# Long Distance Optical Frequency Transfer over Fiber: predicting the frequency stability from the fiber noise

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*Abstract*— **We have recently demonstrated the coherent transfer of an optical signal over a 251 km link of optical fiber by use of the standard Doppler-cancellation approach to remove the effects of the fiber-link noise. The fundamental limit to the frequency instability on the transmitted optical frequency is set by residual phase noise on the optical frequency resulting from the unavoidably imperfect Doppler cancellation of the fiber-link noise. Here we demonstrate that it is possible to quantitatively predict the phase noise and instability of the Doppler-cancelled transmitted optical frequency directly from the measured fiber-link noise. The ability to predict the frequency instability from the measured fiber noise can be a useful tool in evaluating whether a coherent fiber optic link is operating at its fundamental limit, or whether there is additional excess noise from the measurement system present in the link.**

## I. Introduction

As atomic clocks continue to improve in their stability, it has become increasingly difficult to compare the frequency of two remote clocks, or to faithfully deliver the clock frequency to an end user. The difficulty arises from the fact that any variation in the path length connecting end sites results in a Doppler shift of the transmitted frequency, thereby degrading its phase noise and stability to levels well below those of the original clock signal. A variety of techniques have been demonstrated to transport frequencies faithfully despite these Doppler shifts. A "common-view" global positioning satellite system (GPS) has traditionally been used that allows flexibility in the receiver's position and fractional frequency stabilities of a few parts in $10^{-15}$ with one-day averaging.[1]. However, the latest generation of optical clocks operate at significantly lower instabilities. In several recent articles, we have explored the long-distance transport of an optical frequency over a fiber optic link of up to 251 km in length [2, 3], using the previously developed techniques for Doppler cancellation of Ref. [4-6]. We find that it is indeed possible to transport an optical frequency over such lengths of optical fiber with sufficient stability to support the current and next generation of optical clocks. At the end sites, the optical frequency can be translated to a different optical frequency, or to the RF, through an optical frequency comb.[7]

The limitation to the frequency instability of the delivered optical frequency is set by two factors. For very long gate times, the instability is limited by the system noise of the transmitter or receiver. In contrast, at short gate periods the instability is limited by the actual noise imposed on the optical frequency from the variations in the fiber link length (through the Doppler shifts). Hereafter, we refer to this noise as "fiber-link phase noise". Because of the long millisecond delay involved in transporting the signal over the fiber, the fiber-link phase noise cannot be completely suppressed through feedback, and the residual phase noise on the delivered optical frequency will cause a corresponding instability in the delivered frequency. This second effect is the subject of this paper.

In Refs. [2, 3], we outline in some detail the various important effects for the long-distance transport of an optical frequency. We presented results on the achievable phase noise and instability over a series of fiber link lengths up to 251 km, as well as simple expressions for the timing jitter (integrated phase noise) and instability. These simple expressions relied on the assumption that the fiber-link phase noise falls off as $1/f^2$ up to some cutoff frequency, where $f$ is the Fourier frequency. In general, of course, the fiber-link phase noise will have a more complicated structure. The basic theory presented in Appendix A of Ref. [3] is in fact completely general and can be applied to any fiber-link phase noise. In fact, if one measures the free-running fiber-link phase noise for any link, then one can use the theory to predict the quality of the ultimate Doppler-cancelled optical frequency in terms of both its residual phase noise and instability. This prediction capability is useful because it is relatively simple to measure fiber-link phase noise. One can then use the predicted instability to determine whether the link is sufficiently quiet, or to evaluate the performance of the fully operational Doppler-cancelled coherent link. Here we demonstrate such a calculation, and compare it to actual measurements on an 80 km fiber link. The experimental setup was identical to that described in Refs. [2, 3].

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## II. FIBER COMB

### A. Free-running fiber-link phase noise

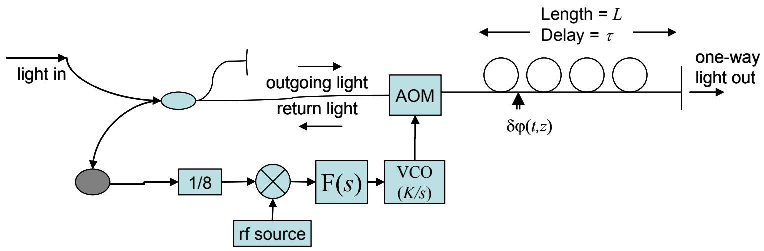


Figure 1. Simplified schematic of the phase-locked loop for the Doppler cancellation, which defines the various contributions to the open-loop gain.

Fig. 1 gives a schematic of the phase-locked loop (PLL) for the Doppler cancellation using standard Laplace notation.[8] Assuming a phase perturbation, $\delta\varphi(z,t)$, on the fiber at time $t$ and position $z$ with respect to some reference path, the phase noise accumulated by light traveling forward in length of optical fiber starting at position $z$=0 and exiting the fiber at $z$=$L$ at time $t$ is

$$\varphi_{fiber}(t)=\int_0^L \delta\varphi\left(z,t-(\tau-z/c_n)\right)dz\,, \qquad (1)$$

where $\tau=L/c_n$ is the propagation delay in the fiber, and $c_n$ is the speed of light in the fiber. The phase noise on the round-trip light exiting the fiber at a time $t$ is

$$\varphi_{fiber,RT}(t)=\int_0^L \left[\delta\varphi\left(z,t-z/c_n\right)+\delta\varphi\left(z,t-(2\tau-z/c_n)\right)\right]dz\,. \qquad (2)$$

The Fourier transforms of the above equations are

$$\tilde{\varphi}_{fiber}(\omega)=\int_0^L e^{i\omega z c_n^{-1}}e^{-i\omega\tau}\delta\tilde{\varphi}(z,\omega)dz,$$
$$\tilde{\varphi}_{fiber,RT}(\omega)=2\int_0^L \cos\left(\omega(\tau-z/c_n)\right)e^{-i\omega\tau}\delta\tilde{\varphi}(z,\omega)dz \qquad (3)$$

Assuming that the noise is uncorrelated with position, the fiber-link phase noise power spectral density (PSD) on the one-way transmitted light is

$$S_{fiber}(\omega)=\left\langle\left|\tilde{\varphi}_{fiber}(\omega)\right|^2\right\rangle=\int_0^L\left\langle\left|\delta\tilde{\varphi}(z,\omega)\right|^2\right\rangle dz. \qquad (4)$$

The round-trip light has a phase noise PSD

$$S_{fiber,RT}(\omega)=\left\langle\left|\tilde{\varphi}_{fiber,RT}(\omega)\right|^2\right\rangle=2S_{fiber}(\omega)\left(1+\mathrm{sinc}(2\tau\omega)\right), \qquad (5)$$

under the simplifying assumption that the noise is independent of position. Similar expressions were derived in Ref. [9]. To simplify the math, we will continue to make the assumption that the noise is independent of position. The fiber-link noise, $S_{fiber}(\omega)$, is fairly easily measured for a test system where the fiber link begins and ends in the same place. In a real system, it is the round trip noise, $S_{fiber,RT}(\omega)$, which is easily measured. In any case, for the discussion here, we will assume that $S_{fiber}(\omega)$ is known. We demonstrate that from this quantity, one can predict the resulting instability and residual phase noise on the transmitted optical frequency with the full PLL of Figure 1 active.

### B. Phase-locked loop for Doppler cancellation: Theory

Now we consider the PLL. Using Laplace notation, the open loop gain is

$$G(s)=G_0F(s)s^{-1}K\left(1+e^{-2s\tau}\right), \qquad (6)$$

where $G_0$ is an overall gain including the divider and phase-to-voltage conversion of the phase detector, $F(s)$ is the loop filter gain, $K$ is the VCO conversion from volts to frequency, and $\tau$ is the one-way delay down the fiber. Letting $s\rightarrow i\omega$ to convert to frequency, and assuming the loop filter is a simple proportional-integral circuit with corner frequency $\omega_c$, we can rewrite this as

$$G(\omega)=-G_0\frac{\omega_c+i\omega}{\omega^2}\cos(\omega\tau)e^{-i\omega\tau}, \qquad (7)$$

where $G_0$ is a redefined frequency-independent overall gain. The usual algebra [8] yields the phase noise on the “local” or round-trip signal with the phase-locked loop active as

$$\varphi_{local}(\omega)=\frac{1}{1+G(\omega)}\varphi_{fiber,RT}(\omega), \qquad (8)$$

so that the power spectral density (PSD), $S_{local}$, on the phase-locked round-trip (local) signal compared to the noise on the free-running round-trip signal is

$$S_{local}(\omega)=\left|\frac{1}{1+G(\omega)}\right|^2 S_{fiber,RT}(\omega)\,. \qquad (9)$$

Now consider the one-way transmitted signal. The correction signal applied to the AOM for the one-way light is $G_0F(\omega)(i\omega)^{-1}Ke^{-i\omega\tau}\tilde{\varphi}_{RT}(\omega)$. Again, following the usual algebra,[8], the phase of the locked “remote” transmitted light is

$$\tilde{\varphi}_{remote}(\omega)=\tilde{\varphi}_{fiber}(\omega)-\left(\frac{G(\omega)}{1+G(\omega)}\right)\frac{\tilde{\varphi}_{fiber,RT}(\omega)}{2\cos(\omega\tau)}\,. \qquad (10)$$

From (3), (4), and (5), the squared magnitude of (10) gives the PSD of the phase-locked transmitted signal $S_{remote}$ compared to the fiber-link noise (i.e., the noise on the free-running transmitted signal) as

$$S_{remote}(\omega) = S_{fiber}(\omega)\left\{1+\left|\frac{G}{1+G}\right|^2 \frac{1+\mathrm{sinc}(2\omega\tau)}{2\cos^2(\omega\tau)} - \mathrm{Re}\left[\left(\frac{G}{1+G}\right)\frac{\left(e^{-i\omega L}+\mathrm{sinc}(\omega\tau)\right)}{\cos(\omega\tau)}\right]\right\}. \quad (11)$$

In Refs. [2, 3], we take the limit of infinite gain and low frequency, for which this equation reduces to

$$\lim_{\omega\to 0} S_{remote}(\omega) \approx \frac{(\omega\tau)^2}{3} S_{fiber}(\omega), \quad (12)$$

from which follow the simplified expressions for timing jitter and instability of Refs. [2, 3].

### *C. Phase-locked loop: Comparison with Experiment*

Next, we compare the above expressions to measurements taken for an 80 km fiber link. For this 80 km link, we measured the phase noise on the round-trip (local) and one-way transmitted (remote) optical frequencies without the PLL (equivalent to $S_{fiber,RT}(\omega)$ and $S_{fiber}(\omega)$), and with the PLL active (equivalent to $S_{local}(\omega)$ and $S_{remote}(\omega)$). These four quantities are related through Eqs. (5), (9), and (11). From these equations, knowledge of $S_{fiber}(\omega)$ (and the gain *G)* should be sufficient to predict the final phase noise on the transmitted signal with PLL active.

Figure 2 shows the free-running fiber link noise, $S_{fiber}(\omega)$ and $S_{fiber,RT}(\omega)$, along with the prediction of Eq. (5). The agreement is very good, and it would have been sufficient to measure only one or the other quantity in a real system.

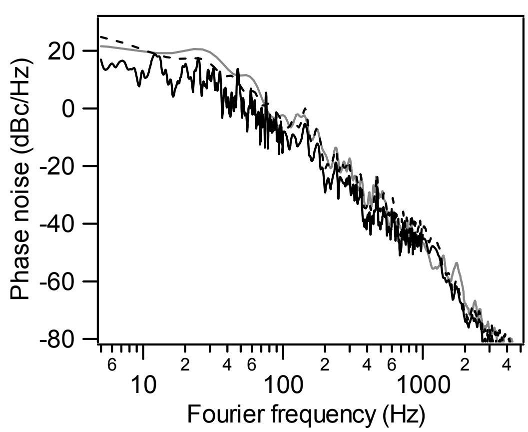


Figure 2. Measured fiber-link noise on the transmitted signal, $S_{fiber}(\omega)$ (solid black) as well as the measured (solid gray) and predicted (dashed black) fiber-link noise on the round-trip signal $S_{fiber,RT}(\omega)$.

In order to compare the measured and calculated phase noise with the PLL active, we need to know the open loop PLL gain, Eq. (7). For low corner frequency, the phase of the gain, $\arg\left[G(i2\pi f)\right] = -(\pi/2+\omega\tau)$, passes through $\pi$ at $\omega = 2\pi f = 2\pi/(4\tau)$, and one would expect to observe oscillations. However, at exactly the same frequency, the cosine term in the gain, Eq. (7), vanishes, and so the oscillations do not grow, as would be the case for a resonant system. Therefore, we can, and do, actually operate with unity gain far above the characteristic frequency $\omega = 2\pi f = 2\pi/(4\tau)$. Figure 3a plots the open loop gain for $G_0 = 4\times10^4$, $\omega_c = 100$ rad/sec and $\tau = 0.38$ msec corresponding to an 80 km link length. In principle, these values could have been independently measured. However, here using Eq. (9) we chose them to get reasonable agreement between the free-running and phase-locked round trip signal, shown in Fig. 3. The agreement on the shape is good.

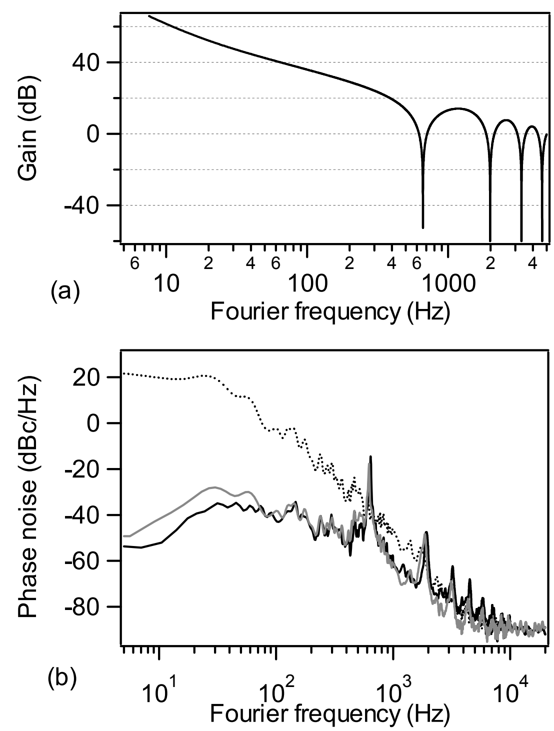


Figure 3. (a) Calculated open loop gain for PLL. (b) Measured unlocked round-trip phase noise, $S_{fiber,RT}(\omega)$ (dotted line) and measured locked round-trip (local) phase noise $S_{local}(\omega)$ (black line) along with the predicted locked round-trip phase noise from Eq. (9) (gray line).

Finally, Figure 4 compares the predicted and calculated residual phase noise on the one-way transmitted (remote) signal by use of Eq. (11). As discussed in Ref. [2, 3], it approaches white phase noise at low frequencies because in that limit Eq. (12) holds and we get a maximal suppression of the free-running phase noise of only $(\omega\tau)^2/3$. Since the free-running fiber-link phase noise increases as $\omega^{-2}$, the resulting residual phase noise is flat. By using the full expression (Eq. (11)) , we find good agreement between the measured and predicted phase noise on the transmitted optical frequency at all Fourier frequencies.

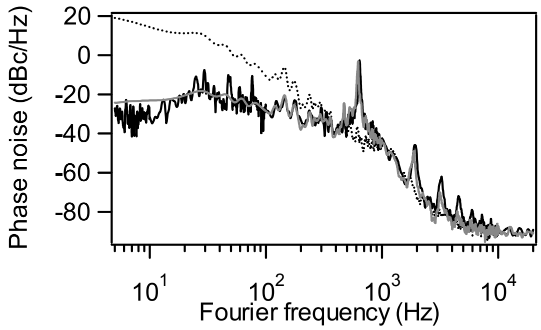


Figure 4. Measured fiber-link noise, $S_{fiber}(\omega)$ (dotted line) and measured residual transmitted (remote) phase noise $S_{remote}(\omega)$ (black line) along with the predicted phase noise from Eq. (11) (gray line).

The integral of this phase noise over frequency will give the total timing jitter, and this integral can be taken for Eq. (11), although we do not do that integration here.

### *D. Frequency Instability: Theory*

For clock comparison, the relevant quantity is the frequency instability, as measured by the Allan deviation. As discussed in much greater detail in Refs. [10, 11], we can directly calculate the Allan deviation from the measured phase noise spectrum. Using the integral expression appropriate for our frequency counters (Eqs. (14) and (16) of Ref. [11]), we have calculated the expected Allan deviation. In Figure 5, it is compared to the measured Allan deviation. We find good agreement at short gate periods. At longer gate periods, the measured Allan deviation is much higher. The reason for this is that there is an increase in the phase noise at low Fourier frequencies (below those shown in Figs. 3 and 4), whereas in the calculation, we assume a flat phase noise at low Fourier frequencies. This excess phase noise at low frequencies arises from the "out-of-loop" fiber path sections in the transmitter and receiver, as discussed in Ref. [3].

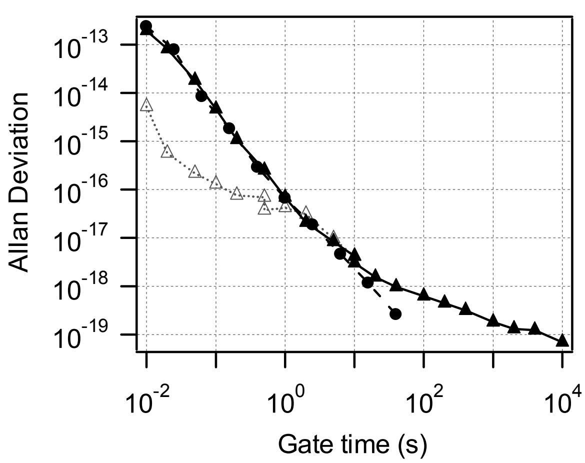


Figure 5. The measured "triangle" Allan deviation over the 80 km link (solid line and triangles) compared to the predicted Allan deviation from the phase noise (solid circles and dashed line). The system noise floor measured for a 0 km link are also shown (open triangles, dotted line). The measured Allan deviation follows the prediction from the phase noise until a gate period of about 10 seconds, after which it follows the system noise floor.

## III. Conclusion

Very low Allan deviations can be achieved by sending an optical carrier over a long fiber optic link. For the 80 km link here, the Allan deviation drops close to $10^{-19}$ at a gate period of $10^4$ seconds. The fundamental limit to the Allan deviation is set by the residual fiber-link noise. If the free-running fiber noise is measured, the residual fiber-link noise can be calculated directly from the open-loop gain. This residual noise can then be used in the known integral formulae to calculate the Allan deviation. Therefore, from the measured fiber-link noise, we can immediately infer the ultimate frequency stability that could be achieved for a fully implemented coherent frequency transfer setup using Doppler cancellation.

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