# A DELTA-DISTRIBUTION DERIVATION OF THE ELECTRIC FIELD IN THE SOURCE REGION

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The expression of the electric field in terms of the electric dyadic Green's function requires, in the source region, the specification of a limiting "principal volume" $V_\delta$ and the associated source dyadic $\bar{\bar{L}}_\delta$ [1-5]; in particular

$$\bar{E}(\bar{r}) = i\omega\mu_0 \lim_{\delta\to 0} \int_{V_J - V_\delta} \bar{\bar{G}}_e(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')dV' + \frac{\bar{\bar{L}}_\delta \cdot \bar{J}(\bar{r})}{i\omega\varepsilon_0} , \qquad (1a)$$

where

$$\bar{\bar{L}}_\delta \equiv \frac{1}{4\pi} \int_{S_\delta} \frac{\hat{n}\,\hat{e}_R}{R^2}\, dS , \qquad (1b)$$

$S_\delta$ being the surface of the principal volume $V_\delta$ [3]. The source dyadic $\bar{\bar{L}}_\delta$ is a depolarizing dyadic which expresses mathematically the physics that the internal electric field of a volume element of current density (or polarization) remains finite as the size of the element shrinks to zero. Moreover, in the limit as the volume element shrinks to zero, the value of this internal electric field depends on but only on the shape of and position within the volume element [3].

Equations (1) have been derived rigorously from standard theorems of vector and dyadic calculus [3], and through regularization of the singular part of the dyadic Green's function operator [5]. However, the commonly used (nonrigorous) delta-function method, when applied to the problem of finding the electric field in the source region from the dyadic Green's function, fails to reveal the principal volume dependence and its corresponding depolarizing dyadic $\bar{\bar{L}}_\delta$. This failure has been traced to an invalid interchange of operators in the commonly used delta-function method making it inapplicable in the source region [3b, sec. III; 6, p. 278]. The purpose of this paper is to use the delta function rigorously through a delta-distribution approach to demonstrate that equations (1) result.

The fundamental reason for the above-mentioned invalid interchange of operators is an ambiguous dyadic delta function existing at $\bar{r} = \bar{r}'$ in the solution $\bar{\bar{G}}_e(\bar{r},\bar{r}')$ to

$$\nabla \times \nabla \times \bar{\bar{G}}_e - k^2 \bar{\bar{G}}_e = \delta(\bar{r}-\bar{r}')\, \bar{\bar{I}} . \qquad (2)$$

Physically, this ambiguity can be understood from the fact, explained above and contained in (1), that the internal field $\bar{\bar{G}}_e$ at

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$\bar{r}=\bar{r}'$ produced by the delta-function source on the right side of (2) also equals a dyadic delta function, but its dyadic cannot be determined uniquely unless the limiting sequence used to define the delta function on the right side of (2) is specified. In other words, the sifting property alone of the delta function on the right side of (2) is not sufficient to define uniquely the dyadic delta function in $\bar{G}_e$ at the source point $\bar{r}=\bar{r}'$. In the following delta-distribution derivation of the electric field in the source region, this ambiguity in $\bar{G}_e$ at $\bar{r}=\bar{r}'$ is removed by defining the delta function rigorously in terms of delta sequences--yielding the results (1) directly.

Consider the equation for the electric field $\bar{E}$ with source $\bar{J}$ written in linear operator notation ($D = \nabla x \nabla x - k^2$);

$$D[\bar{E}] = i\omega\mu_0 \bar{J} \tag{3}$$

The equation (3) holds in an interior or exterior region with closed finite surfaces allowed on which the tangential component of $\bar{E}$ or $\bar{H}$ is specified zero; and for exterior regions the fields must satisfy the radiation condition. Except at the resonant frequencies inside cavities, (3) has a unique solution, i.e., D has a unique inverse linear operator $D^{-1}$ such that

$$\bar{E}(\bar{r}) = i\omega\mu_0 D^{-1}(\bar{r}) \, [\bar{J}(\bar{r}_0)] \, , \tag{4}$$

where $D^{-1}$ operates on $\bar{J}$ with respect to the $\bar{r}_0$ coordinates.

Using the sequential approach to the definition of distributions, we can express $\bar{J}(\bar{r}_0)$ in terms of any delta-sequence of well-behaved functions $\delta_n$ [7]:

$$\bar{J}(\bar{r}_0) = \lim_{n\to\infty} \int_{V_J} \delta_n(\bar{r}_0-\bar{r}') \, \bar{\bar{I}}\cdot\bar{J}(\bar{r}') \, dV' \, . \tag{5}$$

Of course, (5) is merely a rigorous definition of the "sifting property" of the Dirac delta distribution. In most engineering practice the limit in (5) is brought inside the integral and $\delta_n$ is replaced by the symbolic function $\delta(\bar{r}_0-\bar{r}')$ even though the limit at $\bar{r}=\bar{r}'$ of $\delta_n$ is infinity and the resulting integral does not exist in the Riemann or Lebesgue sense. This common practice provides concise, symbolic notation but, as any number of texts on distribution theory warn, when operations with expressions involving distributions are performed, rigorous defintions such as (5) must replace the convenient shorthand notation. And only after a rigorous derivation is performed is it permissible to re-express, if possible, the final result in more concise symbolic notation (e.g., see (14) below).

Insert $\bar{J}(\bar{r}_0)$ from (5) into (4), and bring $D^{-1}$ inside the integral to get

$$\bar{E}(\bar{r}) = i\omega\mu_o \lim_{n\to\infty} \int_{V_J} \bar{\bar{G}}_n(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')dV' \tag{6}$$

where $$\bar{\bar{G}}_n \equiv D^{-1}(\bar{r}) \, [\delta_n(\bar{r}_o-\bar{r}')\bar{\bar{I}}] \text{ , i.e.,}$$

$$\nabla x \nabla x \bar{\bar{G}}_n - k^2\bar{\bar{G}}_n = \delta_n(\bar{r}-\bar{r}')\bar{\bar{I}}. \tag{7}$$

(By writing out $D^{-1}$ explicitly for the free space problem, it can be proven that bringing $D^{-1}$ under the integral sign is a valid interchange.) Two important features of (6) and (7) are that 1) the nonsingular functions $\delta_n$ allow a unique solution $\bar{\bar{G}}_n$ to (7) at all values of $\bar{r}$, and 2) the limit in (6) as $n\to\infty$ must remain outside the integral.

The last part of our derivation is to show that (6) and (7) reduce to (1). This can be proven for an arbitrary delta sequence; however, the analysis simplifies if we first choose a set of spherical delta-sequence functions $\delta_n$. Specifically, choose $\delta_n(\bar{r})$ as follows:

$$\delta_n(\bar{r}) = \begin{cases} \frac{3n^3}{4\pi} & r < \frac{1}{n} \\ 0 & r > \frac{1}{n} \end{cases} \tag{8}$$

Then for large n the solution to (7) approaches

$$\bar{\bar{G}}_n(\bar{r},\bar{r}') = \begin{cases} \frac{-n^3\bar{\bar{I}}}{4\pi k^2} & |\bar{r}-\bar{r}'| < \frac{1}{n} \\ \bar{\bar{G}}_e(\bar{r},\bar{r}') & |\bar{r}-\bar{r}'| > \frac{1}{n} \end{cases} , \tag{9}$$

and (6) becomes upon substitution from (9),

$$\bar{E}(\bar{r}) = i\omega\mu_o \lim_{n\to\infty} \left[ \int_{V_J-V_n} \bar{\bar{G}}_e \cdot \bar{J}(\bar{r}')dV' - \frac{\bar{J}(\bar{r})}{3k^2} \right] , \tag{10}$$

where $V_n$ is a sphere of radius 1/n centered at the point $\bar{r}'=\bar{r}$. Define $\delta$ as the diameter of the sphere ($\delta=2/n$), and (10) can be rewritten

$$\bar{E}(\bar{r}) = i\omega\mu_o \lim_{\delta\to 0} \int_{V_J-V_\delta} \bar{\bar{G}}_e \cdot \bar{J}(\bar{r}')dV' + \frac{\bar{J}(\bar{r})}{3i\omega\varepsilon_o} . \tag{11}$$

To convert (11) to the more general formula (1a) holding for an arbitrary principal volume $V_\delta$, let $V_\delta^{\text{sphere}}$ circumscribe $V_\delta$, and invoke the following formula easily provable from a standard dyadic integral formula [2b, App. 3, eq. 52]:

$$\lim_{\delta \to 0} \int_{V_{sphere} - V_\delta} \bar{\bar{G}}_e dV' = (\bar{\bar{I}}_\delta - \frac{\bar{\bar{I}}}{3})/k^2 \ . \qquad (12)$$

Q.E.D.

Note that if the limit and integral in (6) were interchanged, as in the commonly-used nonrigorous delta-function method, one would have missed entirely the subtle, yet all-important principal-volume behavior of $\bar{\bar{G}}_n$, and thereby obtain the following familiar expression ill-defined in the source region,

$$\bar{E}(\bar{r}) = i\omega\mu_o \int_{V_J} \bar{\bar{G}}_e(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')dV' \ . \qquad (13)$$

Finally, to conform to the notation Van Bladel introduced [2a, App.], we can express symbolically a dyadic Green's function $\bar{\bar{G}}_e'$ to be used in the sense of (1a),

$$\bar{\bar{G}}_e' = PV_\delta[\bar{\bar{G}}_e] - \frac{\bar{\bar{I}}_\delta \ \delta(\bar{r}-\bar{r}')}{k^2} \ . \qquad (14)$$

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