Compensation of Markov Estimator Errors in Time-Jittered Sampling of Nonmonotonic Signals

Guangqiu Tong and T. Michael Souders

Abstract—**The so-called Markov estimator is sometimes used to estimate signals from their time-jittered samples [1]. The estimates are unbiased for monotonic signals, but exhibit errors in regions of nonmonotonicity. A method of compensation is presented to reduce this error. It requires a knowledge of the PDF of the time jitter, and a proposed method for determining the PDF based on the generalized Markov estimator has been verified through simulations. The performance of the compensation approach is presented for four different nonmonotonic waveforms.**

I. Introduction

Timing jitter generally causes systematic errors in the amplitude estimates of sampled waveforms. This is true for both real- and equivalent-time sampling processes. Consider the case of equivalent-time sampling. Multiple samples are often taken at each nominal sample time and, to reduce the effects of noise, the mean of each sample set is taken as the estimated value at the corresponding sample time. If the noise has zero mean and is strictly additive, then the resulting estimates are unbiased. However, if the noise is the result of timing jitter, then the mean generally gives rise to systematic errors in the estimates [1]. If, on the other hand, the median is used rather than the mean to compute the estimates, then a bias occurs only for signals which are nonmonotonic. This is illustrated in Fig. 1 for a sinusoidal waveform. The figure shows the actual waveform and the error that results when the waveform is sampled with 10° (1σ) time jitter and the median is used as the estimator. Note that the errors only occur in nonmonotonic regions.

It has been shown in [2] that the median of a large sample set can be computed very efficiently by operating on the samples with a Markov process. This so-called Markov estimator is naturally implemented in a class of equivalent-time sampling systems called sampling voltage tracker (SVT) systems [1], [3]. In this paper, a method of compensation is presented which can be used to reduce the bias associated with the Markov estimator (or the median) that occurs in the regions where the signal is nonmonotonic. In addition, a generalized form of the Markov estimator proposed by Souders *et al.* as a means for computing any percentile of a statistical sample, has been verified by simulation. The algorithm was proposed without supporting proof in [1]. Use of this algorithm provides a very convenient means of computing the cumulative distribution function (CDF) of the jitter which must be known in order to compute the compensation function.

II. The Markov Estimator

The Markov estimator is based on a Markov process which operates on the statistical sample set. Referring to Fig. 2, the Markov process works as follows. The samples, $y_t(k)$, all taken at nominal sample time, t, on the input waveform, $Y(t)$, are compared in turn with a reference value, $y_t'(k)$. This reference value is incremented

Manuscript received December 6, 1990; revised May 19, 1993.

G. Tong is with the National Institute of Metrology, Beijing, Republic of China.

T. Michael Souders is with the Electricity Division, National Institute of Standards and Technology, Gaithersburg, MD 20899.

IEEE Log Number 9211482.

3.00 © 1993 IEEE

If the probability distribution of the time jitter is known, then the equivalent time shifts can be calculated from (3)–(6), and the Markov estimates, $Y'(t)$, can be compensated by an amount $\Delta Y(t)$ to yield an improved estimate, $Y''(t)$, as follows:

$$Y''(t) = Y'(t) + \Delta Y(t) \qquad (t_1 < t < t_2) \tag{7}$$

$$\Delta Y(t) = Y'(t + t_d(t)) - Y'(t) \qquad (t_1 < t < t_p) \tag{8}$$

$$\Delta Y(t) = Y'(t - t_d(t)) - Y'(t) \qquad (t_p < t < t_2). \tag{9}$$

As a practical matter, the critical points, t_p, which appear in (3)–(6), must be determined from the Markov estimates, which will still contain some residual noise. The effects of the noise on the critical point determination can be minimized by using a multipoint discrete difference equation.

IV. The Cumulative Distribution of the Jitter

In order to calculate the compensation function, the cumulative distribution function of the jitter must be known. Equations (3)–(6) each involve the calculation of the cumulative distribution between prescribed limits, based on the PDF; a direct determination of the CDF simplifies each of these computations. Probably the most direct and accurate method of determining the effective jitter distribution is to use the sampling process itself, under the conditions of actual use. The specific implementation depends on the type of sampling system used. One method suggested by Gans [4] is to input a fast-slewing waveform with known rate of change, and to accumulate a set of samples all taken at the same nominal sample time. The PDF and CDF are readily calculated from the sample set, provided that additive noise is negligible (Additive noise can be measured separately by inputting a dc or slowly varying signal.). However, Gans' method is not feasible for implementation in SVT systems because the Markov estimate is computed online and the samples are not stored. For these systems, the generalized Markov estimator proposed in [1] for finding any percentile of a distribution, provides an attractive approach. This generalized form is implemented by setting the ratio of increment amount, $\delta\uparrow$, to decrement amount, $\delta\downarrow$, equal to $p/(1-p)$, where p is the percentile desired. This approach can be used as follows to estimate the cumulative distribution of the time jitter, $P_c(\Delta t)$:

1) Set the nominal sampling time to coincide with a fast-slewing point on the input waveform in a monotonic region. (Such a point can be found by examining the normal Markov estimate, $Y'(t)$, of the input waveform which will be unbiased in the monotonic regions.)
2) Choose a set of percentiles, $P_{cj}(\Delta t)$, spanning the range $\{0, 1\}$, at which the measurements will be made. Set the increment and decrement amounts by

$$\frac{\delta\uparrow}{\delta\downarrow} = \frac{P_{cj}(\Delta t)}{(1 - P_{cj}(\Delta t))} \tag{10}$$

and determine the corresponding generalized Markov estimate, $Y'_j(t)$, for each. The $\delta\uparrow$ and $\delta\downarrow$ can be determined by the ratio and the required measurement accuracy.
3) The jth value of Δt corresponding to the jth percentile is computed as

$$\Delta t_j = \frac{Y'_j(t) - Y'(t)}{\dot{Y}(t)} \tag{11}$$

where $\dot{Y}(t)$ is the derivative of $Y(t)$, obtained by numerical differentiation of $Y'(t)$.

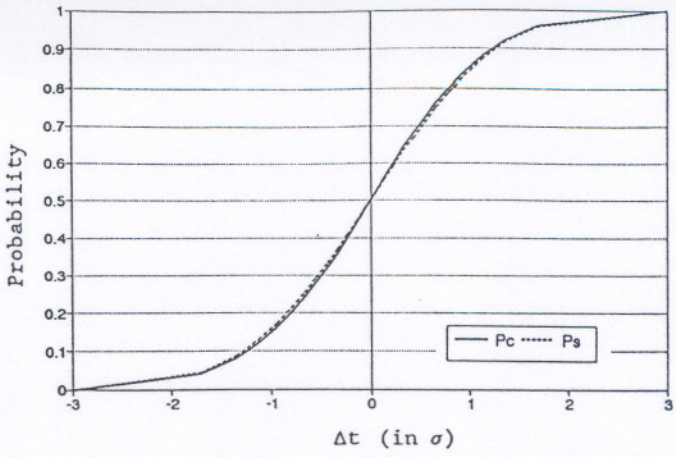

Fig. 4. Plot of the CDF of Gaussian time jitter, computed using the generalized Markov estimator (solid), and ideal (dashed), with a sine wave input signal. The simulated jitter was 10° (1σ).

TABLE I
Results of Tests of Percentile Method for Computing CDF

P_c	P_s	$P_s - P_c$ (error)
0.0400	0.0436	0.0036
0.0800	0.0869	0.0069
0.1200	0.1292	0.0092
0.1600	0.1711	0.0111
0.2000	0.2119	0.0119
0.2400	0.2546	0.0146
0.2800	0.2926	0.0126
0.3200	0.3336	0.0136
0.3600	0.3707	0.0107
0.4000	0.4077	0.0077
0.4400	0.4463	0.0063
0.4800	0.4814	0.0014
0.5200	0.5181	0.0019
0.5600	0.5542	0.0059
0.6000	0.5904	−0.0096
0.6400	0.6302	−0.0098
0.6800	0.6676	−0.0124
0.7200	0.7088	−0.0122
0.7600	0.7473	−0.0127
0.8000	0.7881	−0.0119
0.8400	0.8296	−0.0104
0.8800	0.8714	−0.0086
0.9200	0.9133	−0.0067
0.9600	0.9554	−0.0046

The ordered pairs, $[P_{cj}, \Delta t_j]$, define the CDF of the jitter, assuming that the additive noise is negligible. As in Gans' method, the additive noise can similarly be estimated by performing the same test on a dc signal, or on a portion of the input waveform that is only slowly changing.

Since this generalized Markov estimator was offered only as a conjecture [1], it was first verified through simulations. The test used a sine wave as the input signal, and a Gaussian probability distribution for the jitter, whose CDF is given by

$$P_s(\Delta t_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Delta t_j} e^{-\Delta t^2/2}\, d(\Delta t). \tag{12}$$

The simulation results, i.e., the estimated CDF, are plotted in Fig. 4, and are compared with $P_s(\Delta t_j)$ from (10) above in Table I. Each percentile value was calculated using 360 000 samples. The max-

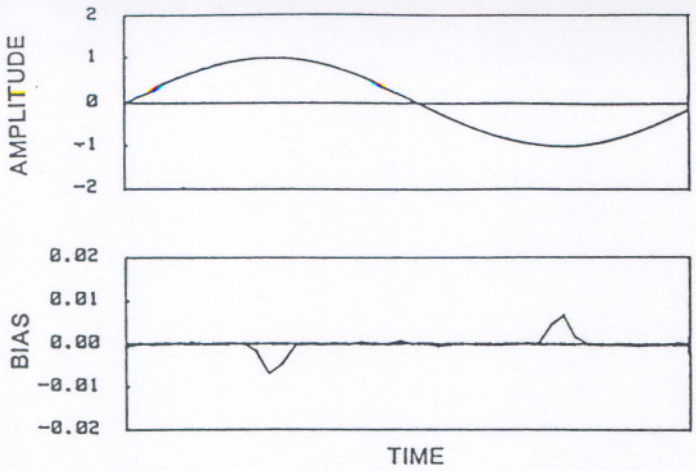

Fig. 1. Simulation results using the Markov estimator for a sine wave sampled with Gaussian jitter where $\sigma = 10°$. (top) The original waveform. (bottom) The bias using Markov estimator, with $\delta = 10^{-4}$.

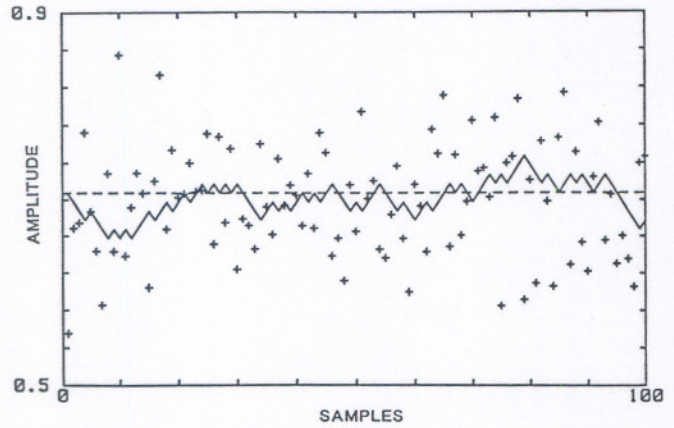

Fig. 2. Illustration of the Markov process. The solid curve is the Markov chain response, $y_t'(k)$, to the samples, $y_t(k)$, designated by (+). The samples were taken from a sine wave at 45°, with Gaussian jitter where $\sigma = 36°$.

or decremented by a small, fixed amount, δ, after each sample, depending on the state of the previous comparison. Under these conditions, $y_t'(k)$ follows a random walk described by a Markov chain, defined as

$$\left.\begin{aligned} y_t'(k+1) &= y_t'(k) + \delta \quad \text{for} \quad y_t'(k) \le y_t(k) \\ y_t'(k+1) &= y_t'(k) - \delta \quad \text{for} \quad y_t'(k) > y_t(k) \end{aligned}\right\}. \tag{1}$$

The Markov estimate, $Y'(t)$, is the mean value of the Markov chain, $y_t'(k)$:

$$Y'(t) = \frac{1}{n+1} \sum_{k=0}^{n} y_t'(k). \tag{2}$$

As noted above and shown in [2], $Y'(t)$ asymptotically converges to the population median of $y_t(k)$, as $n \to \infty$ and then $\delta \to 0$.

III. Analysis and Compensation

The source of the errors produced by the Markov estimator in the presence of time jitter is illustrated in Fig. 3. A nonmonotonic region of the input signal, $Y(t)$, is shown, sampled at time t. The sampling is subject to time jitter having a probability density function (PDF) given by $p(\Delta t)$ with a median of zero. For the following

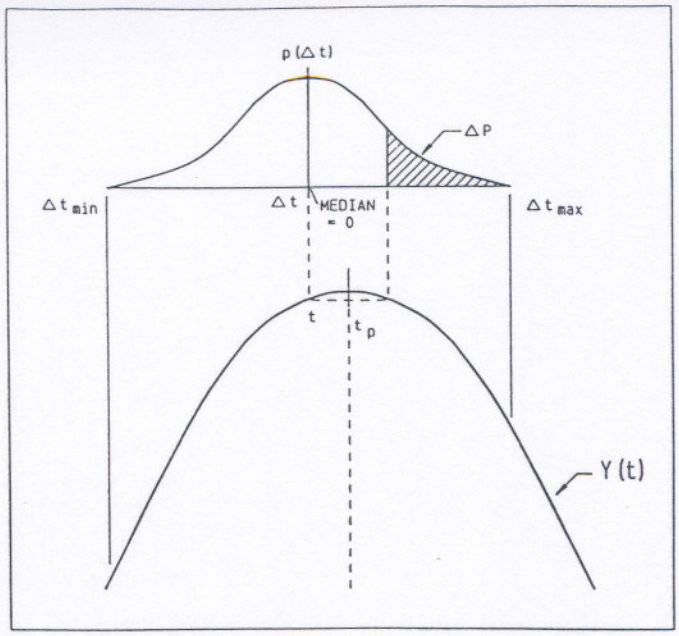

Fig. 3. Illustration of the source of bias associated with the Markov estimator in a nonmonotonic region of the input waveform, $Y(t)$, with time jitter PDF given by $p(\Delta t)$. Point t_p is the stationary point where $d(Y(t))/dt = 0$.

analysis, it will be assumed that $Y(t)$ is symmetric about time t_p (the stationary point where $dY(t)/dt = 0$) over the region defined by $t_p \pm \Delta t_{pk}/2$, where Δt_{pk} is the larger of $|\Delta t_{\max}|$ or $|\Delta t_{\min}|$, and that all stationary points are separated by at least Δt_{pk}. It can be seen from the figure that, if point t is at least $\Delta t_{pk}/2$ away from t_p, then half of the samples, $Y(t + \Delta t)$, will be greater than $Y(t)$, and half will be less. Therefore, the Markov estimate, $Y'(t)$, defined as the median of samples $Y(t + \Delta t)$, will converge to $Y(t)$ in the limit, i.e., it will be asymptotically unbiased. However, when the nominal sample time, t, is within $\Delta t_{\max}/2$ on the left side, or $\Delta t_{\min}/2$ on the right side of time t_p, some of the samples taken on one side of the probability distribution (designated by ΔP in the figure) will be greater that $Y(t)$ and some will be less. Thus the median, $Y'(t)$, of the samples, i.e., the Markov estimate, will not be an unbiased estimate of $Y(t)$.

The fraction of the samples, $y_t(k)$, that contribute to the bias is given by

$$\Delta P = \int_{2(t_p - t)}^{\Delta t_{\max}} p(\Delta t)\, d(\Delta t) \qquad (t_1 < t < t_p) \tag{3}$$

for $t < t_p$, where t_1 is $t_p - \Delta t_{pk}/2$ and by

$$\Delta P = \int_{\Delta t_{\min}}^{2(t_p - t)} p(\Delta t)\, d(\Delta t) \qquad (t_p < t < t_2) \tag{4}$$

for $t > t_p$, where t_2 is $t_p + \Delta t_{pk}/2$.

The Markov estimate, $Y'(t)$, is the same value as the level $Y(t + t_d)$, where the time shift t_d satisfies one of the following equations:

$$\int_{t_d(t)}^{0} (p(\Delta t) + p(|\Delta t| + 2(t_p - t)))\, d(\Delta t) = \Delta P \qquad (t_1 < t < t_p) \tag{5}$$

or

$$\int_{0}^{t_d(t)} (p(\Delta t) + p(-|\Delta t| + 2(t_p - t)))\, d(\Delta t) = \Delta P \qquad (t_p < t < t_2). \tag{6}$$

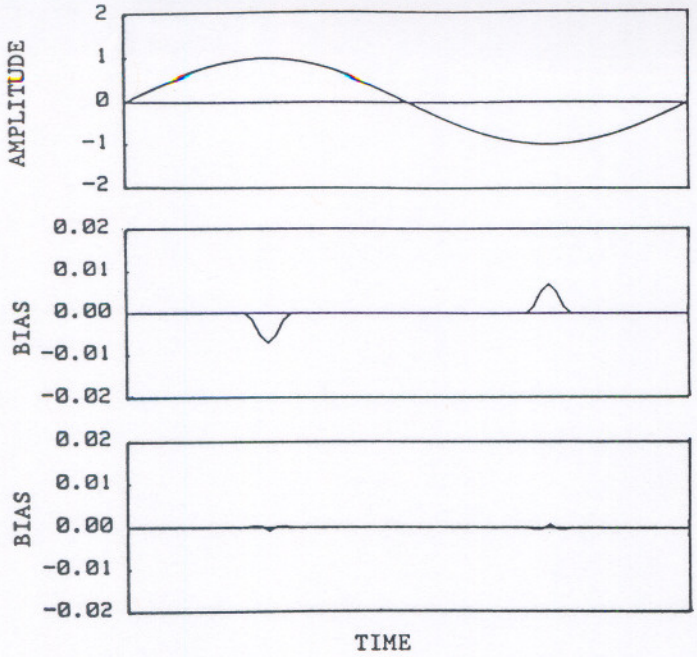

Fig. 5. Comparison of compensated and uncompensated Markov estimators for a sine wave input signal, using ideal CDF of time jitter ($1\sigma = 10°$) for computations. Top plot is original waveform, middle plot is the error in the Markov estimator, and bottom plot is the error after compensation.

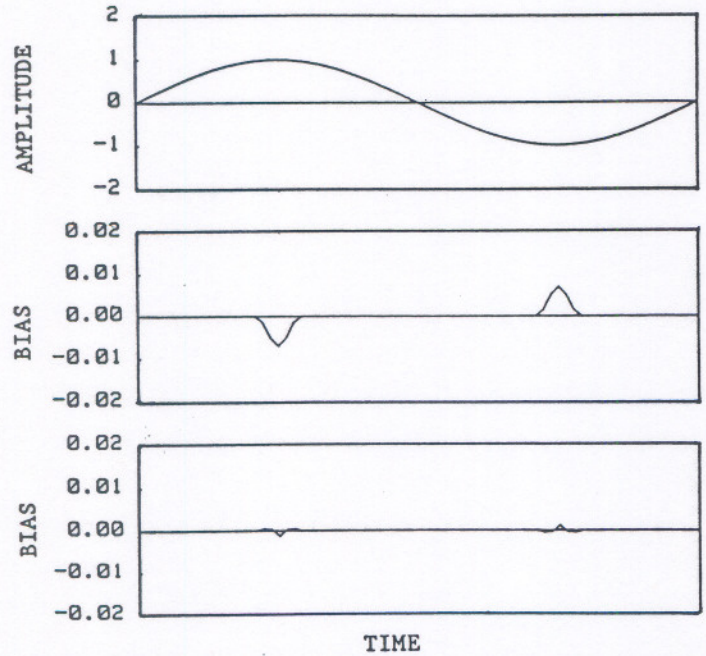

Fig. 6. Comparison of compensated and uncompensated Markov estimators for a sine wave input signal, using the percentile method to estimate the time-jitter CDF.

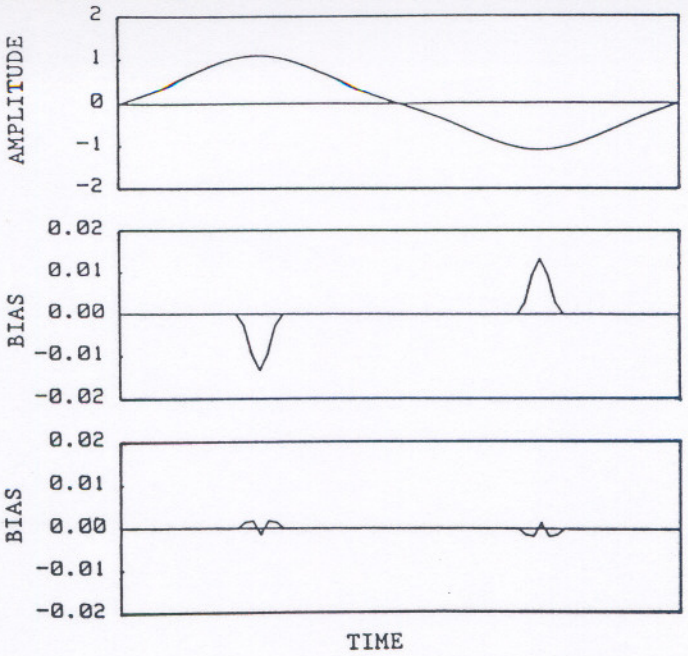

Fig. 7. Comparison of compensated and uncompensated Markov estimators for an input signal of the form, $\sin(\omega t) - 0.1 \sin(3\omega t)$, using the percentile method to estimate the time-jitter CDF.

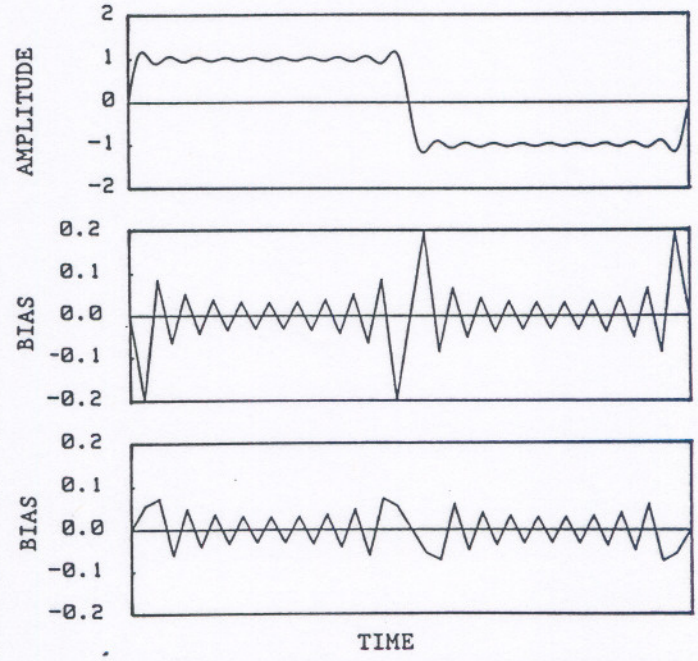

Fig. 8. Comparison of compensated and uncompensated Markov estimators for an input signal comprised of the sum of the first ten harmonic components of a square wave, using the ideal CDF of the time jitter.

imum error in the estimated CDF is 0.0146, which is considered adequate for this work.

V. Examples

Figures 5–8 show the results from computer simulations of this compensation approach, applied under four different conditions. In each of the four figures, the top plot is the input waveform, the middle plot is of the errors of the Markov estimator, and the bottom plot is of the errors after compensation. In each example, the jitter is Gaussian with a σ of 10° of the fundamental component of the input waveform. At each time point, t, 360 000 observations were simulated. In Fig. 5, the input signal is a sine wave, and the ideal CDF of a Gaussian distribution was used. The compensation reduced the peak error from 0.7% to 0.08%. Using the percentile method to first determine the distribution of the jitter, the peak error after compensation increased somewhat to 0.12% as shown in Fig. 6. In Fig. 7, the input signal is $\sin(\omega t) - 0.1 \sin(3\omega t)$, the

percentile method has been applied, and the peak error has been reduced from 1.3% to 0.17%. Finally, in Fig. 8, the input signal is the sum of the first 10 harmonic components of a square wave, the ideal CDF has been used, and the peak error has been reduced from 20% to 7.3%. Note that the original assumptions concerning symmetry and the required separation of critical points are not met in this last example.

VI. Conclusion

A method has been described for reducing the error of the Markov estimator when sampling nonmonotonic signals in the presence of timing jitter. The method involves computing a compensation term derived from the Markov estimate of the signal and the cumulative distribution function of the time jitter. A generalization of the Markov estimator, proposed earlier as an algorithm for finding any percentile of a distribution, has been demonstrated as an effective means for determining the CDF. The accuracy of the compensation has been determined through simulation for several practical waveforms. The resulting errors after compensation are only about 1/3–1/8 of the errors of the original Markov estimates. These errors could probably be further reduced if a multipoint spline fit were used to interpolate between sample points in computing the value $Y'(t + t_d(t))$ used in (8). In the examples given, linear interpolation was used.

The use of the Markov estimator with compensation is an attractive alternative to waveform averaging in equivalent-time sampling systems when timing jitter is a major source of error, and a more accurate estimate of the waveform is required. The compensation approach is particularly appropriate for the sampling voltage tracker type of sampling system since the SVT automatically implements the Markov estimator.

References

[1] T. M. Souders, D. R. Flach, C. Hagwood, and G. L. Yang, "The effects of timing jitter in sampling systems," *IEEE Trans. Instrum. Meas.*, vol. 39, pp. 80–85, Feb. 1990.

[2] R. K. Tsutakawa, "Random walk design in bio-assay," *J. Am. Statistical Assoc.*, vol. 62, pp. 842–856, 1967.

[3] T. M. Souders, H. K. Schoenwetter, and P. S. Hetrick, "Characterization of a sampling voltage tracker for measuring fast, repetitive signals," *IEEE Trans. Instrum. Meas.*, vol. IM-36, pp. 956–960, Dec. 1987.

[4] W. L. Gans, "The measurement and deconvolution of time jitter in equivalent-time waveform samplers," *IEEE Trans. Instrum. Meas.*, vol. IM-32, pp. 126–133, Mar. 1983.

A Method of Measuring Electrode Temperature for Arc Power Calculation

H. Nouri

Abstract—**A technique for the construction of the thermocouple probe used for the measurement of electrode temperature under the influence of arcing in a low-power circuit breaker at break is reported.**

The problem of appreciable heat diffusion to the surroundings during the short time of arcing at break, typically 3–6 ms (depending on the speed of break), is dealt with by placing the thermocouple probe as near as possible to the electrode's surface, and in the center where, in most cases, the arc occurs.

Under dc test conditions the amounts of the arc power received by each electrode for various parameters such as current and electrode spacing are calculated.

Manuscript received July 19, 1991; revised May 11, 1993.

H. Nouri is with the School of Electronic and Electrical Engineering, University of Bath, Bath BA2 7AY, United Kingdom.

IEEE Log Number 9211483.

I. Introduction

One kind of electrode erosion is caused by joule heating during arcing, so that electrode material is boiled away. The repeated operation of break and make results in metal being transferred from one electrode to the other which eventually leads to the disfigurement of the electrode surface and causes a circuit breaker to operate defectively.

Customarily the erosion is assessed from the observation of the electrode surface by the scanning electron microscope, by weighing, or from measurement of the surface roughness.

Up to now, the assessment of the erosion from the power balance relation, which indicates the degree of the difference in the correlation between the electrical power and thermal power (the amount of power calculated from the thermocouple output during arcing), has not received a great deal of attention because of the complexity involved in measuring the electrode's temperature at such a short duration of arcing at break (typically 3–6 ms).

Recently, theoretical attempts have been made to determine a relation between the electrode's temperature and erosion [1], but due to the complexity of the arc mechanism and the sintered nature of the materials of the electrode surface, the relation becomes very complex.

To support such research, the construction of a probe for an accurate electrode temperature measurement of a low-power circuit breaker, and also the calculations of the amount of the arc power received by each electrode with respect to current (4–10 A), and electrode spacing (0.05–1 mm), are presented.

II. Method of Construction

A type T thermocouple insulated with a covering of Poly Tetra Fluoro Ethylene (PTFE) is chosen because of its high sensitivity, linearity of characteristics, and negligible thermal mass. Because of the small physical size of the electrode, as shown in Fig. 1, the thermocouple wires chosen were as small as possible (0.075 mm core and 0.25 mm outer diameter sleeve).

The thermocouple's junction was welded by twisting one centimeter of the bared wires together and discharging a capacitor momentarily between a carbon electrode and the twisted end of the thermocouple. The twisted end is placed vertically to form a ball with a diameter in the range of 150–200 μm. On completion of the weld, the bared wires were untwisted and insulated from each other.

A blind hole was then machined axially into the stub of the electrode, and the end faced off, leaving 200 μm thickness of material behind the contact surface, as shown in Fig. 2.

To ensure the thermocouple weld was centered in the hole, and to achieve maximum thermal conductivity, a jig was made from a brass tube 4 mm in length with an outer diameter of 1.35 mm and inner diameter 1 mm. This tube was two-thirds filled with an epoxy resin adhesive known as Araldite MY750,[1] with hardener HY917

[1]Evode Ltd., Stafford, UK.

0018-9456/93$03.00 © 1993 IEEE