Theory of the Frequency Comb Output from a Femtosecond Fiber Laser

Nathan R. Newbury, *Senior Member, IEEE* and Brian R. Washburn, *Member, IEEE*

***Abstract*—The output of a femtosecond fiber laser will form a frequency comb that can be phase-locked through feedback to the cavity length and pump power. A perturbative theory is developed to describe this frequency comb output, in particular for a solitonic fiber laser. The effects of resonant dispersion, saturation of the self-amplitude modulation, cavity loss, third-order dispersion, Raman scattering, self-phase modulation, and self-steepening on the spacing and offset of the fiber-laser frequency comb are given. The mechanisms by which the pump power, cavity length and cavity loss control the frequency comb spacing and offset are identified. Transfer functions are derived for the comb response to change in cavity length, pump power or cavity loss. This theory can potentially be applied to other frequency comb sources as well.**

***Index Terms*— Frequency measurement, Laser stability, Optical fiber lasers, Optical fiber measurement applications.**

I. Introduction

A mode-locked laser will emit a pulse train in time, the Fourier transform of which is a frequency comb. The spacing of the comb is set by the laser repetition frequency, f_r, and the offset frequency of the comb is set by the rate of change of the carrier-envelope offset (CEO) phase, f_{ceo}. Based on the original demonstration by Udem *et al.* [1], extremely stable frequency combs have been realized using mode-locked Ti:Sapphire lasers [2-4]. Recently, this concept has been extended to produce self-referenced phase-locked frequency combs further into the near-infrared using mode-locked Er fiber lasers [5-9] and a Cr:forsterite laser [10]. These combs use the same basic technique of the original Ti:Sapphire laser-based combs, namely spectral broadening in nonlinear fiber [11, 12] followed by rf detection of the f_{ceo} [13, 14].

A phase-stabilized fiber-laser frequency comb provides a series of frequency markers across the near infrared from 1 to 2 μm that are directly referenced to a known rf frequency. Fiber-laser frequency combs have the potential of providing this stabilized frequency comb in a fully fiber-optic, robust, power-efficient package that could enable a variety of applications; they should prove useful in frequency metrology, telecommunications, and remote sensing applications. A simplified schematic of a mode-locked fiber laser and the resulting frequency comb is given in Fig. 1.

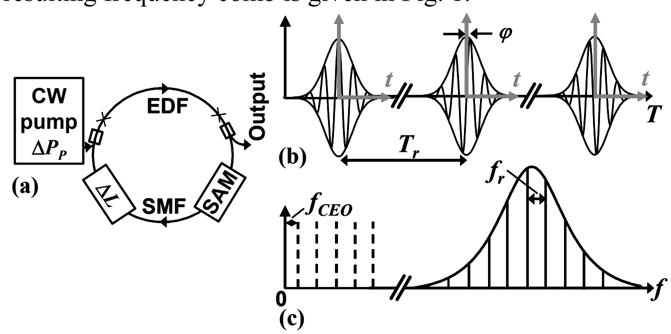

Fig. 1. (a) Simplified schematic of a fiber ring laser. The Erbium-doped fiber (EDF) region is pumped by a cw pump at 980 nm and provides gain. The cavity length change (ΔL) might be through a piezoelectric transducer (PZT) fiber stretcher or air delay line. For a soliton laser, the net cavity dispersion is anomalous. For a stretched-pulse laser the net cavity dispersion is normal. The effective self amplitude modulation (SAM) is provided by nonlinear polarization rotation. Other laser designs are possible. The laser output forms a pulse train in time (b) and a frequency comb in frequency space (c), with a spacing set by the repetition rate, $f_r = 1/T_r$ and offset frequency set by the CEO frequency, f_{ceo}.

The stabilized frequency comb is established by phase-locking the frequency spacing, f_r, and comb offset f_{ceo} to a known microwave (or optical) reference. The repetition rate is stabilized through feedback to the cavity length [5-8, 15] and the offset frequency is stabilized through feedback to the pump power [5-9, 15, 16]. (Alternatively, the repetition frequency can be stabilized by the pump power [17].) In any system developed to date, the feedback bandwidth for the phase-lock of the offset frequency has been limited by the laser response to ~2 to 10 kHz [5, 15, 17]. This limited bandwidth, combined with the rather large phase noise on the f_{ceo} measurement, is a limiting factor on the comb phase quality.

So far the development of fiber-laser based frequency combs has been experimental in nature. This paper presents a theory to describe the frequency comb output of a fiber laser that includes the basic fiber propagation of the laser pulse as well as the effects of the resonant Er contribution to the pulse propagation, self-phase modulation (SPM), third-order dispersion (TOD), stimulated Raman scattering (SRS), self-steepening (SS), perturbations to the cavity loss, and saturation of the self-amplitude modulation. The goal of the theory is to answer three basic questions: first, what

Manuscript received June 14 2005. This work was supported by the U.S. Government and is not subject to copyright.

The authors are with the National Institute of Standards and Technology, Boulder, CO 80305 USA (N. R. Newbury's phone: 303-497-4227; fax: 303-497-3387; e-mail: nnewbury@ boulder.nist.gov).

parameters of the laser determine the position of the frequency comb teeth? Second, how does feedback to the cavity length and pump power control the position of the frequency comb teeth? Third, what sets the bandwidth of the feedback response of the laser? To answer these questions we derive transfer functions relating a change in the control parameters, *i.e.* pump power, cavity length, or cavity loss, to a change in the frequency comb output, specifically the comb spacing, f_r, and comb offset f_{ceo}

Given the analogy with the Ti:Sapphire laser-based frequency combs, it might be tempting to apply the theory for these combs to the fiber laser-based frequency combs. There have been a number of investigations of the control of Ti:Sapphire frequency combs beginning with the early work of Xu *et al.* [18], which pointed out the effects of SPM and intensity-dependent spectral shifts on the control of f_{ceo},. The contribution of various nonlinear effect, such as SPM, self-steepening, and nonlinear refraction have been discussed in detail in [19-24]. Recent empirical observations suggest that the intensity-dependent spectral shifts (which give rise to shifts in the repetition rate when coupled with cavity dispersion) can dominate [25, 26]. Similar shifts have been observed in fiber laser systems [17]. However, with the exception of the combined numerical and analytical model for the noise of mode-locked lasers developed by Paschotta [27, 28], there has not been any comprehensive theory developed to describe the comb output. In particular, the control bandwidth has not been discussed, partly because it has been limited by the transducers and not the laser system [24, 26]. Moreover, the fiber-laser based comb will operate with different physical parameters than the Ti:Sapphire laser-based comb. For all these reasons, the interesting work related to Ti:Sapphire-laser combs cannot be directly translated to fiber-laser based frequency combs.

A much more fruitful starting point to describe the frequency comb output of a fiber laser is through the Master equation for mode-locked fiber lasers that has, in particular, been developed and applied by the MIT group of Haus and Ippen [29-32]. In the main body of the paper, we develop an analytical perturbative theory for the comb output of a fiber laser, beginning with the Master equation. Rather than use the solitonic perturbation theory employed by Haus and Mecozzi in their seminal paper [30], we use an alternative perturbative approach based on a moment method similar to [33, 34] that has the advantage of allowing us to treat the more general case of arbitrary chirp. We find that the chirp will modify the coupling constants, but that the change in the chirp can be conveniently removed from the final equations of motion. By including the gain dynamics, we derive the bandwidth of the system response, which is intimately connected with the laser stability. By including the main relevant perturbations, we identify the most significant physical mechanisms involved in controlling the comb: self-phase modulation, self-steepening, third-order dispersion, spectral shift, and resonant group velocity. As discussed above, SPM and SS have received a lot of attention for Ti:sapphire-laser combs; here we derive the numerical factors appropriate to a fiber-laser comb. We find the additional effects of TOD, spectral shifts and resonant gain contribution are all potentially much more significant, depending on the laser parameters. Finally, we show that the pump-induced spectral shift can be explained by a combination of SRS and a frequency-dependent loss coupled with power-broadening of the Erbium gain bandwidth.

Before launching into the derivation of the fiber-laser response based on the Master equation, we develop a simpler heuristic model for the laser response in the next Section. This simpler model includes all the major effects and can provide greater physical intuition. Section III begins the full perturbative treatment by reviewing the Master equation for a mode-locked fiber laser. Section IV connects the perturbed pulse train and frequency comb. Section V describes the seven perturbations considered here. Section VI solves the perturbed Master equation to find the full transfer functions. After a brief discussion in Section VII, Section VIII concludes.

II. Heuristic, Streamlined Derivation of the Laser Response to changes in pump power

The purpose of this section is to provide a relatively short derivation of the comb response to a change in the pump power. The basic response to a change in cavity length can be similarly derived although it is potentially complicated by any associated cavity loss as discussed in Section VI. While the numeric factors for the nonlinear terms are specific to a solitonic laser, the general equations will also apply to the stretched-pulse laser. The system is both highly nonlinear and highly sensitive to any shift in round-trip time (a 0.1 fs shift yields a 1 MHz shift in offset frequency). As a result, a number of mechanisms come into play as summarized in Fig. 2.

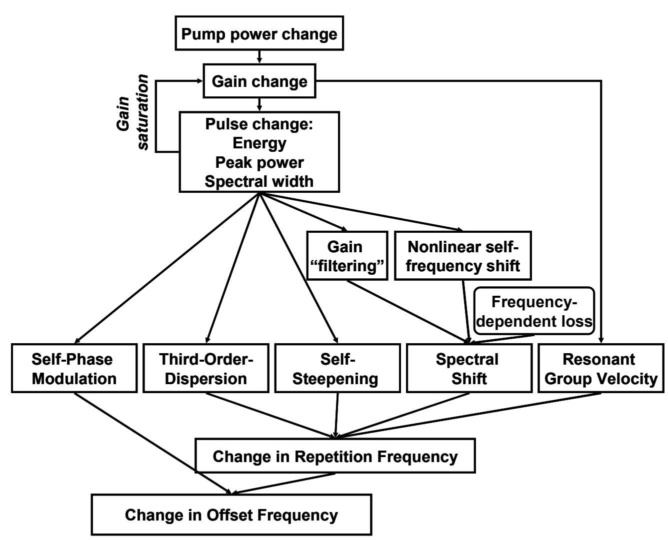

Fig. 2. Basic overview of the system response. A change in the pump power changes both the gain and the pulse parameters with a time constant determined by both the laser parameters and the response of the Erbium gain. The changes in gain and pulse parameters affect the repetition and offset frequencies through a number of mechanisms; the most important ones being self-phase modulation, third-order dispersion, self-steepening, spectral shifts, and the resonant group velocity.

Starting at the top of Fig. 2, we begin with a derivation of the coupling between the gain and pulse energy; this coupling determines the bandwidth of the system response and the magnitude of the pump-induced changes in the pulse. We can derive the coupled differential equations describing the pulse energy and gain through the following argument. Spontaneous relaxation and stimulated pumping from both pump and signal, causes the gain to relax to its steady-state value with a time constant T_g, which is typically much shorter than the spontaneous decay rate of ~ 10 ms for Erbium. If the pump power, P_P, changes, the steady-state value of the gain, g, will also change; this change is quantified by the derivative $g_P = P_P\, dg/dP_P$. Because of gain saturation, any resulting increase in pulse energy, w, drives a decrease in the steady-state gain; this change is quantified by the derivative $g_{sw} = -w\, dg/dw$. Finally, an increase in pulse energy is reduced by the additional loss per pass, denoted η. Putting all of these arguments together yields the coupled differential equations for the change in pulse energy, Δw, and in gain, Δg as a function of time T:

$$
\begin{aligned}
&T_r \partial_T \Delta w = -\eta \Delta w + 2\Delta g\, w, \\
&T_r \partial_T \Delta g = -\frac{T_r}{T_g}\left[\Delta g + g_{sw}\frac{\Delta w}{w} - g_P \frac{\Delta P_P}{P_P}\right],
\end{aligned}
\tag{1}
$$

where T_r ~ 20 ns is the round-trip time around the cavity. A similar set of coupled differential equations has been derived to describe lasers mode-locked with a saturable absorber [35]. Simple self-amplitude modulation (SAM), which provides the necessary energy-dependent gain for mode-locked operation, implies an increase in pulse energy per round-trip or $\eta < 0$, which, if unchecked, would lead to exponential increase in pulse energy. If the gain response is sufficiently rapid, coupling of the pulse energy to the gain saturation can overcome this effect and stabilize the pulse; however for the fiber laser, the stabilizing effect of the gain saturation is too slow and an extra nonlinear loss (or saturation of the SAM) must be included to force $\eta > T_r/T_g$~0 (see Sections V.H, VI.E and Ref.s [35-38]). If (1) is underdamped (i.e. has complex eigenvalues), then relaxation oscillations can result [35]; however, for the fiber laser the system is overdamped, as has been shown experimentally [36], and the system exhibits simple exponential decay governed by two time constants. The slowest time constant is

$$
\frac{1}{\tau_1} \approx \frac{1}{T_g}\left(1 + \frac{1}{\eta}\right) \tag{2}
$$

assuming $T_r / (\eta T_g) << 1$ and g_{sw}~1/2 (see Section V.C), and it determines the system response. The 3-dB bandwidth for phase-locking the system using pump-power modulation, ν_{3dB} is given by $\nu_{3dB} = 1/(2\pi\tau_1)$ and falls in the range of $1/(2\pi T_g) < \nu_{3dB} << \infty$ assuming $\eta < 1$. In terms of τ_1, the solution to the Fourier transform of (1) is

$$
\begin{aligned}
&\Delta\tilde{g}(\Omega) = \left(\frac{\tau_1}{T_g}\right) g_P \frac{\Delta P_P}{P_P}\frac{1}{1 - i\Omega\tau_1}, \\
&\frac{\Delta\tilde{w}(\Omega)}{w} = \left(\frac{2}{\eta}\right)\Delta\tilde{g}(\Omega),
\end{aligned}
\tag{3}
$$

where the tilde represents the Fourier-transform of the quantity with respect to Fourier frequency Ω. For P_P measured with respect to threshold, g_P=1/2 In the limit of $\eta \to 0$, these equations have the physically sensible result that Δg = 0 and $\Delta w/w = \Delta P_P/P_P$. In other words, if there is no additional cavity loss, the gain must remain fixed since the laser operates with gain equal to loss. Higher values of η imply additional loss per pass and therefore a corresponding increase in the gain. *The time constant* (2) *and the resulting 3-dB bandwidth* ν_{3dB} *set the response of the frequency comb to a pump-power change. The relative magnitude of the gain and pulse energy change in* (3) *will impact the relative magnitude of different mechanisms that change the frequency comb.*

Now that we have an expression for the system response bandwidth and for the change in gain and pulse energy we can proceed to derive the effect of these changes on the frequency comb parameters. First we derive an expression for both the repetition frequency and offset frequency in terms of the system parameters. We define a "generalized" propagation constant, $\beta^{gen}(\omega)$, so that the Fourier-component of the pulse at ω after one round trip around the laser cavity of length L is $E(L) = E(0)\exp(i\beta^{gen}(\omega) - i\omega t)$, where the generalized, lumped, propagation constant,

$$
\beta^{gen}(\omega) = \beta(\omega) + \beta^{res}(\omega) + \beta^{NL}(\omega) \tag{4}
$$

includes contributions from the lumped fiber propagation constant, $\beta(\omega)$, the nonlinear contributions, $\beta^{NL}(\omega)$ and the resonant contribution from the gain medium, $\beta^{res}(\omega)$. The nonlinear SPM contribution is given very roughly by $\beta^{NL}(\omega)$ $\sim\delta|A|^2$, where A is the peak electric field of a pulse with total energy w and temporal width τ, and $\delta = L\omega n_2/(c a_{eff})$, where a_{eff} is the effective area and n_2 is the nonlinear index. $\beta^{res}(\omega)$ can be calculated for a Lorentzian model of the Erbium gain with a full width $2\Omega_g$, centered at ω_0 with a peak value g (see Section V.C). To simplify the treatment of $\beta^{res}(\omega)$ in what follows, we expand about the gain peak, ω_0, in which case $\beta^{res}(\omega)$ contributes a term g/Ω_g to the inverse group velocity but nothing to the phase velocity or to second-order dispersion.

The repetition frequency, f_r, is simply the inverse round trip time, T_r, which is the first derivative $d\beta^{gen}/d\omega$ averaged over the pulse spectrum of width ω_{rms} and evaluated at the carrier frequency $\omega_c = \omega_0 + \omega_\Delta$. We find,

$$
f_r^{-1} = T_r \approx \beta_1 + \omega_\Delta\beta_2 + \frac{1}{2}\omega_{rms}^2\beta_3 + \frac{g}{\Omega_g} + \frac{\mu w\delta}{2\tau\omega_0} + Cl_\omega, \tag{5}
$$

where $\beta_n = d^n\beta/d\omega^n|_{\omega_0}$ and assuming $\omega_\Delta << \omega_{rms}$. The first three terms come from the linear fiber propagation. The fourth term is the resonant contribution to the group velocity. The fifth term is the self-steepening effect where μ ~ 1 characterizes the frequency dependence of δ, (see Section V.F). This heuristic deriviation of the self-steepening contribution agrees with the more rigorous perturbative treatment given later. The final term, not derived here, is proportional to the chirp, C, and the spectral derivative of the loss, l_ω (see (60)).

The offset frequency, f_{ceo}, is given by the product of the repetition rate and the phase change per loop, which is $\beta(\omega) - \omega T_r$ (see Section IV). Expanding in a Taylor series,

$$\begin{aligned} 2\pi f_{ceo} &= T_r^{-1}\left(\beta_0 + \omega_\Delta \beta_1 + \varphi_0 - \left(\omega_0 + \omega_\Delta\right) T_r\right) \\ &\approx T_r^{-1}\left(\beta_0 + \varphi_0 - \omega_0 T_r\right) = T_r^{-1}\left(\beta_0 + \varphi_0\right) - \omega_0, \end{aligned} \tag{6}$$

where the phase shift φ_0 arises from the nonlinear contribution $\beta^{NL}(\omega)$ and β_2; it is designated as the "self-phase modulation" contribution. For a chirp-free soliton solution, φ_0 is the well-known soliton phase shift, $w\delta/4\tau$, but it diverges from this value for a chirped pulse (see (21)) or for a stretched-pulse laser. Note the offset frequency has *no explicit dependence on the frequency shift*; any dependencies arise through the repetition rate T_r^{-1}. This same expression can be rewritten in terms of the group and phase velocity as in (67) given later.

Next we derive an expression for the spectral shift from the gain peak, ω_Δ, that appears in (5). Pump-induced spectral shifts have been empirically shown to potentially dominate in both Ti:Sapphire and fiber-laser frequency combs [17, 25, 26]. The carrier frequency is given by the local maximum in the net gain; expanding the gain, g, and loss, l, about the gain peak, ω_0, gives a net gain of $(g - \omega_\Delta^2 D_g) - (l + \omega_\Delta l_\omega)$, where the "gain dispersion", $D_g = (1/2)\, d^2 g(\omega)/d\omega^2$, and the system-dependent loss slope is $l_\omega = dl/d\omega$. The laser operates at the peak gain which occurs at the (linear) carrier frequency shift, $\omega_{\Delta,L} = -l_\omega/(2D_g)$. There are additional nonlinear perturbations that give rise to a frequency shift. Citing (61), the nonlinear contribution to the frequency shift is

$$\omega_{\Delta,NL} = -\frac{w\delta}{5D_g \tau\left(1+C^2\right)}\left(\tau_R + \mu\omega_0^{-1} C\right), \tag{7}$$

where the first term in parentheses arises from the Raman effect with strength τ_R, and the second term arises from the self-steepening effect. (We ignore the contribution from any frequency-dependence of the self-amplitude modulation). Just as with the linear carrier frequency shift, the nonlinear shift depends inversely on the gain filtering factor, D_g.

We now continue on to the lower portion of Fig 2 and calculate the response of the repetition frequency, given in (5) and the offset frequency, given in (6), to a change in pump power. To do this, we need to understand the dependence of each of the system parameters in (5) and (6) on pump power. Equation (3) gives the dependence of gain and pulse energy on pump power. For a solitonic laser, a change in pulse energy will have an accompanying change in pulse duration, τ, that is inversely proportional to w and an accompanying change in spectral width, ω_{rms}, that is proportional to w. The scaling of the spectral shift, $\omega_\Delta = \omega_{\Delta,L} + \omega_{\Delta,NL}$ with pump power is slightly more complicated since the gain filtering parameter D_g depends on both gain and signal power through power-broadening of the homogenous gain bandwidth (see Section V.C). (Power broadening does not similarly affect the resonant group-velocity contribution, g/Ω_g.) Making the simple assumption that D_g scales linearly with gain and inversely with signal power, the change in ω_Δ with pump power is

$$\Delta\omega_\Delta = \left(3\omega_{\Delta,NL} + \omega_{\Delta,L}\right)\frac{\Delta w}{w} - \omega_\Delta \frac{\Delta g}{g} + \omega_\Delta \frac{\Delta P_{cw}}{P_{cw}} \tag{8}$$

using (3), where the first term arises from the pulse-energy dependence of the nonlinear spectral shift and gain filtering, D_g, the second from the gain-dependence of the gain filtering D_g and the last term is added heuristically to account for any cw laser power, P_{cw}, that can additionally broaden D_g.

Finally, using these scaling laws and (5) yields the pump-induced change in the repetition frequency as

$$\frac{\Delta f_r}{f_r} = -f_r\left\{\frac{\Delta g}{\Omega_g} + \left(\omega_{rms}^2 \beta_3 + \frac{\mu w\delta}{\tau\omega_0}\right)\frac{\Delta w}{w} + \beta_2 \Delta\omega_\Delta\right\} \tag{9}$$

From (6), the normalized change in the CEO frequency is

$$\frac{\Delta f_{ceo}}{f_r} = \frac{\varphi_0}{\pi}\left(\frac{\Delta w}{w}\right) + \frac{\beta_0}{2\pi}\left(\frac{\Delta f_r}{f_r}\right). \tag{10}$$

where $\beta_0/2\pi \sim 3\text{x}10^6$ is the number of wavelengths that fit around the fiber cavity. The dependence of the gain and pulse energy on pump power are given in (3) and the dependence of the spectral shift on pump power is given in (8). *These results, (9) and (10), are our final result and summarize the response of the frequency comb to a pump-power change.* They are identical to those derived later in the paper using the Master equation. We can identify each of the terms in (9) and (10) with a specific physical mechanism. The first term in (10) is the SPM term. The second term in (10) is the pump-induced change in repetition frequency that is caused by the mechanisms given in (9). Specifically, the first term in (9) results from the resonant group velocity contribution, the second results from TOD, the third from SS and the fourth from the spectral shift, which has a main nonlinear contribution from stimulated Raman scattering and a linear contribution from the frequency-dependent loss.

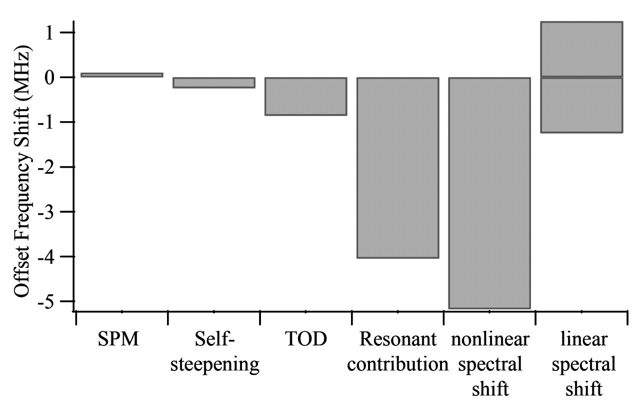

Fig. 3. Summary of contributions to control of f_{ceo} laser from a 1% change in pump power above threshold using Eqs. (9) and (10), the values in Table I and a stability factor of η=1/3. The linear loss, l_ω, is assumed to vary over a range corresponding to ±1%/nm change in loss.

Aside from the linear frequency shift, all the contributions to Δf_r are negative, corresponding to a slowing of the pulse. Since the SPM contribution is typically very small, the change in offset frequency is directly proportional to the change in repetition frequency, with a scaling factor of $\beta_0/2\pi$. The relative contributions are plotted in Fig. 3 using the values from Table 1. For other laser parameters, these relative contributions can vary significantly. Also, the equations should be adjusted for the experimentally-measured scaling (*e.g.* we have observed $\omega_{rms}\sim w^{1/2}$ rather than $\sim w$).

To provide a more rigorous derivation of the above response as well as the response to cavity length changes and loss changes, the remainder of the paper develops a perturbative approach based on the Master equation for mode-locking.

TABLE I
VALUES USED IN CALCULATIONS

Symbol	Quantity	Value
D	Dispersion ($=\beta_2/2$)	-0.025 ps^2
δ	SPM strength	4 kW^{-1}
β_0	Propagation constant	$2\pi(3.5\times10^6)$
v_{gr}/v_{ph}	Group/phase velocity[a]	0.982
L	Loop length	4 m
g	Lumped gain[b]	0.75
Ω_g	Gain bandwidth[c]	2π (0.4 THz)
T_1	Er lifetime	10 ms
l	Lumped loss	$\sim g$
τ_R	Raman slope	5 fs
μ	Self-steepening factor	1.3
ω_0	Carrier frequency	2π (200 THz)
w_0	Pulse energy	0.1 nJ
τ_0	Pulse width[d]	115 fs
D_g	Gain filtering[e]	0.005 ps^2
T_r	Round-trip time ($=1/f_r$)	1 / (50 MHz)

The values are typical for an "all-fiber" soliton laser, for example of [7]. [a]Estimated for an all-fiber laser consisting of 3 m of standard single-mode fiber and 1 m of Erbium doped fiber [39]. [b]Assuming a total power gain of 4. [c]Corresponds to an approximate FWHM homogeneous linewidth of 7 nm at 1560 nm which is an approximation based on the measured values of 3 to 11 nm for different fiber types and wavelengths [40]. [d]Corresponds to 200 fs FWHM. [e]Corresponds to an effective saturation power of 2Ps''=50.

III. THE MASTER EQUATION REVIEWED

The electric field pulse traversing the laser, a, is written in terms of the electric field, E, in the slowly varying envelope approximation as

$$E(z,t')=a(z,t')\exp\left(i\beta_0 z-i\omega_0 t'\right)+c.c., \tag{11}$$

where β is the average propagation constant of the fiber laser and $\beta_0=\beta(\omega_0)$ is the propagation constant evaluated at ω_0. It is convenient to select ω_0 equal to the peak of the erbium gain profile, which is assumed to be Lorentzian with a width Ω_g.

The Master equation is derived as a difference equation for the pulse, a, after one round trip around the laser cavity [30, 32]. Defining the reduced time as

$$t=t'-\left(\beta_1+\frac{g}{\Omega_g}\right)z\,, \tag{12}$$

where $\beta_1=d\beta/d\omega$ evaluated at $\omega=\omega_0$, the Master equation for the mode-locked laser is [30, 32]

$$T_r\frac{\partial a}{\partial T}=\left[g-l\right]a+\left(D_g-iD\right)\frac{\partial^2 a}{\partial t^2}+\left(\gamma+i\delta\right)|a|^2 a+V(a), \tag{13}$$

where the envelope $a(t,T)$ is a function of the reduced time t on the timescale of a pulse and the much slower time T on the timescale of a round-trip time (see Fig. 1). The total length of the laser cavity is L. The propagation constant is redefined to include the fiber laser length, $L\beta(\omega)\rightarrow\beta(\omega)$, so that it is the "lumped" propagation constant. The quantities g and l are the lumped gain and loss, respectively. The round-trip time is $T_r=(\beta_1+g\Omega_g^{-1})$. The gain-dispersion is typically given as $D_g=g\Omega_g^{-2}$, although Section V.C discussed modifications to this term. The lumped fiber dispersion is characterized by $D=\beta_2/2$, where $\beta_2=d^2\beta/d\omega^2$. The lumped SPM term is described by δ and the SAM term, responsible for mode-locked operation, is described by γ. The final term $V(a)$ represents any perturbations, considered later. Here we employ the sign convention of [40] so that the Master equation is the complex conjugate of that appearing in [30].

The solution to the Master equation is of the form [41]

$$a_0(t,T)=a_c(t-t_A)e^{ip(t-t_A)}e^{i\theta}, \tag{14}$$

where the arrival time, t_A, and phase, θ, can depend on T but otherwise the circulating laser pulse should reproduce itself every round trip. The carrier frequency offset from gain resonance is $-p$ (which equals ω_Δ given in Section II). For a solitonic laser, the chirped envelope is given by

$$a_c(t)=A\left[\operatorname{sech}\left(\frac{t}{\tau}\right)\right]^{1-iC}, \tag{15}$$

with amplitude A, pulse width τ (where the full-width half maximum is 1.76τ), and pulse chirp C. For the stretched pulse laser, the solution is a chirped Gaussian rather than a chirped sech [42].

Since the arrival time and phase are permitted to change during each pass of the laser (corresponding to a modification to the round-trip time, T_r, and CEO phase), it is more useful to make the general definition,

$$\begin{aligned}\varphi(T)&\equiv T_r\partial_T\theta(T)\\ \Delta T_r(T)&\equiv T_r\partial_T t_A(T),\end{aligned} \tag{16}$$

for the phase change per round trip, φ, and shift in the repetition time, ΔT_r. (These definitions are strictly correct since the Master equation was originally derived as a difference equation for the change per round trip.) For the unperturbed Master equation, $\Delta T_r=0$ since there are no first derivative with respect to time. Also, since ω_0 was chosen to be equal to the gain peak, the unperturbed solution has a carrier frequency shift $p_0=0$. The remaining parameters describing the unperturbed pulse A_0, C_0, τ_0, and θ_0 (or equivalently φ_0) can be found by substitution into the appropriate Master equation. Following [30, 32] for a soliton laser the pulse width is given by

$$\tau_0^2 = \frac{D_g - 2C_0 D - C_0^2 D_g}{l - g}, \tag{17}$$

where stability requires $\tau_0^2 > 0$. The pulse amplitude, A_0, is

$$(\gamma + i\delta) A_0^2 \tau_0^2 = -(D_g - iD)(2 - 3iC_0 - C_0^2). \tag{18}$$

The chirp is

$$C_0 = q \pm \sqrt{q^2 + 2}, \tag{19}$$

where

$$q = \frac{-3(D\delta - \gamma D_g)}{2(\delta D_g + \gamma D)}. \tag{20}$$

The phase shift per pulse can be written as

$$\varphi_0 = \frac{\delta A_0^2}{2} + \frac{-C_0 D_g + C_0^2 D}{2\tau_0^2}, \tag{21}$$

which reduces to the well known soliton phase shift $\varphi_0 = \delta A_0^2/2$ for zero chirp. Finally, for completeness, the total pulse energy is

$$w = 2A^2\tau. \tag{22}$$

IV. Connection between the Laser Pulse Train and the Frequency Comb

The train of pulses given by Eqs. (14) and (15) is

$$E_0(t') = a_c(t) e^{-i\omega_0 t'} \otimes \sum_m \delta(t' - mT_r)\, e^{im(\varphi_0 + \beta_0 - \omega_0 T_r)}, \tag{23}$$

using (11) and (12),and where the index m denotes the pulse with arrival time mT_r and $\otimes$ denotes convolution. From this equation, the CEO phase is identified as $\phi_{ceo,0} = \varphi_0 + \beta_0 - \omega_0 T_r$. The Fourier transform of this expression gives the unperturbed frequency comb,

$$\tilde{E}_0(f) = f_{r,0}\tilde{a}_c(f - \omega_0/2\pi) \sum \delta(f - nf_{r,0} + f_{ceo,0}), \tag{24}$$

where the unperturbed repetition frequency is $f_{r,0}=T_r^{-1}$ and the unperturbed CEO frequency is $f_{ceo,0} = f_{r,0}\phi_{CEO,0}/(2\pi)$.

Equation (24) is the standard expression for the frequency comb [3, 4, 43]. Our interest is in the effects of the perturbations, $V(a)$, on the frequency comb. These perturbations will generally result in a different T-dependent function for the output pulses than Eq. (14) so that, in general, the pulse train cannot even be written in the simple form of Eq. (23). However, following [44] we assume that the perturbations are either independent of T or very slowly varying with the T (on the timescale of T_r) so the pulse train is constant over the measured interval, yielding a well defined comb.

In any experiment, the frequency comb of Eq. (23) or (24) is never directly measured since it is at optical frequencies. Rather the comb is heterodyned either with itself, a similar separate comb, or a cw laser to generate an RF frequency comb that will have frequencies corresponding to those in Eq. (24), depending on the exact experimental setup. For a typical bandwidth receiver, only changes to the lowest-order moments of the pulse are detectable [44]. This statement is effectively identical to assuming that the pulse retains its basic shape of Eq. (14) but with possible changes in the total pulse energy, Δw, the phase shift per pulse, $\Delta\varphi$, the carrier frequency, $-\Delta p$, the arrival time, $m\Delta T_r$ and the pulse chirp, ΔC. The pulse train in Eq. (23) becomes

$$E_0(t') \approx a_c(t') e^{-i(\omega_0 - \Delta p)t'} \otimes \sum_m \sqrt{1 + \frac{\Delta w}{w}}\, \delta(t' - m(T_r + \Delta T_r))\, e^{im\phi_{CEO}} \tag{25}$$

to lowest order, where $\phi_{CEO} = \bar{\varphi} + \beta_0 - \omega_0(T_r + \Delta T_r)$. The quantity $\bar{\varphi}$ is the intensity-averaged phase shift per pulse rather than the quantity φ of (16). It is defined, in terms of the pulse-averaged phase $\bar{\theta}$ (see Eq. (50)) as

$$\bar{\varphi} \equiv T_r \partial_T \bar{\theta}. \tag{26}$$

For the hyperbolic secant solution, Eq. (15), the pulse-averaged phase is

$$\bar{\theta} \equiv \theta + (1 - \ln(2))C, \tag{27}$$

so that $\bar{\varphi} = \varphi + (1 - \ln(2)) T_r \partial_T C$. For the unperturbed pulse, $\bar{\varphi}_0 = \varphi_0$. The choice of $\bar{\varphi}$ over φ is made for several reasons. First, the equations of motion are simpler for $\bar{\varphi}$. Second, this quantity is what would be measured in a typical experiment involving frequency combs [44]. Finally, it is arguably the quantity preserved in supercontinuum formation. This choice effectively defines the CEO phase as the average phase across the pulse rather than the phase at either the temporal or spectral peak of the pulse.

The Fourier transform of (25) is identical to Eq. (24) except for a modified amplitude, modified repetition frequency,

$$f_r = \frac{1}{T_r} - \frac{\Delta T_r}{T_r^2}, \tag{28}$$

and modified CEO frequency,

$$2\pi f_{ceo} = \frac{1}{T_r}\left[\bar{\varphi}_0 + \beta_0 - \omega_0 T_r\right] + \frac{1}{T_r}\left[\Delta\bar{\varphi} - (\bar{\varphi}_0 + \beta_0)\frac{\Delta T_r}{T_r}\right] \tag{29}$$

where the second terms in (28) and (29) are respectively the perturbations to the repetition frequency and CEO frequency. Neither changes in the pulse energy nor in the carrier frequency appear directly in (28) or (29); however, changes in both will indirectly affect the comb spacing, f_r, and offset, f_{ceo}.

V. The Perturbations

A. Introduction

The perturbations, $V(a)$, in the Master equation (13) will give rise to shifts in the pulse energy, Δw; the phase shift per pulse, $\Delta\bar{\varphi}$; the carrier frequency, $-\Delta p$; the round trip time, ΔT_r; and the pulse chirp, ΔC. These shifts can directly or indirectly impact the frequency comb output of the laser through (28) and (29). A total of seven perturbations are considered here: length fluctuations, gain fluctuations, spectrally-dependent loss and loss fluctuations, the delayed Raman term, the self-steepening term, third-order dispersion, and saturation of the SAM:

$$V = V_{length} + V_{gain} + V_{loss} + V_{Raman} + V_{SS} + V_{TOD} + V_{SAT}. \quad (30)$$

B. Length Fluctuations

Experimentally, the cavity length might be varied, for example, through a piezo-electric transducer (PZT) fiber stretcher or a variable air delay line. The length perturbation term is easily derived directly from the Master equation as [30]

$$V_{Length} = \Delta L(T)\left(i\beta_0^L - \beta_1^L\partial_t - i\beta_2^L\partial_t^2/2\right)a(t,T), \quad (31)$$

ignoring any extra SPM or SAM from the added length change. This term is found to drive a change in pulse arrival time through the middle term, average phase through the first and third terms, and chirp through the third term. The L superscript on the β's indicate that these quantities take on the values appropriate to the medium providing the added length – rather than the average values of the fiber laser used in the Master equation. (As will be seen later, this distinction is important in calculating the effect of a length change on the CEO frequency.) In other words, for a variable air delay $\omega_0^{-1}\beta^L{}_0 \approx \beta^L{}_1 \approx c^{-1}$ and $\beta^L{}_2 \approx 0$, whereas for a fiber stretcher the values are those appropriate for the fiber being stretched. (For a uniform temperature-induced length change the values would be identical to the average values in the Master equation.) Note that this term is directly proportional to one already in the Master equation and quite small on the time scale of T_r. As a result, the full perturbed solution will retain the shape of the basic solution to a very high order.

C. Gain Fluctuations

Within the approximation of the Master equation, the gain fluctuation term is simply

$$V_{gain} = \Delta g(T)\left(1 - \Omega_g^{-1}\partial_t + \Omega_g^{-2}\partial_t^2\right)a(x). \quad (32)$$

This term is found to drive a change in pulse energy (through the first and last terms), pulse timing through the middle term, and average phase and possibly chirp through the third term. Just as with the length perturbation, this perturbation represents a slow change in a term already present in the Master equation, and the solution will retain its shape to very high order. However, unlike the length term, there is no direct experimental control over the gain, and this equation must be supplemented with an equation describing the gain dynamics. To derive the appropriate dependence on signal and pump power, it is useful to start with the complex susceptibility. We will continue to assume a simple three-level model for Erbium (ignoring excited-state absorption) with the gain provided by a two-level transition with equal absorption and emission cross sections and a width Ω_g. In the presence of strong signal and pump beams, the resonant contribution to the complex susceptibility is [45]

$$\chi(\omega) = \left(\frac{2ncg_0}{\omega}\right)\left(\frac{P_p' - 1}{P_p' + 1 + 2\dfrac{P_s'}{\left(1+\omega_\Delta^2\Omega_g^{-2}\right)}}\right)\left(\frac{\omega_\Delta\Omega_g^{-1} - i}{1+\omega_\Delta^2\Omega_g^{-2}}\right) \quad (33)$$

where g_0 is a constant, ω_Δ is the shift from resonance as in Section II, and the normalized signal and pump powers, $P_s'=P_s/P_{s,sat}$ and $P_P'=P_P/P_{P,sat}$ are expressed in terms of the saturation powers $P_{s,sat}$ and $P_{P,sat}$. In reality, the fiber is of finite length so that the effect of this complex susceptibility should be described by an exponential of the integral over length. However, in the Master equation approach, we treat the susceptibility as a single lumped quantity so that the gain is $g(\omega) = (-\omega/2nc)\mathrm{Im}(\chi)$ and the lumped propagation constant is $\beta^{res}(\omega) = (\omega/2nc)\mathrm{Re}(\chi)$. (For high gain, the total power gain $G=e^{2g}$, but in so much as this is exactly balanced by the loss this adjustment the Master equation need not be modified; only changes in the gain or loss are important.) A Taylor expansion about resonance, ω_0, gives $\beta^{res}(\omega) = \beta_0^{res} + \omega_\Delta\beta_1^{res} + \omega_\Delta^2\beta_2^{res}/2 + \omega_\Delta^3\beta_3^{res}/6$, where $\beta_0^{res}= 0$, $\beta_1^{res}= g/\Omega_g$ (as already appears in the Master equation), $\beta_2^{res} = 0$, and

$$\beta_3^{res} = -\frac{6g}{\Omega_g^3\left(1+2P_s''\right)}, \quad (34)$$

where the generalized saturation parameter $P_s'' = P_s'/(1+P_P')$. his term will be incorporated in V_{TOD} in Eq. (44).

A similar Taylor expansion of the gain gives $g(\omega) = g - D_g\omega_\Delta^2$ (exactly the terms expected from the Master equation) where ,

$$g = g_0\left(\frac{P_p' - 1}{1 + P_p' + 2P_s'}\right) \quad (35)$$

and the gain filtering is modified to include a power-broadening factor as

$$D_g = \frac{g}{\Omega_g^2\left(1+2P_s''\right)} \quad (36)$$

Note that the perturbation to the group velocity, scales with the fundamental resonant width while the gain filtering effect, (36), scales with the power-broadened width. The last term in (32) is therefore incomplete as it only includes the effect of the gain change on the overall gain filtering, whereas the signal power change will also affect the gain filtering term. We nevertheless will carry the 1st order perturbation theory through using (32), finding that the last term is of negligible importance. The signal-power dependence of the gain filtering coefficient, D_g, will be included as a 2nd-order perturbation in the final expression since it does potentially affect the spectral shifts. Note that the above expressions assume a very simple model of the Erbium gain. In reality, the gain is not a simple Lorentzian and, moreover, because of the strong signal saturation the Kramers-Kronig relation cannot be used to infer the dispersive properties from the measured gain [45]. The main deviation from the Lorentzian model is probably $\beta_0^{res}\neq 0$.

Finally, the dynamics of the system arise from the dynamics of the population difference and follow an exponential decay with the time constant $T_g^{-1} = T_1^{-1}\,(1 + 2P_s{}' + P_P{}')$. T_g can be easily determined experimentally by measuring the dynamics of an EDFA with identical operating conditions. Perturbations to the gain, arising from perturbations in either the signal or pump powers follow

$$\partial_T \Delta g = -\frac{1}{T_g}\left(\Delta g + g_s \frac{\Delta P_s}{P_s} - g_P \frac{\Delta P_P}{P_P}\right), \tag{37}$$

where the coefficients $g_s = -P_s\partial_{Ps}g$ and $g_P = P_P\partial_{Pp}g$. The total power gain of the fiber is expressed as $P_{s,out} = e^{2g}P_{s,in}$, where $g=\int g(z)\mathrm{d}z$ is the integrated value of the gain over the fiber length. Since the laser operates in the strongly saturated regime, *i.e.,* $P_s/P_{s,sat}>>1$, the total output power will be a linear function of the pump power, $P_{s,out} \propto (P_P - P_{P,sat})$. In that limit, $g_s=1/2$ and $g_p = 1/2 + P_{P,sat}/2P_P \sim 1/2$.

Equation (37) assumed a cw signal and needs to be rewritten in terms of the previously defined pulse moments. The gain is assumed to be a function of the ratio $P_s/P_{s,sat}$; for the spectrally broad signal, this quantity is replaced by its spectral average, $<P_s(\omega)/P_{s,sat}(\omega)> = w_0 T_r^{-1}\left(1-\kappa\right)/P_{s,sat}$, where κ is the ratio of the mean-squared pulse bandwidth to the gain width:

$$\kappa = \omega_{rms}^2 / \Omega_g^2 \,. \tag{38}$$

For a soliton laser with a chirped sech pulse shape,

$$\omega_{rms}^2 = \frac{1+C^2}{3\tau^2} \tag{39}$$

After the substitution $P_s \rightarrow w_0 T_r^{-1}(1-\kappa)$ and the appropriate partial derivatives, Equation (37) becomes

$$\partial_T \Delta g = -\frac{1}{T_g}\left(\Delta g + g_{sw}\frac{\Delta w}{w} - g_{sC}\frac{\Delta C}{C} - g_{sT}\frac{\Delta T_r}{T_r} - \frac{\Delta P_P}{P_P} g_P\right), \tag{40}$$

where $g_{sw} = g_s\left(1-3\kappa\right)/(1-\kappa) \sim g_s(1-2\kappa)$, $g_{sT} = g_s$ and $g_{sC} = g_s 2C^2\left(3\tau^2\Omega_g^2(1-\kappa)\right)^{-1}$, with g_s evaluated at an equivalent cw power of $w_0 T_r^{-1}(1-\kappa)$. This is the differential equation that will be used to describe the gain dynamics.

D. Spectrally-dependent loss and loss fluctuations

We consider two different types of loss perturbations. First, there can be a change in loss, Δl, associated with, for example, a length change. Second, there can be a static frequency dependent loss, $l_\omega = dl/d\omega$. The perturbation is,

$$V_{loss} = -\Delta l a - i l_\omega \partial_t a \tag{41}$$

The first term will clearly lead to a change in pulse energy and the second will lead to a frequency shift.

E. Delayed Raman Effect

The delayed, Raman portion of the nonlinearity gives [40],

$$V_{Raman} = -i\delta\tau_R a \partial_t \left|a\right|^2 , \tag{42}$$

where τ_R~5 fs. This term leads to a carrier frequency shift, *i.e.*, the Raman self-frequency shift of a pulse. For the laser, this shift is strongly damped by gain filtering (otherwise the pulse would eventually shift outside the Er gain bandwidth entirely).

F. Self-Steepening Term

The self-steepening (SS) term is [40, 46, 47]

$$V_{SS} = -\mu\frac{\delta}{\omega_0}\partial_t\left(\left|a\right|^2 a\right), \tag{43}$$

where $\mu = 1+\omega_0 \mathrm{d}N/\mathrm{d}\omega$ and N is the modal shape scale factor [47] and ignoring dispersion of n_2. The frequency dependence of the modal shape is related to the frequency-dependence of the effective area [48] and can be similarly estimated using Bessel-functions to describe the transverse mode profile[40]. For single-mode step-index fiber with a cutoff wavelength of 1.2 μm, $\omega_0 \mathrm{d}N/\mathrm{d}\omega$ ~0.3 giving $\mu \sim 1.3$. From (55), this term drives a timing shift, since it effectively changes the group velocity [19], and a frequency shift for a chirped pulse. The strength of this term is comparable to the Raman term since $\omega_0^{-1} \sim 1$ fs.

G. Third-Order Dispersion

For a short pulse, third-order dispersion,

$$V_{TOD} = \frac{1}{6}\beta_3^{eff}\partial_t^3 a \,, \tag{44}$$

can be important, where the effective third-order dispersion β_3^{eff} has a contribution from the fiber, β_3, and from the Er gain. In the Lorentzian gain model, the resonant contribution from the Er is given by (34) , while the contribution from the single-mode fiber is ~ 0.13 ps^3/km and typically dominates.

H. Saturation of the Self-Amplitude Modulation

The condition for passive mode-locking requires the rapid buildup of a pulse. This implies a negative time constant for the exponential growth of the pulse energy; and indeed one finds a times constant of ~ -4(l–g) as shown below for the unperturbed Master equation. However, once mode-locking is achieved, there must exist some mechanism to stabilize the pulses [29, 38]. For gain media with short decay times, this mechanism is provided by the gain saturation. However, for a fiber laser, the gain saturation occurs on a very slow time scale ($T_g>>T_R$), and the stabilization is provided by either rollover of the interferometric SAM or by increasing energy loss to Kelly sidebands[29]. The effective saturation is heuristically given as [29, 37] ,

$$V_{SAT}\left(a\right) = -\gamma_5\left|a\right|^4 a \,. \tag{45}$$

Substitution into the equations of motion (55) gives a shift in the pulse energy and, for a chirped pulse, the average phase, and chirp. These first-order terms, however, are of little interest since they do not affect the time-dependent behavior of the pulse energy; calculation of that effect requires a second-order treatment which rapidly becomes a cumbersome. Instead we find the expression for the pulse energy evolution through (50) and the Master equation (13),

$$T_r\partial_T w = 2w\left[\left(g-l\right) - \omega_{rms}^2 D_g + \frac{2}{3}\gamma A^2 - \frac{8}{15}\gamma_5 A^4\right]. \tag{46}$$

Steady-state operation occurs when the expression in square brackets vanishes. This term can be rewritten explicitly in terms of the balance between the effective gain and loss as

$$g + \frac{2}{3}\gamma A^2 = l + \omega_{rms}^2 D_g + \frac{8}{15} A^4 \gamma_5 \,. \tag{47}$$

From (46), (47), and the scaling $A \sim \tau^{-1} \sim \omega_{rms} \sim w$, the time dependence for a perturbation to the pulse energy is,

$$T_r \partial_T \Delta w = -\eta \Delta w - \frac{4wCD_g}{3\tau^2}\Delta C \,, \tag{48}$$

where

$$\eta = 4(g-l) + \frac{32}{15} A^4 \gamma_5 \tag{49}$$

which is stable to perturbations provided $\eta > 0$.

Mode-locked operation requires $l > g$ (otherwise the laser would operate cw). Stability then requires the second term in (49) to exceed the first. An upper limit of $\eta < 5$ can be obtained from Eq. (47) assuming $g=l$, $\omega_{rms}^2 = 0$ and a maximum value for nonlinear polarization rotation of $\gamma A^2 \sim 0.6\pi$ [29]. Including the spectral broadening and differential between gain and loss will reduce η significantly. A more realistic estimate for a solitonic laser must include the fact that Kelly sidebands will dominate the saturation [29] reducing the possible peak powers. If we assume a chirp-free solution, so that $\gamma = \delta D_g / |D|$, the parameters in Table I and use (47) and (49), we find $\eta < 0.4$ for $g = l$. Again the differential between gain and loss will further reduce η. These arguments, combined with the relatively quiet pulse train observed out of mode-locked fiber lasers [36], suggest that η is less than unity but not much less than unity. As we find later, η is experimentally accessible from a comparison of the laser response bandwidth and the Er gain response bandwidth.

VI. The Perturbed Master Equation and Solution

The next step is to solve the perturbed Master equation (13) . In [30], Haus and Mecozzi used soliton perturbation theory to solve the perturbed Master equation assuming zero chirp, so that soliton perturbation theory was valid, and ignored gain dynamics, by effectively setting the gain relaxation rate to infinity. (In [49], the gain dynamics were included for the case of a solid-state semiconductor laser.) Here, in order to derive the response bandwidth of the laser to a pump-power change, we do include gain dynamics of the Er fiber gain. Also, in order to treat the general case of nonzero chirp (which will almost inevitably be realized experimentally), we use a perturbative method based on moments of the electric field as used by Agrawal and coworkers in [33, 34] rather than invoking soliton perturbation theory. Finally, in order to identify the mechanisms responsible for control of the comb, we include a wide range of perturbations. We solve for the equations of motion for a soliton laser although the same approach could be applied to the stretched-pulse laser.

A. Solution for the Moments of the Perturbed Laser Pulse

The pulse energy, pulse arrival time, carrier frequency shift, chirp, and averaged phase can be defined for a general pulse as

$$\begin{aligned}
w(T) &= \int \left|a(t,T)\right|^2 dt, \\
t_A(T) &= \frac{1}{w(T)} \int t \left|a(t,T)\right|^2 dt, \\
p(T) &= \frac{i}{2w(T)} \int \left[a^*(t,T)\partial_t a(t,T) - a(t,T)\partial_t a^*(t,T)\right] dt, \\
C(T) &= \frac{-i}{w(T)} \int (t - t_A)\left[a^*(t,T)\partial_t a(t,T) - a(t,T)\partial_t a^*(t,T)\right] dt, \\
\bar{\theta}(T) &= \frac{-i}{2w(T)} \int \left(\ln(a) - \ln(a^*)\right)|a|^2 dt.
\end{aligned} \tag{50}$$

Let us denote the unperturbed pulse solution as $a_0(t,T)$. Substituting the full solution $a = a_0 + \Delta a$, the changes in the moments to first order in $\Delta a(t,T)$ are

$$\begin{aligned}
\Delta w(T) &= \mathrm{Re}\left[2\int a_0^* \Delta a dt\right], \\
\Delta t_A(T) &= \mathrm{Re}\left[\frac{2}{w_0}\int (t - t_{A,0}) a_0^* \Delta a dt\right], \\
\Delta p(T) &= \mathrm{Re}\left[\frac{2i}{w_0}\int \left[\partial_t a_0^*\right] \Delta a dt\right], \\
\Delta C(T) &= \mathrm{Re}\left[\frac{2i}{w_0}\int \left[(1 + iC_0) a_0^* + 2t\partial_t a_0^*\right] \Delta a \, dt\right], \\
\Delta\bar{\theta}(T) &= \mathrm{Re}\left[\frac{1}{w_0}\int \left\{-2\bar{\theta}_0 - i\left(1 + \ln\left(\frac{a_0}{a_0^*}\right)\right)\right\} a_0^* \Delta a \, dt\right].
\end{aligned} \tag{51}$$

In the calculation of both the average frequency and chirp, the intensity was assumed to fall off to zero at long times, thereby ignoring the contribution from continuum radiation. These equations all in the form

$$\Delta M(T) = \mathrm{Re}\left[\int f^*_{\_M}(t) \Delta a(t) dt\right], \tag{52}$$

where M can be w, t_A, p, C, or $\bar{\theta}$, and the projection functions $f^*_{\_M}$ are purposefully written using the notation of [30]. Here they are defined for a general pulse through Eq. (51) without recourse to soliton perturbation theory. Assuming the perturbation is independent of the phase, it is simplest to pull out the constant phase from the perturbation (corresponding to θ_0) so that $a = (a_C + \Delta a)\exp(i\theta_0)$. Substitution into (13) and dropping higher-order terms yields the perturbed Master equation,

$$T_r \partial_T \Delta a = \hat{N}(\Delta a) + V(a_C) \,, \tag{53}$$

where the nonlinear operator is

$$\begin{aligned}
\hat{N}(\Delta a) = &\left(g - l + (D_g - iD)\partial_t^2 - i\varphi_0\right)\Delta a + \\
&2(\gamma + i\delta)|a_C|^2 \Delta a + (\gamma + i\delta) a_C^{\,2} \Delta a^* .
\end{aligned} \tag{54}$$

Multiplication by the projection function, $f_{\_M}^*$, gives the basic equation of motion for the change in the general moment ΔM,

$$T_r \partial_T \Delta M(T) = \mathrm{Re}\left[\int f^*_{\_M} \hat{N}(\Delta a) dt + \int f^*_{\_M} V(a_c) dt\right]. \tag{55}$$

These equations are completely general. Introducing the specific solution, (14) and (15), the projection functions, defined by Eq. (51) and (52) become

$$
\begin{aligned}
f^*_{\_t}(x) &= \left(\frac{2}{w_0}\right)\tau_0 x a_c^*(x), \\
f^*_{\_p}(x) &= \left(\frac{-2i}{w_0}\right)\frac{(1+iC_0)}{\tau_0}\tanh(x)\, a_c^*(x), \\
f^*_{\_w}(x) &= 2a_c^*(x), \\
f^*_{\_C}(x) &= \frac{2i(1+iC_0)}{w_0}\big(1-2x\tanh(x)\big)a_c^*(x), \\
f^*_{\_\bar{\theta}}(x) &= \frac{-i}{w_0}\Big\{\big[1-2iC_0\left(1-\ln(2)\right)\big]- \\
&\qquad 2iC_0\ln\left(\operatorname{sech}(x)\right)\Big\}a_c^*(x),
\end{aligned}
\tag{56}
$$

where $x=t/\tau_0$ and using the fact that $t_{A,0} = p_0 = 0$. In the limit of zero chirp, the first three equations are identical to the projection operators derived from soliton perturbation theory. The final two projection operators do not appear in the previous work [30]. (Instead the projection operator for the phase is given as $f^*_{\_\theta} = f^*_{\_\bar{\theta}} - f^*_{\_C}/2$; however, for nonzero chirp the projection operator for the phase is correctly given as $f^*_{\_\theta} = f^*_{\_\bar{\theta}} - (1-\ln(2))\, f^*_{\_C}$.)

The second integral of Eq. (55) is easily carried out using the definitions (56) for a given perturbation. The first integral is not so straightforward, since it generally requires knowing something about the form of the perturbation Δa, which is heretofore unspecified. In both the moment approach [33, 34] or the soliton perturbation theory approach, the assumption is made that the pulse will retain the basic soliton shape (14) and the perturbation can be written in terms of derivatives of Eq. (14) as

$$
\Delta a = \sum \Delta M\, \partial_M a_0(t)\Big|_{\theta=0,p_0=0,t_A=0,w=w_0,C=C_0}, \tag{57}
$$

where the sum is over the five moments $M = \{\theta, p, w, t_A, C\}$. In writing (57), the further assumption is made that the pulse amplitude and width will not vary independently but will satisfy Eq. (18), leaving only a perturbation in pulse energy. This assumption is justified for the unchirped solution on the basis of the soliton area theorem. The chirp of the autosoliton is not expected to significantly alter this assumption. It can also be justified from a variational approach on the basis that none of the perturbations considered here, and in particular the perturbations to the length or gain, will affect the validity of Eq. (18).

B. Full Equations of Motion

Finally, solving the perturbation equation (55) with the seven perturbations (30), the assumed solution (57), and the definitions (16) and (26), the equations of motion are

$$
\begin{aligned}
\Delta T_r(T) &= k_{Tp}\Delta p(T) + k_{Tg}\Delta g(T) + k_{TL}\Delta L(T) \\
&\quad + \Delta T_{TOD} + \Delta T_{SS} + \Delta T_l, \\
T_r\partial_T \Delta p(T) &= k_{pp}\left(\Delta p(T) - \Delta p_R - \Delta p_{SS} - \Delta p_l\right), \\
\Delta\bar{\varphi}(T) &= k_{\varphi w}\Delta w(T) + k_{\varphi C}\Delta C(T) + k_{\varphi g}\Delta g(T) + k_{\varphi L}\Delta L(T), \\
T_r\partial_T \Delta C(T) &= k_{CC}\Delta C(T) + k_{Cg}\Delta g(T) + k_{CL}\Delta L(T), \\
T_r\partial_T \Delta w(T) &= k_{ww}\Delta w(T) + k_{wC}\Delta C(T) + k_{wg}\Delta g(T) + k_{wl}\Delta l(T), \\
T_r\partial_T \Delta g(T) &= k_{gg}\Delta g(T) + k_{gw}\Delta w(T) + k_{gT}\Delta T_r(T) + \\
&\quad k_{gC}\Delta C + k_{gPp}\Delta P_p(T),
\end{aligned}
\tag{58}
$$

where the dependence on T is given explicitly. After considerable algebra, the coupling constants k_{xx} are

$$
\begin{aligned}
&k_{Tp} = -2\left(D + CD_g\right);\; k_{Tg} = \Omega_g^{-1};\; k_{TL} = \beta_1^L; \\
&k_{pp} = -4\omega_{rms}^2 D_g; \\
&k_{\varphi w} = \frac{2\varphi_0}{w};\; k_{\varphi C} = -\frac{4C^2 D_g}{9\tau^2};\; k_{\varphi g} = \frac{2D_g C\left(2-C^2\right)}{9g\tau^2};\; k_{\varphi L} = \beta_0^L; \\
&k_{CC} = -\frac{2}{3\tau^2}\left(CD + 3D_g + 2C^2 D_g\right);\; k_{Cg} = -2C\kappa;\; k_{CL} = -2\omega_{rms}^2\beta_2^L; \\
&k_{ww} = -\eta;\; k_{wC} = -\frac{4wCD_g}{3\tau^2};\; k_{wg} = 2w(1-\kappa);\; k_{wl} = -2w \\
&k_{gg} = -\frac{T_r}{T_g};\; k_{gw} = -\frac{T_r g_{sw}}{T_g w};\; k_{gT} = \frac{g_{sT}}{T_g};\; k_{gC} = \frac{T_r g_{sC}}{T_g}; \\
&k_{gP_p} = \frac{T_r g_P}{T_g P_p}.
\end{aligned}
\tag{59}
$$

The identities (17), (18), and (21) were used to simplify some of the coupling constants, the subscript 0 was dropped on the unperturbed pulse parameters, and Eq. (48) was used for k_{ww}. The shifts in the round-trip time arise from TOD, self-steepening and frequency-dependent loss:

$$
\Delta T_{TOD} = \frac{\beta_3^{eff}}{2}\omega_{rms}^2,\quad \Delta T_{SS} = \mu\omega_0^{-1}\delta A_0^2,\quad \Delta T_l = Cl_\omega. \tag{60}
$$

The shift in the carrier frequency arise from the Raman self-frequency shift, the self-steepening effect and the frequency-dependent loss:

$$
\Delta p_R = \frac{\tau_R w\delta}{5\tau D_g\left(1+C^2\right)},\; \Delta p_{SS} = \frac{\mu\omega_0^{-1}Cw\delta}{5\tau D_g\left(1+C^2\right)},\; \Delta p_l = \frac{l_\omega}{2D_g}. \tag{61}
$$

The damping of the Raman self-frequency shift from the gain filtering is clearly evident if this term is compared to the frequency shift per round-trip of $(4w\delta\tau_R)/(15\tau^3)$, as calculated from the overlap integral of $f_{\_p}^*$ and V_{Raman} in (55).

Before analyzing Eq. (58), it is useful to simplify these equations further. First, the expression for $\Delta T_r(T)$ can be substituted into equation for $\Delta g(T)$ and the term $k_{Tg}k_{gT}\Delta g$ dropped since $k_{Tg}k_{gT} \sim (\Omega_g T_g)^{-1} \sim 10^{-10}$. Also the steady-state value of Δp can be substituted since it decouples from the other moments. The gain then depends only on the pulse energy, cavity length, pump power, and chirp.

The main simplification of these equations is through the removal of the dependence of the gain, pulse energy, and phase on the change in chirp, ΔC. (Note that the unperturbed value of the chirp, C, is still important as it enters into a number of the coupling constants.) First, let us drop the coupling of chirp to length since $k_{cL}\sim 0$. Then the chirp depends only on the gain. The two inequalities,

$$\left|\frac{k_{wC}k_{Cg}}{k_{CC}}\right|\times\left|k_{wg}\right|^{-1} << 1 \tag{62}$$

and

$$\left|\frac{k_{gC}k_{Cg}}{k_{CC}}\right|\times\left|\frac{k_{gw}k_{wg}}{k_{ww}}\right|^{-1} << 1, \tag{63}$$

can be shown to hold for any stable pulse (defined as $\tau_0^2 > 0$ from (17)) in which case the terms proportional to ΔC can be dropped from both the pulse energy and gain equations of motion. These inequalities are shown in Fig. 4 as a function of the ratio D/D_g for several different ratios of γ/δ, assuming $\eta\sim 1$. The dependence of the phase on ΔC can also be dropped since $k_{cL}\sim 0$ and

$$\left|\frac{k_{\varphi C}k_{Cg}}{k_{CC}}\right|\times\left|k_{\varphi g}\right|^{-1} << 1\,. \tag{64}$$

This inequality is also plotted in Fig. 4.

Figure 4 shows that the coupling of the parameters to the change in the chirp value is always negligible compared to the other couplings in (58) for *any* stable pulse (not just a chirp-free pulse). As a result, it is valid to drop terms proportional to ΔC from the equations of motion.

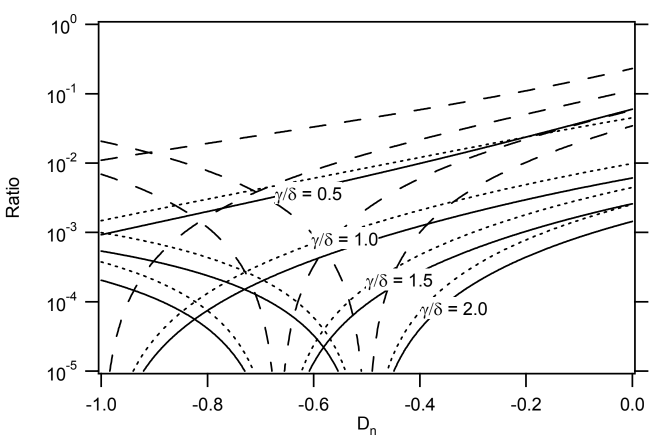

Fig. 4. The relative magnitude of the coupling to ΔC versus D/D_g for different ratios of SPM to SAM (γ/δ) with l-g=0.1. The ratio (62) is a solid line, the ratio (63) is dotted, and the ratio (64) is dashed. The largest values occur for γ/δ=0.5. Below that value (not shown), unstable pulses are obtained for low values of D/D_g.

C. Simplified Equations of Motion

With the above simplifications, the equation of motion for the chirp can be dropped and the remaining equations of motion can be rewritten in the significantly simpler form of

$$\begin{aligned}
&\Delta T_r = k_{Tp}\Delta p + k_{Tg}\Delta g + k_{TL}\Delta L + \Delta T_{TOD} + \Delta T_{SS} + \Delta T_l,\\
&\Delta\overline{\varphi} = k_{\varphi w}\Delta w + k_{\varphi g}\Delta g + k_{\varphi L}\Delta L,\\
&T_r\partial_T\Delta p = k_{pp}\left(\Delta p - \Delta p_R - \Delta p_{SS} - \Delta p_l\right),\\
&T_r\partial_T\Delta w = k_{ww}\Delta w + k_{wg}\Delta g + k_{wl}\Delta l,\\
&T_r\partial_T\Delta g = k_{gg}\Delta g + k_{gw}\Delta w + k_{gPp}\Delta P_p +\\
&\qquad k_{gT}\left(k_{Tp}\Delta p + k_{TL}\Delta L + \Delta T_{TOD} + \Delta T_{SS} + \Delta T_l\right),
\end{aligned} \tag{65}$$

where the dependence on T is implicit. In these simplified equations, the gain and pulse energy are coupled directly only to each other and are driven by source terms related to the cavity loss or pump power change. Ignoring the last source term for Δg and setting κ =0, these two coupled differential equations reduce to (1) derived in Section II.

The physical interpretation of each term is clear, given the definition of the coupling constants in Eq. (59). For example, the round trip time can change either directly from a change in the gain contribution to the group velocity, indirectly from a frequency shift combined with the net fiber dispersion or, trivially, from a length change. The additional shifts from self-steepening, loss and third-order dispersion are second order. Similarly, the gain relaxes at a rate given by the gain relaxation time T_g to a steady state value that depends the pump power and on gain saturation through either a change in pulse energy or repetition rate.

D. Steady- State Solution

The steady-state solution to the simplified equations of motion (65) in the absence of length or pump-power changes for f_r is,

$$\begin{aligned}
f_{r,DC}^{-1} &= \beta_1 + g\Omega_g^{-1} + \Delta T_{TOD} + \Delta T_{SS} + \Delta T_l\\
&\quad + k_{Tp}\left(\Delta p_{Raman} + \Delta p_{SS} + \Delta p_l\right),
\end{aligned} \tag{66}$$

after dropping second-order terms and for $CD_g<<D$. This equation is exactly (5) derived early. The magnitude of each of the contributions can be estimated using the typical values listed in Table 1. The first term is simply the standard transit time around the loop (in the absence of gain) and is ~20 ns. The second term accounts for the reduction in group velocity from the active gain medium and is ~300 fs. The remaining three terms give, respectively, the shift from third-order dispersion with a value of ~7 fs, the self-steepening term with a value of ~3 fs, and frequency-shift induced changes in the group velocity with a value of ~30 fs. (The Raman-induced frequency shift is ~ -2π×100 GHz.) Although the resonant contribution from the gain is by far the largest perturbation, its contribution to the pump-induced shift in the repetition frequency is somewhat reduced by the fact that the pump-induced change in gain is suppressed depending on the value of η as shown later.

The steady-state CEO frequency from (65), (29), and noting that $\Delta\varphi << \varphi_0 << \beta_0$ is

$$2\pi f_{ceo,DC} = -\omega_0\left(1-\frac{v_{gr}}{v_{ph}}\right) + f_r\varphi_0 - f_r^2\beta_0\left(f_{r,DC}^{-1} - \beta_1\right) \tag{67}$$

where the phase and group velocities are defined in the usual way as $v_{ph}=L(\beta_0/\omega_0)^{-1}$ and $v_{gr}=L\beta_1^{-1}$. This is equivalent to (6) given in Section II. The first term is the standard expression for the CEO frequency and is on the order of 1% to 2 % of the carrier frequency or $2\pi \times 2$ THz for a fiber laser. The second term is the SPM contribution and will be on the order of f_r. The value of the remaining terms is simply $f_r^2\beta_0$ times the values given above for the timing shifts. The self-steepening term, although small, is of some interest since it actually arises from the same underlying perturbation as the nonlinear phase term, namely the intensity dependence of the index of refraction. The sum of the two has a value of $f_r(\varphi_0 - \omega_0\Delta T_{SS}) = f_r(\varphi_0 - \mu A_0^2\delta\omega_0) = -f_r\mu\varphi_0/2$ for a chirp free pulse. In other words, the self-steepening term is larger than the nonlinear phase shift term and effectively reverses the sign dependence of the CEO frequency. This fact was previously pointed out for Ti:Sapphire laser-based combs [19].

E. Characteristic Time Constants

Before solving the equations, it is useful to write down the characteristic time constants describing the time-evolution of the five parameters: pulse energy, gain, phase, round-trip time and carrier frequency. The characteristic time constants for the coupled differential equations that describe the evolution of the gain and pulse energy determine the basic response of the system to pump-power changes. Defining $T_r/T_g = \varepsilon$ and, for the moment approximating $k_{wg}k_{gw} = \varepsilon$ (by setting $\kappa = 0$ and $2g_{sw}\sim 1$ as noted earlier), the eigenvalues describing the evolution of the gain and pulse energy from (65) are,

$$\frac{T_r}{\tau_{1,2}} = \frac{(\varepsilon+\eta) \pm \sqrt{(\varepsilon+\eta)^2 - 4(\eta\varepsilon+\varepsilon)}}{2} \qquad (68),$$

where $\mathrm{Re}(\tau_{1,2}) > 0$ is required for stability. If the radicand is negative, the eigenvalues are complex and relaxation oscillations occur in response to perturbations [35], otherwise the system is overdamped and only simple exponential decay occurs in response to perturbations. For the fiber laser, $T_r/T_g = \varepsilon \sim 10^{-4} << 1$ (see Table 1); this strong difference in timescales has important consequences for the laser behavior. To lowest order in ε, one finds the pulse is unstable for $\eta < -\varepsilon$. Stable relaxation oscillations occur over the narrow window $-\varepsilon < \eta < 2\varepsilon^{1/2}$ or $-10^{-4} < \eta < +2\times10^{-2}$. Stable exponential decay occurs for $\eta > 2\varepsilon^{1/2}$ or $\eta > +2\times10^{-2}$. From (49), $\eta \approx 0$ falls within the required range for stable relaxation oscillations for $g \approx 1$ and negligible nonlinear effects, as expected for cw operation. However, for mode-locked operation $l > g$ and saturation of the SAM is required to drive η to sufficiently high values for stability ($\eta > -10^{-4}$). Over any reasonable operating range, the SAM saturation term in (49) will vary significantly with pulse energy so that $\eta > 2\varepsilon^{1/2}\sim 2\times10^{-2}$ and the system is overdamped. Indeed previous experiments have explored the absence of relaxation oscillations in mode-locked fiber laser. [36]. In that limit, and removing the approximation $\kappa \approx 0$ and $2g_{sw}\sim 1$,

$$\begin{aligned} \frac{1}{\tau_1} &\approx \frac{1}{T_g}\left(1 + \frac{2g_{sw}(1-\kappa)}{\eta}\right) \\ \frac{1}{\tau_2} &\approx \frac{1}{T_r}\left(\eta - \frac{T_r}{T_g}\frac{2g_{sw}(1-\kappa)}{\eta}\right) \approx \frac{\eta}{T_r}, \end{aligned} \qquad (69)$$

to lowest order in $T_g/(\eta T_r)$.

The phase and round trip time have no characteristic time constant describing their relaxation to a steady state (a result of the fact that the laser pulse is permitted to vary in round-trip time and phase shift per pass). The carrier frequency relaxes with a characteristic time constant

$$\frac{1}{\tau_3} = 4\omega_{rms}^2 D_g f_r \qquad (70)$$

which was also derived in [30] for zero chirp.

F. Response of pulse energy and gain to pump power, cavity length and cavity loss

In the spirit of Fig. 2, we first solve the equations of motion for the change in the gain and pulse energy that results from a change in the pump power, cavity length or cavity loss. While there are no systems that current modulate the cavity loss to control the laser, this possibility remains attractive. In addition, any change in the cavity length almost assuredly results in a change in the cavity loss.

To solve for the fractional change in pulse energy and gain, we Fourier Transform the equations of motion (65). In terms of the frequency coordinate, Ω, the conjugate variable to T the solution can be expressed as

$$\begin{aligned} \frac{\Delta\tilde{w}}{w} &= H_{wP}\Delta\tilde{P} + H_{wL}\Delta\tilde{L} + H_{wl}\Delta\tilde{l} \\ \frac{\Delta\tilde{g}}{g} &= H_{gP}\Delta\tilde{P} + H_{gL}\Delta\tilde{L} + H_{gl}\Delta\tilde{l} \end{aligned} \qquad (71)$$

where we have suppressed the dependence of the Fourier transforms (denoted with a tilde) and the transfer functions, H_{xx}, on Ω. Using the definition of the coupling constants in (59), the transfer functions to first order are found to be

$$\begin{aligned} H_{gP}(\Omega) &= \left(\frac{g_P}{gP_P}\right)\left(\frac{\tau_1}{T_g}\right)\frac{1}{(1-i\Omega\tau_1)} \\ H_{wP}(\Omega) &= (1-\kappa)\left(\frac{2g_P}{P_P}\right)\left(\frac{\tau_1\tau_2}{T_gT_r}\right)\frac{1}{(1-i\Omega\tau_1)(1-i\Omega\tau_2)} \\ H_{gL}(\Omega) &= \left(\frac{2g_{sT}\beta_1^L}{gT_r}\right)\left(\frac{\tau_1}{T_g}\right)\frac{1}{(1-i\Omega\tau_1)} \\ H_{wL}(\Omega) &= (1-\kappa)\left(\frac{2g_{sT}\beta_1^L}{T_r}\right)\left(\frac{\tau_1\tau_2}{T_gT_r}\right)\frac{1}{(1-i\Omega\tau_1)(1-i\Omega\tau_2)} \\ H_{gl}(\Omega) &= \frac{2g_{sw}}{g}\left(\frac{\tau_1\tau_2}{T_gT_r}\right)\frac{1}{(1-i\Omega\tau_1)(1-i\Omega\tau_2)} \\ H_{wl}(\Omega) &= -2\left(\frac{\tau_1\tau_2}{T_gT_r}\right)\frac{(1-i\Omega T_g)}{(1-i\Omega\tau_1)(1-i\Omega\tau_2)} \end{aligned} \qquad (72)$$

All the transfer functions, with the interesting exception of

H_{wl} are dominated by the first pole at τ_l. In other words, this time constant sets the bandwidth of the response. The last two transfer functions describe the change in gain or pulse energy that results from a change in cavity length. Typical cavity length changes are quite small and the corresponding shifts in gain and energy are quite small. More importantly, experimentally there is a change in cavity loss associated with the length change that is expected to be much more significant. In other words, for a given cavity length change, one expects $\Delta l >> \Delta L/L$ because of unavoidable change in coupling or fiber loss. Therefore in the remainder of the analysis, we will assume $H_{gL} = H_{wL} = 0$ to avoid further increasing the number of transfer functions.

G. *Response of repetition frequency and offset frequency to a change in gain and pulse energy*

Again, in the spirit of Fig. 2, we continue to find the effects of the change in gain and pulse energy on the other pulse parameters. The equations of motion (65) are valid to first order; they do not include the potentially significant changes in the timing or frequency shifts given in (60) and (61) resulting from change in the gain or pulse energy. These effects must be included to model the system accurately. Fortunately, modifications to the loss, cavity length or pump power all modify existing terms in the Master equation; as a result the pulse retains it shape and changes adiabatically. With this assumption, it is simple to modify the equations of motion.

First, we consider the carrier frequency offset. Dropping the fixed offset to the carrier offset,

$$T_r \partial_T \Delta p = k_{pp}\left(\Delta p - \left(3\Delta p_R + 3\Delta p_{SS} + \Delta p_l\right)\frac{\Delta w}{w} + \left(\Delta p_R + \Delta p_{SS} + \Delta p_l\right)\frac{\Delta g}{g}\right) \tag{73}$$

To derive (73) we assumed the temporal width scales as $1/w$, spectral width as w and the power-broadened gain filtering D_g scales with g/w. (In fact, D_g will scale more weakly with w than inversely because of the form of P_s''.) Defining the transfer functions for the frequency offset as

$$\Delta\tilde{p}(\Omega) = H_{pP}(\Omega)\Delta\tilde{P}_P(\Omega) + H_{pl}(\Omega)\Delta\tilde{l}(\Omega) \tag{74}$$

we find,

$$H_{pP} = H_{wP}\frac{3\Delta p_R + 3\Delta p_{SS} + \Delta p_l}{1 - i\Omega\tau_3} - H_{gP}\frac{\Delta p_R + \Delta p_{SS} + \Delta p_l}{1 - i\Omega\tau_3} \tag{75}$$

where the dependence of the H_{xx} on Ω is suppressed. H_{pl} is defined as in (75) with the subscript substitution of $P \rightarrow l$.

Finally, we consider the repetition frequency and the offset frequency. From (28) and (29),

$$\begin{aligned} \frac{\Delta f_r}{f_r} &= -\frac{\Delta T_r}{T_r} \\ \frac{\Delta f_{ceo}}{f_r} &= \frac{\Delta\bar{\varphi}}{2\pi} + \frac{\beta_0}{2\pi}\frac{\Delta f_r}{f_r}, \end{aligned} \tag{76}$$

using the fact that $\beta_0 >> \bar{\varphi}_0$. The CEO frequency is normalized with respect to the repetition rate since this is consistent with typical frequency comb metrology experiments, where the CEO frequency is defined modulo the repetition frequency. Defining the transfer functions as

$$\begin{aligned} \frac{\Delta\tilde{f}_r(\Omega)}{f_r} &= H_{repP}\Delta\tilde{P}_P + H_{repL}\Delta\tilde{L} + H_{repl}\Delta\tilde{l}, \\ \frac{\Delta\tilde{f}_{ceo}(\Omega)}{f_r} &= H_{ceoP}\Delta\tilde{P}_P + H_{ceoL}\Delta\tilde{L} + H_{ceol}\Delta\tilde{l}. \end{aligned} \tag{77}$$

As with (73), we can modify the equations of motion for the round-trip time and the phase (65) to read,

$$\begin{aligned} \Delta T_r &= k_{Tp}\Delta p + \Omega_g^{-1}\Delta g + \beta_1^L \Delta L + 2\left(\Delta T_{SS} + \Delta T_{TOD}\right)\frac{\Delta w}{w}, \\ \Delta\bar{\varphi} &\approx 2\varphi_0\frac{\Delta w}{w} + \beta_0^L \Delta L, \end{aligned} \tag{78}$$

where we assume $gk_{\varphi g} << wk_{\varphi w}$. The transfer functions are then easily written down from (76), (72), (75) and (78) as

$$\begin{aligned} H_{repP} &= -f_r\left[k_{Tp}H_{pP} + \frac{g}{\Omega_g}H_{gP} + 2\left(\Delta T_{SS} + \Delta T_{TOD}\right)H_{wP}\right] \\ H_{repL} &= -\frac{\beta_1^L}{T_r} \\ H_{ceoP} &= \frac{\varphi_0}{\pi}H_{wP}(\Omega) + \frac{\beta_0}{2\pi}H_{repP} \\ H_{ceoL} &= \frac{\beta_0^L}{2\pi}\left(1 - \frac{v_{ph}^L v_{gr}}{v_{gr}^L v_{ph}}\right) \end{aligned} \tag{79}$$

where the dependence of the H_{xx} on Ω is suppressed. H_{repl} and H_{ceol} are defined as in (79) with the subscript change $P \rightarrow l$. In deriving the transfer function, H_{ceoL}, the phase and group velocities associated with the added propagation length ΔL are defined $v^L_{ph} = (\beta^L_0/\omega_0)^{-1}$ and $v^L_{gr} = \beta^{L\,-1}_1$. Also as discussed earlier, we drop any direct dependence of Δw or Δg on ΔL.

H. *Orthogonality of controls*

Typically, the pump power is used to control the CEO frequency, and the length change is used to control the repetition frequency. Ignoring any coupling between the cavity length change and the cavity loss, the orthogonality of the controls is given by the ratio

$$\frac{H_{ceoP}H_{repL}}{H_{ceoL}H_{repP}} = \left(1 - \frac{v_{ph}^L v_{gr}}{v_{gr}^L v_{ph}}\right)^{-1} \sim 50 - 300. \tag{80}$$

for non-zero values of H_{repP}. (If $H_{repP} = 0$, clearly even high numbers are reached.) In the anomalous, but possible situation that $H_{ceoP} = 0$ this orthogonality clearly breaks down due to the canceling of the first and second term. This situation can presumably be avoided by appropriate choice of laser parameters. The estimated value of 300 corresponds to a fiber stretcher where we estimate $v^L_{gr}/v^L_{ph} \approx 0.985$ compared to the estimate for the fiber laser of $v_{gr}/v_{ph} \approx 0.982$ (see Table 1). For an air gap, $v^L_{gr}/v^L_{ph} \approx 1.0$, yielding a value of ~50.

VII. Discussion

Both transfer functions relating to a cavity length change,

H_{repL} and H_{ceoL}, are directly proportional to the length change; their bandwidth is therefore limited only by the bandwidth of the transducer causing the length change. For a PZT-based fiber stretcher or free-space delay, this bandwidth will be low, but an electro-optic phase modulation device could provide much higher bandwidths. With the exception of H_{wl}, all the remaining transfer functions are limited in bandwidth by the same time constant, τ_1, given in (69). This time constant sets the 3-dB bandwidth of the phase-lock as $\Omega_{3dB}/2\pi = 1/(2\pi\tau_1)$. The typical spontaneous decay rate of Er gain is ~ 10 ms, which has a corresponding 3-dB bandwidth of only $1/(2\pi\times10\text{ms}) = 20$ Hz. Fortunately, $1/\tau_1$ is considerably larger than 10 ms for two reasons. First, the basic relaxation time of the gain is actually T_g, which includes stimulated transition rates from both the signal and pump beams. Typically, the saturation levels are quite high ($2P_s/P_{s,sat}$ ~ 100), giving a dramatic shortening to the relaxation time. Second, the coupling to the pulse energy further reduces τ_1 as given in (69) depending on the system-dependent value of η. Note that the increased bandwidth with decreasing η that can potentially provide a tighter phase-lock of the comb comes at the cost of decreased damping for energy fluctuations and therefore at the cost of increased noise.

The transfer functions of (79) are written in a form that is relative compact and is conducive to tracking the origin of the effects. However, it is useful to write them in a more explicit form. Keeping only the lowest order pole (τ_1), the response of the carrier envelope offset frequency to a change in pump power, ΔP, can be written exactly as in (10) except that $\eta \rightarrow \eta/(1-\kappa)$ in the limit g_P=1/2, CD_g<<D and using the fact that $\Delta\omega_{\Delta,NL} = -(\Delta p_R+\Delta p_{SS})$, and $\Delta\omega_{\Delta,L} = -\Delta p_l$. See Fig. 3 and the related discussion in Section II for identification of all the relevant terms.

The dominant effect of a length change, ΔL, on either repetition frequency or CEO frequency is clear from inspection of (79) – the added fiber length directly affects the repetition frequency through an increase in the round trip time and directly affects the offset frequency through a modification to the relative phase and group velocities. This simple picture is modified if there is a change in cavity loss associated with the length change. Assuming that there is a coupling between the cavity loss and length change given by,

$$\Delta l = k_{lL}\Delta L \tag{81}$$

the response of the carrier envelope offset frequency to a change in cavity length can be quite complicated. However, let us assume that the resonant gain contribution dominates H_{repl} and therefore H_{ceol}, in that case

$$\frac{\Delta\tilde{f}_{ceo}(\Omega)}{\Delta\tilde{L}f_r} \approx \frac{\beta_0^L}{2\pi}\left(1-\frac{\text{v}_{ph}^L\text{v}_{gr}}{\text{v}_{gr}^L\text{v}_{ph}}\right) - k_{lL}\frac{\beta_0 f_r}{2\pi(1+\eta)\Omega_g}\frac{1}{(1-i\Omega\tau_1)}. \tag{82}$$

From the discussion after (80), the first term in paranthesis is ~1/50 to 1/300 depending on whether a fiber or free-space delay line is used. For the fiber delay, the second loss-induced change in (82) dominates if $k_{lL} > \Omega_g/(300\ L\ f_r)$ ~ 50 ppm/μm. For an air delay, the crossover is at 300 ppm/μm.

The results here make a number of predictions regarding the behavior of fiber-laser frequency combs that can be compared with the reported measurements in the literature. (A more detailed comparison with experiment is underway.) The bandwidth of the system response to a pump-power change was first measured in [15] to be a few kilohertz for a stretched-pulse laser system. Similar values of 7 kHz and 4.5 kHz were found in [17] and [5], respectively. These values are consistent with (76). A relative response of $\Delta f_{ceo}/\Delta f_r$~ 5×10^6 reported in [5] is in reasonable agreement with the expected value from (80) of $H_{ceoP}(0)/H_{repP}(0)$ ~ $\beta_0/(2\pi)$ ~ 4×10^6. The response of f_r to a change in pump power has been reported as a few ppm [17], which is in reasonable agreement with estimates of $H_{repP}(0)$. Finally, [7, 16] report values for the response of f_{ceo} to a pump power change of 2 to 15 MHz/mW, which is in reasonable agreement with estimates of $H_{ceoP}(0)$. In general practical systems suffer from additional effects not included in this theory. For example, the gain is not Lorentzian and there will therefore be some resonant contribution to the phase velocity. As another example, cw breakthrough can occur and modify the results.

This analysis has not focused on the noise present on the frequency comb. Noise can either come directly from the laser, or result from the conversion and amplification of the intrinsic laser noise during the supercontinuum generation that is associated with detection of the CEO frequency [44, 50, 51]. The stronger the stabilization of all four important pulse parameters (the pulse energy, carrier frequency, repetition rate, and CEO frequency) the lower the expected comb noise. By controlling the pump power and cavity length, one can stabilize only two of these quantities, and naturally f_r and f_{ceo} are chosen. Additional stabilization of either the pulse energy or carrier frequency requires an additional control parameter. One intriguing additional control parameter is the modulation of the loss in the fiber laser. Control of the SAM action through the polarizers in the cavity is another possibility.

VIII. Conclusion

We have presented a complete perturbative treatment for the frequency comb output of a solitonic fiber laser. We have identified several effects including resonant gain contribution and third-order-dispersion that have not been discussed in detail for frequency combs. We have identified the cause of the spectral shifts in the laser output that can modify the comb spacing. We have given the correct numerical factors for the SPM and SS contributions in a fiber laser comb. We have related the bandwidth of the laser response to the stability of the laser pulse train. Finally, we have developed the response of the comb within one consistent framework where we can easily incorporate other perturbations. It would be interesting to extend this theory to other frequency combs such as the stretched-pulse fiber laser-base comb, or the Ti:Sapphire-laser based comb. It would also be interesting to compare these results to a full numerical treatment as discussed in [27, 28].

IX. References

[1] T. Udem, J. Reichert, R. Holzwarth, and T. W. Hänsch, "Absolute optical frequency measurement of the cesium D_1 line with a mode-locked laser," *Phys. Rev. Lett.*, vol. 82, pp. 3568-3571, 1999.

[2] S. A. Diddams, D. J. Jones, S. T. C. J. Ye, J. L. Hall, J. K. Ranka, R. S. Windeler, R. Holzwarth, T. Udem, and T. W. Hänsch, "Direct Link between Microwave and Optical Frequencies with a 300 THz Femtosecond Laser Comb," *Phys. Rev. Lett.*, vol. 84, pp. 5102, 2000.

[3] D. J. Jones, S. A. Diddams, J. K. Ranka, A. Stentz, R. S. Windeler, J. L. Hall, and S. T. Cundiff, "Carrier-envelope phase control of femtosecond mode-locked lasers and direct optical frequency synthesis," *Science*, vol. 288, pp. 635-9, 2000.

[4] T. Udem, R. Holzwarth, and T. W. Hänsch, "Optical Frequency Metrology," *Nature*, vol. 416, pp. 233, 2002.

[5] B. R. Washburn, S. A. Diddams, N. R. Newbury, J. W. Nicholson, M. F. Yan, and C. G. Jørgensen, "Phase-locked erbium-fiber-laser-based frequency comb in the near infrared," *Opt. Lett.*, vol. 29, pp. 250-252, 2004.

[6] T. R. Schibli, K. Minoshima, F.-L. Hong, H. Inaba, A. Onae, H. Matsumoto, I. Hartl, and M. N. Fermann, "Frequency metrology with a turnkey all-fiber system," *Opt. Lett.*, vol. 29, pp. 2467-2469, 2004.

[7] B. R. Washburn, R. Fox, N. R. Newbury, J. W. Nicholson, K. Feder, P. S. Westbrook, and C. G. Jørgensen, "Fiber-laser-based frequency comb with a tunable repetition rate," *Opt. Express*, vol. 12, pp. 4999-5004, 2004.

[8] F. Adler, K. Moutzouris, A. Leitenstorfer, H. Schnatz, B. Lipphardt, G. Grosche, and F. Tauser, "Phase-locked two-branch erbium-doped fiber laser system for long-term precision measurements of optical frequencies," *Opt. Express*, vol. 12, pp. 5872-5880, 2004.

[9] P. Kubina, P. Adel, F. Adler, G. Grosche, T. W. Hänsch, R. Holzwarth, A. Leitenstorfer, B. Lipphardt, and H. Schnatz, "Long term comparison of two fiber based frequency comb systems," *Opt. Express*, vol. 13, pp. 904-909, 2005.

[10] K. Kim, B. R. Washburn, G. Wilpers, C. W. Oates, L. Hollberg, N. R. Newbury, S. A. Diddams, J. W. Nicholson, and M. F. Yan, "Stabilized frequency comb with a self-referenced femtosecond Cr:forsterite laser," *Opt. Lett.*, vol. 30, pp. 932-935, 2005.

[11] J. W. Nicholson, M. F. Yan, P. Wisk, J. Fleming, F. DiMarcello, E. Monberg, A. Yablon, C. G. Jørgensen, and T. Veng, "All-fiber, octave-spanning supercontinuum," *Opt. Lett.*, vol. 28, pp. 643, 2003.

[12] J. W. Nicholson, P. S. Westbrook, K. S. Feder, and A. D. Yablon, "Supercontinuum generation in UV irradiated fibers," *Opt. Lett.*, vol. 29, pp. 2363-2365, 2004.

[13] F. Tauser, A. Leitenstorfer, and W. Zinth, "Amplified femtosecond pulses from an Er:fiber system: Nonlinear pulse shortening and self-referencing detection of the carrier-envelope phase evolution," *Opt. Express*, vol. 11, pp. 594-600, 2003.

[14] F.-L. Hong, K. Minoshima, A. Onae, H. Inaba, H. Takada, A. Hirai, H. Matsumoto, T. Sugiura, and M. Yoshida, "Broad-spectrum frequency comb generation and carrier-envelope offset frequency measurement by second harmonic generation of a mode-locked fiber laser," *Opt. Lett.*, vol. 28, pp. 1-3, 2003.

[15] J. Rauschenberger, T. M. Fortier, D. J. Jones, J. Ye, and S. T. Cundiff, "Control of the frequency comb from a mode-locked Erbium-doped fiber laser," *Opt. Express*, vol. 10, pp. 1404-1410, 2002.

[16] H. Hundertmark, D. Wandt, N. Haverkamp, and H. R. Telle, "Phase-locked carrier-envelope-offset frequency at 1560 nm," *Opt. Express*, vol. 12, pp. 770-775, 2004.

[17] N. Haverkamp, H. Hundertmark, C. Fallnich, and H. R. Telle, "Frequency stabilization of mode-locked Erbium fiber lasers using pump power control," *Appl. Phys. B*, vol. 78, pp. 321-324, 2004.

[18] L. Xu, C. Spielmann, A. Poppe, T. Brabec, F. Krausz, and T. W. Hänsch, "Route to phase control of ultrashort light pulses," *Opt. Lett.*, vol. 21, pp. 2008-2010, 1996.

[19] H. A. Haus and E. P. Ippen, "Group Velocity of Solitons," *Opt. Lett.*, vol. 26, pp. 1654-1656, 2001.

[20] P. M. Goorjian and S. T. Cundiff, "Nonlinear effects on the carrier-envelope phase," *Opt. Lett.*, vol. 29, pp. 1363-1365, 2004.

[21] M. J. Ablowitz, B. Ilan, and S. T. Cundiff, "Carrier-envelope phase slip of ultrashort dispersion-managed solitons," *Opt. Lett.*, vol. 29, pp. 1808-1810, 2004.

[22] F. W. Helbing, G. Steinmeyer, U. Keller, R. S. Windeler, J. Stenger, and H. R. Telle, "Carrier-envelope offset dynamics of mode-locked lasers," *Opt. Lett.*, vol. 27, pp. 194-196, 2002.

[23] F. W. Helbing, G. Steinmeyer, J. Stenger, H. R. Telle, and U. Keller, "Carrier-envelope offset dynamics and stabilization of femtosecond pulses," *Appl. Phys. B*, vol. 74, pp. S35-S42, 2002.

[24] J. Stenger and H. R. Telle, "Intensity-induced mode shift in a femtosecond laser by a change in the nonlinear index of refraction," *Opt. Lett.*, vol. 25, pp. 1553-1555, 2000.

[25] K. W. Holman, R. J. Jones, A. Marian, S. T. Cundiff, and J. Ye, "Intensity-related dynamics of femtosecond frequency combs," *Opt. Lett.*, vol. 28, pp. 851-853, 2003.

[26] K. W. Holman, R. J. Jones, A. Marian, S. T. Cundiff, and J. Ye, "Detailed studies and control of intensity-related dynamics of femtosecond frequency combs from mode-locked Ti:Sapphire lasers," *IEEE Journ. Select. Topics Quant. Electron.*, vol. 9, pp. 1018-1024, 2003.

[27] R. Paschotta, "Noise of mode-locked lasers (Part I): numerical model," *Appl. Phys. B*, vol. 79, pp. 153-162, 2004.

[28] R. Paschotta, "Noise of mode-locked lasers (Part II): timing jitter and other fluctuations," *Appl Phys. B*, vol. 79, pp. 163-173, 2004.

[29] L. E. Nelson, D. J. Jones, K. Tamura, H. A. Haus, and E. P. Ippen, "Ultrashort-pulse fiber ring lasers," *Appl. Phys. B*, vol. 65, pp. 277-94, 1997.

[30] H. A. Haus and A. Mecozzi, "Noise of Mode-Locked Lasers," *IEEE J. Quantum Electron.*, vol. 29, pp. 983-996, 1993.

[31] S. Namiki, C. X. Yu, and H. A. Haus, "Observation of nearly quantum-limited timing jitter in an all-fiber ring laser," *J. Opt. Soc. Am. B*, vol. 13, pp. 2817-23, 1996.

[32] H. A. Haus, J. G.Fujimoto, and E. P. Ippen, "Structures for additive pulse mode locking," *J. Opt. Soc. Am. B*, vol. 8, pp. 2068-2076, 1991.

[33] J. Santhanam and G. P. Agrawal, "Raman-induced timing jitter in dispersion-managed optical communication systems," *J. Sel. Topics in Quant. Electron.*, vol. 8, pp. 632-639, 2002.

[34] J. Santhanam and G. P. Agrawal, "Reduced timing jitter in a dispersion managed light-wave system through parametric amplification," *J. Opt. Soc. Am. B*, vol. 20, pp. 284-291, 2003.

[35] A. Schlatter, S. C. Zeller, R. Grange, R. Paschotta, and U. Keller, "Pulse-energy dynamics of passively mode-locked solid-state lasers above the Q-switching threshold," *J. Opt. Soc. Am. B*, vol. 21, pp. 1469-1478, 2004.

[36] S. Namiki, E. P. Ippen, H. A. Haus, and K. Tamura, "Relaxation oscillation behavior in polarization additive pulse mode-locked fiber ring lasers," *Appl. Phys. Lett.*, vol. 69, pp. 3969-3971, 1996.

[37] J. D. Moores, "On the Ginzburg-Landau laser mode-locking model with fifth-order saturable absorber term," *Opt. Commun.*, vol. 96, pp. 65-70, 1993.

[38] F. Krausz, M. E. Fermann, T. Brabec, P. F. Curley, M. Hofer, M. H. Ober, C. Spielmann, E. Wintner, and A. J. Schmidt, "Femtosecond solid-state lasers," *IEEE J. of Quantum Electron.*, vol. 28, pp. 2097-2121, 1992.

[39] J. Nicholson, " Approximate dispersion values for the Er fiber provided by Jeff Nicholson at OFS.," 2004.

[40] G. P. Agrawal, *Nonlinear Fiber Optics*, 3rd ed. San Francisco: Academic Press, 2001.

[41] O. E. Martinez, R. L. Fork, and J. P. Gordon, "Theory of passively mode-locked lasers for the case of a nonlinear complex propagation coefficient," *J. Opt. Soc. Am. B*, vol. 2, pp. 753, 1985.

[42] H. A. Haus, K. Tamura, L. E. Nelson, and E. P. Ippen, "Stretched-Pulse Additive Pulse Mode-Locking in Fiber Ring Lasers: Theory and Experiment," *IEEE J of Quantum Electronics*, vol. 31, pp. 591-598, 1995.

[43] S. T. Cundiff, J. Ye, and J. L. Hall, "Optical frequency synthesis based on modelocked lasers," *Rev. Science Instrum.*, 2001.

[44] B. R. Washburn and N. R. Newbury, "Phase, timing, and amplitude noise on supercontinua generated in microstructure fiber," *Opt. Express*, vol. 12, pp. 2166, 2004.
[45] E. Desurvire, *Erbium-Doped Fiber Amplifiers*. New York: Wiley, 1994.
[46] Y. Kodama and A. Hasegawa, "Nonlinear pulse propagation in a monomode dielectric guide," *IEEE J. Quantum Electron.*, vol. QE-23, pp. 510-24, 1987.
[47] K. J. Blow and D. Wood, "Theoretical description of transient stimulated Raman scattering in optical fibers," *IEEE J. Quantum. Electron.*, vol. 25, pp. 2665-73, 1989.
[48] N. R. Newbury, "Pump-wavelength dependence of Raman gain in single-mode optical fibers," *J of Lightwave Tech*, vol. 21, pp. 3364-3373, 2003.
[49] L. A. Jiang, M. E. Grein, H. A. Haus, and E. P. Ippen, "Noise of Mode-Locked Semiconductor Lasers," *IEEE J. Sel. Topics in Quantum Electron.*, vol. 7, pp. 159-167, 2001.
[50] N. R. Newbury, B. R. Washburn, K. L. Corwin, and R. S. Windeler, "Noise amplification during supercontinuum generation in microstructure fiber," *Opt. Lett.*, vol. 28, pp. 944, 2002.
[51] T. M. Fortier, J. Ye, S. T. Cundiff, and R. S. Windeler, "Nonlinear phase noise generated in air-silica microstructure fiber and its effect on carrier-envelope phase," *Opt. Lett.*, vol. 27, pp. 445-447, 2002.

Nathan R. Newbury (M'03-SM'05) received his BA, magna cum laude in physics from Williams College in 1986 and his Ph.D in physics from Princeton University in 1992.

At Princeton, he conducted research on laser-polarized alkali atoms and spin-exchange with noble gas atoms as well as co-authoring *Princeton Problems in Physics*. From 1993-1995, he was a postdoctoral associate at JILA in Boulder CO working toward the production of Bose-Einstein condensation in laser cooled atoms. He then joined MIT Lincoln Laboratory in 1996, where he worked in a number of areas including bio-detection and LIDAR. In 2001, he joined the Fiber Optics and Components group in the Optoelectronics Division at NIST, where he is currently working in the area of nonlinear fiber optics and frequency combs.

Dr. Newbury is a member of the Optical Society of America and the IEEE Lasers and Electro-Optics (LEOS) Society.

Brian R. Washburn (S'98-M'03) was born in Racine, WI, in 1972. He received the B.S. degree (*summa cum laude*) from the University of Wisconsin-Parkside in 1994 and his Ph. D. degree in physics at the Georgia Institute of Technology, Atlanta GA. His research included ultrafast semiconductor spectroscopy, nonlinear fiber optics, photonic crystal fibers, and ultrashort pulse characterization.

In 2002, he started a postdoctoral position at the National Institute of Standards and Technology, Boulder CO studying noise properties of supercontinuum generation. This work has led to the development of a phase-locked all-fiber-based supercontinuum source for frequency metrology. In August 2005 he will join Kansas State University, Manhattan KS, as an assistant professor in physics. His current research interests include ultrafast semiconductor spectroscopy, nonlinear fiber optics, and mode-locked fiber lasers.

Dr. Washburn is a member of the Optical Society of America, the IEEE Lasers and Electro-Optics (LEOS) Society, and the American Physical Society.