a Cauchy noise channel. Here,

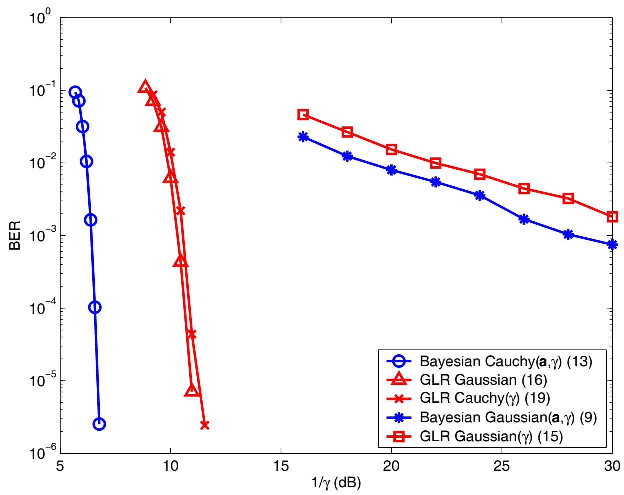


**Fig. 2**. Performance in S($\alpha = 1$)S (Cauchy) noise

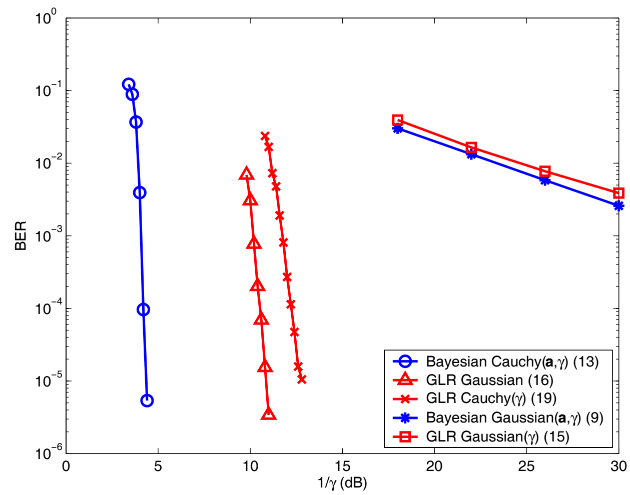


**Fig. 4**. Performance in S($\alpha = 0.5$)S noise

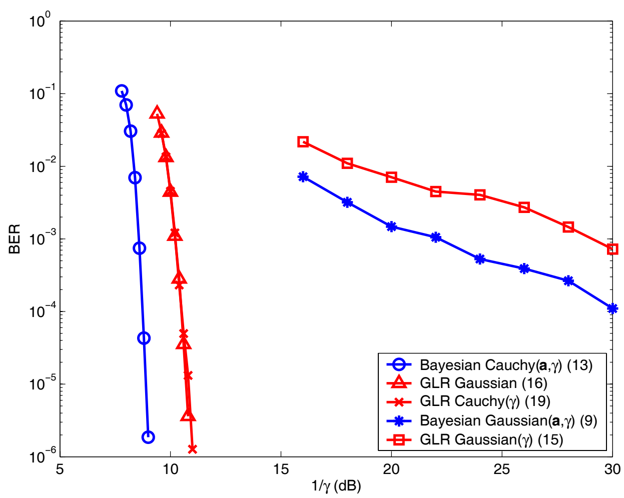


**Fig. 3**. Performance in S($\alpha = 1.5$)S noise

as expected, the Bayesian Cauchy metric outperforms the other metrics, but by much larger margins than those in the Gaussian channel (4 dB and up). Interestingly, knowledge of the dispersion appears to degrade the performance of the Gaussian GLR metric, which is mismatched to the actual noise distribution. Similar trends are observed on channels with S($\alpha = 1.5$)S noise (Fig. 3) and S($\alpha = 0.5$)S noise (Fig. 4), but with smaller and larger margins, respectively, as might be expected in these less and more severe impulsive noise environments. In general, while the Bayesian Gaussian metric degrades performance in non-Gaussian stable noise, the Bayesian Cauchy metric gives much improved performance over a wide range of $\alpha < 2$. Furthermore, the Gaussian GLR metric (without knowledge of $\gamma$) provides substantial gains relative to the Bayesian Gaussian metric over this range, with no side information and lower complexity than either of the Bayesian metrics.

## 5. SUMMARY

New soft decision metrics were derived for coded orthogonal signaling with non-coherent detection in symmetric $\alpha$-stable noise. In addition to the optimum metrics for Gaussian and Cauchy noise, a class of generalized likelihood ratio metrics was derived requiring less (or no) side information (signal amplitudes, noise dispersion). Performance was evaluated for a turbo code example by Monte Carlo simulation. While all the studied metrics perform closely (within 1 dB) for $\alpha = 2$ (Gaussian noise), the Bayesian-derived Cauchy metric performs best for a wide range of $\alpha < 2$, consistent with findings in [2, 3] for uncoded systems. Moreover, a GLR metric with lower complexity has been found that provides a substantial performance improvement over the Gaussian metric over this range while requiring no side information. Future work will evaluate the performance of those metrics that rely on side information when these parameters are estimated from noisy measurements rather than being assumed known *a priori*.

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