Methods for Aligning the NIST Watt-Balance

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***Abstract*—The NIST watt-balance has been developed to explore the possibility of monitoring the stability of the mass standard by means of electrical quantum standards. The mass standard is the last basic standard that is kept as an artifact. The watt-balance uses a movable coil in a radial magnetic field to compare the mechanical energy required to lift a kilogram mass in earth's gravity with the electrical energy required to move the coil the same distance in a magnetic field. The electrical energy is monitored in terms of quantized Hall resistance and Josephson junction voltage standards. The accuracy of this experiment depends on a large number of factors. Among them are the ability to align the apparatus so that the movable coil and magnet are coaxial and aligned to the local vertical. Misalignments of the coil and magnet result in forces and torques on the coil. The coil is suspended like a pendulum and responds easily to these torques and horizontal forces. This paper describes a computer program that was written to calculate the shape of the magnetic field and the torques and forces on the movable coil that result from any misalignments. This information is being used to develop an alignment procedure that minimizes misalignments and the errors they cause. This program has enhanced our understanding of the cause of torques about the vertical axis on the coil and the dependence of this torque on the magnetic field gradient.**

I. Introduction

The fundamental definitions of the volt and ohm are in terms of mechanical lengths and forces. This allows the mechanical and electrical units to be tied to the same standards. However, with the development of quantized electrical standards, more practical definitions are based on the Josephson junction voltage and the quantized Hall resistance standards. The values chosen for the parameters in these standards were adjusted as recently as 1990 to keep the basic relation between electrical and mechanical units as close as possible.

One of the devices used to relate the mechanical and electrical units is the watt-balance apparatus [1]–[4]. This apparatus compares the force of a mass in gravity to the force on a current in a magnetic field. Now that the long-term stability for maintaining the electrical units is greater than that for maintaining the mass unit ($\sim 10^{-9}$ for voltage and resistance versus $\sim 10^{-8}$ for the kg), can the watt-balance be used, in reverse, to monitor the stability of the mass standard? That is the proposal of B. Kibble in [2] and the purpose of a project in the UK [1], [2] and one in the U.S. [3], [4].

The NIST watt-balance apparatus uses a superconducting magnet to create a stable magnetic field (see Fig. 1). The watt-balance experiment is performed in two parts. In the first part, a suspended coil is moved through the magnetic field at a nearly constant velocity, and the voltage induced across the coil is integrated over measured vertical intervals. In the second part, the current in the suspended coil required to balance a mass is determined for a number of vertical positions of the coil. These data are then analyzed to equate the electrical energy in units of volts, amperes and seconds to mechanical energy in units of newtons and meters. The electrical quantities are calibrated against Josephson junction voltages and quantized Hall resistances. Thus, via this experiment a mass can be calibrated in terms of quantized electrical standards.

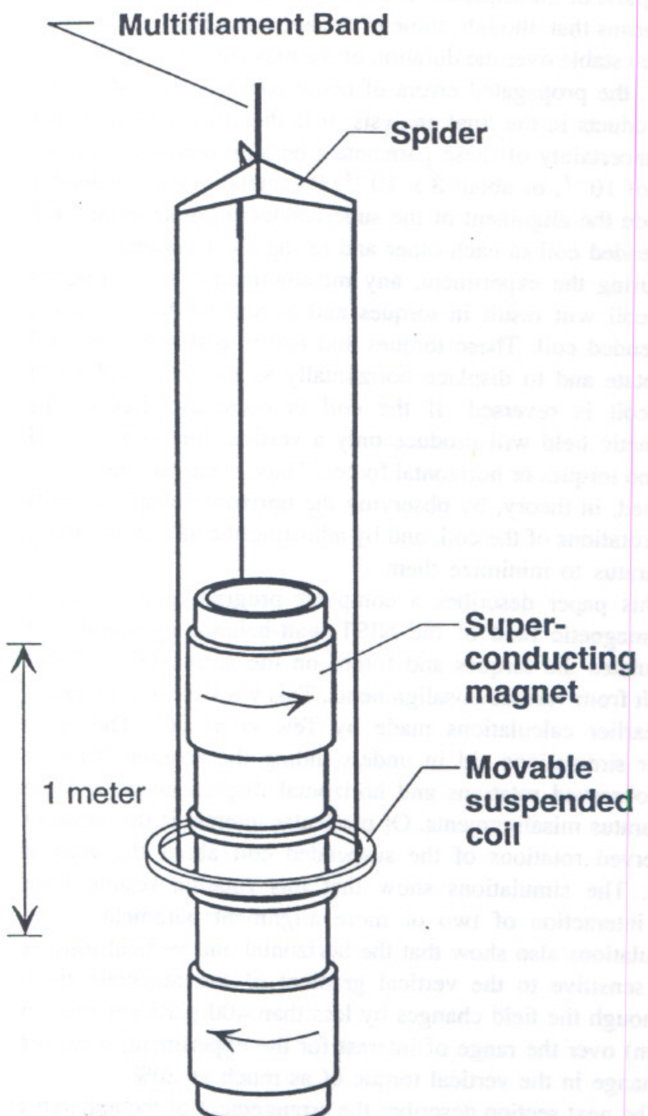

Fig. 1. Arrangement and relative size of suspended coil and superconducting magnet.

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To achieve the desired relative uncertainty of 10^{-7} (and ultimately 10^{-8}) requires that many of the parameters of the experiment be measured with an even lower uncertainty. For example, the local acceleration of gravity, the current in the coil during a balance, the voltage-time integral, and the velocity of the coil all must be measured to less than 10^{-7} relative uncertainty. The use of the same apparatus to perform both parts of the experiment and a special magnetic field shape [5] means that, though, some of the apparatus parameters must remain stable over the duration of the experiment to better than 10^{-7}, the propagated errors of many parameters appear only as products in the final analysis. It is therefore sufficient that the uncertainty of these parameters be kept below the square root of 10^{-7}, or about 3×10^{-4}. Parameters in this category include the alignment of the superconducting magnet and the suspended coil to each other and to the local vertical.

During the experiment, any misalignments of the magnet and coil will result in torques and horizontal forces on the suspended coil. These torques and forces will cause the coil to rotate and to displace horizontally as the current through the coil is reversed. If the coil is correctly aligned, the magnetic field will produce only a vertical force on the coil and no torques or horizontal forces. Thus, the apparatus can be aligned, in theory, by observing the horizontal displacements and rotations of the coil, and by adjusting the alignment of the apparatus to minimize them.

This paper describes a computer program that simulates the magnetic field of the NIST watt-balance apparatus and calculates the torques and forces on the suspended coil that result from various misalignments. This work is a continuation of earlier calculations made by Tew *et al.* [6]. The computer simulations aid in understanding the relation between the observed rotations and horizontal displacements and the apparatus misalignments. Of particular interest is the cause of observed rotations of the suspended coil about the vertical axis. The simulations show that this rotation results from the interaction of two or more alignment parameters. The simulations also show that the horizontal and vertical torques are sensitive to the vertical gradient of the magnetic field. Although the field changes by less than 400 parts per million (ppm) over the range of interest for the experiment, it causes a change in the vertical torque of as much as 20%.

The next section describes the arrangement of the apparatus coils. Following sections describe the formulas used in the force computations of the program, the alignment parameters modeled, and the results of the simulation. The simulation results are compared with measurements taken on the apparatus.

II. Arrangement of Apparatus Coils

The suspended coil is hung like a pendulum from a three-spoked "spider" frame hung from a multifilament ribbon or "band" as shown in Fig. 1. Any horizontal force on the coil will cause it to shift in position. The force is countered by the pendulum action of the suspension. The spider is designed with two modes of stiffness. In the loose mode, the torques on the suspended coil about the horizontal axis cause a rotation of the coil and the spider frame. In the stiff mode the spider frame resists rotations, so these torques cause primarily a shift in the coil with only a small amount of rotation. These two modes of stiffness will be used to help separate alignment parameter errors. A torque on the coil about the vertical axis will cause the coil to rotate about its axis. From the symmetry of the coil this torque is the smallest, but the restoration force is also small, so significant rotations are observed.

During the experiment the suspended coil is moved about 35 mm above and below the center plane ($z = 0$) of the superconducting magnet. A laser interferometer is used to measure this vertical travel using a corner cube attached to the bottom of the spider frame. This arrangement is being replaced with a laser using three corner cubes attached directly to the suspended coil. Two other corner cubes and a plane mirror on the coil are used with a laser to measure the horizontal displacements of the coil and the rotation about the three axes.

The suspended coil is made of three concentric coils that can be connected independently. This feature is used to adjust the magnetic field as described below. The superconducting magnet is made of 22 coils. These coils are grouped as two stacks of ten coils each, called the upper and lower main coils and two independent coils called the upper and lower compensation coils. The coils are wired in series opposition, so that the field of the upper coils opposes the field of the lower coils. This results in a radial field at the center plane where the suspended coil is positioned. A small correction current is added to the two compensation coils so that the field at the suspended coil decreases as the reciprocal of the radius. This field shape makes the apparatus insensitive to changes in the diameter of the suspended coil due to temperature changes [5]. The suspended coil has an average radius of 350 mm and a cross section approximately 20 mm high and 33 mm wide. The superconducting magnet is approximately 1.5 m high with a nominal radius of 240 mm.

III. Magnetic Field Program

The computer program calculates the forces and torques on the suspended coil by computing the magnetic field vector, H, at the suspended coil. Each of the 22 superconducting coils is approximated as a single circular loop. The magnetic field around a circular loop can be expressed in terms of complete elliptic integrals of the first kind, $K(m)$, and the second kind, $E(m)$, where m is the parameter of the elliptic integral. For a circular coil of radius a in the x, y plane centered on the z axis, as shown in Fig. 2, the magnetic field only has components in the z and radial directions. For a point with cylindrical coordinates z and ρ, the magnetic field components are given by [7], [8]

$$H_z = \frac{i}{2\pi\sqrt{z^2+(\rho+a)^2}} \cdot \left\{K(m) - E(m)\left[1 - \frac{2a(a-\rho)}{z^2+(a-\rho)^2}\right]\right\} \quad (1)$$

and

$$H_\rho = \frac{iaz}{\pi[z^2+(\rho+a)^2]^{3/2}} \cdot \left\{\frac{E(m)}{1-m} - 2\left[\frac{K(m)-E(m)}{m}\right]\right\} \quad (2)$$

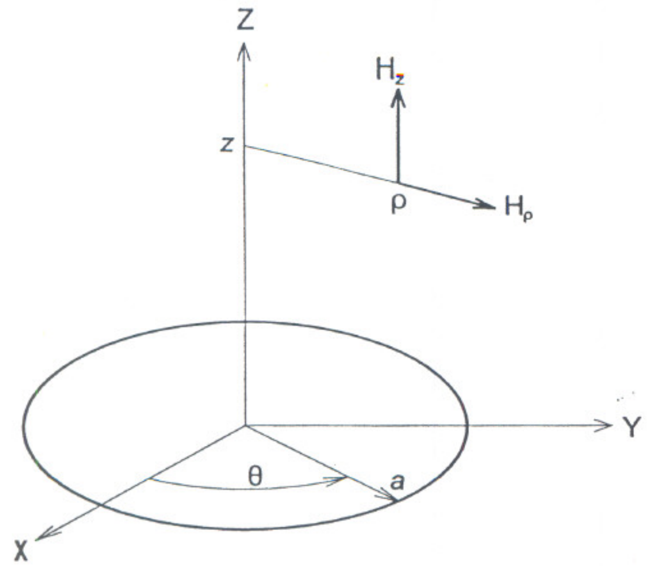

Fig. 2. Coordinate system to reference magnetic field at point z and ρ, around single loop of radius a.

where i is the current in the coil and m is given by

$$m = \frac{4a\rho}{z^2 + (\rho + a)^2}. \tag{3}$$

Note: when calculating the elliptic integrals, some texts use a modulus k rather than a parameter m where $k^2 = m$.

The suspended coil is usually approximated as a single circular loop, but for increased accuracy it can be approximated as an array of coaxial loops. For each loop, the magnetic field, $\boldsymbol{H}_c(\theta)$, at the point on the loop referenced by angle θ, is calculated by summing the fields from all 22 superconducting coils. The current in each of the main coils is taken to be unity, and the current in the compensation coils is set as described below.

The vector force on a segment of the suspended coil, $d\boldsymbol{F}(\theta)$, is calculated using the vector cross product as

$$d\boldsymbol{F}(\theta) = i_c|\boldsymbol{r}_c|\boldsymbol{d}_c(\theta) \times \boldsymbol{H}_c(\theta)\, d\theta \tag{4}$$

where i_c is the current in the suspended coil, $|\boldsymbol{r}_c|$ is the magnitude of the coil radius vector, and $\boldsymbol{d}_c(\theta)$ is a unit vector in the direction of the coil at θ. The total force vector on the coil, $\boldsymbol{F}$, is calculated by integrating $d\boldsymbol{F}(\theta)$ over all θ.

Similarly the torque vector on a segment of the suspended coil is calculated as

$$d\boldsymbol{\gamma}(\theta) = \boldsymbol{r}_c(\theta) \times d\boldsymbol{F}(\theta). \tag{5}$$

The total torque vector on the coil, $\boldsymbol{\gamma}$, is calculated by integrating $d\boldsymbol{\gamma}(\theta)$ over all θ.

For the simulations the current in the compensating coils is adjusted by calculating F_z, the vertical force on two coils, one with a radius of 360 mm and one with a radius of 340 mm at $z = 0$. The compensating coil current that results in the same force on both coils is the one used. This method is similar to the one used for the NIST watt-balance. There the inner and outer coils of the suspended coil are connected in series opposition. Then the current in the compensation coils is adjusted so that there is no change in the force on the suspended coil when its current is reversed. For the arrangement of the NIST superconducting coils, the correct compensation coil current (ampere-turns) is slightly less than 0.3 of the main coil current (ampere-turns).

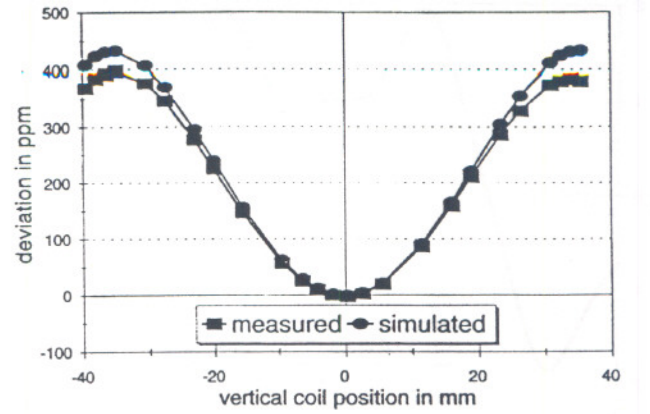

Fig. 3. Deviation of measured force profile and simulated magnetic field strength relative to value at center versus position above or below center plane.

The misalignments that the program models for the suspended coil are displacement of the coil center from the midpoint of the axis of the superconducting magnet in the three directions, c_x, c_y, and c_z, and rotations ϕ_{cx} and ϕ_{cy} of the coil about the x and y axes. The program models rotations ϕ_{mx} and ϕ_{my} of the superconducting magnet about the x and y axes away from the local vertical. The local vertical is always taken to be along the z axis. Thus (1) and (2) must be transformed to the coil coordinates to calculate the magnetic field. There are also four internal misalignments of the upper superconducting coils relative to the lower coils, that are not discussed in this paper. The program calculates forces and torques for any combination of these alignment parameters.

IV. Simulation Results

The program results compare well with measured and analytic results and provide useful insights into the complicated interactions between the suspended coil and the superconducting magnet field. Fig. 3 shows a comparison of measured and simulated results. One curve is the measured force profile for the apparatus. This shows the change in vertical force on the suspended coil as a function of the vertical position of the coil. This force is proportional to the integral of the radial magnetic field over the volume of the coil. The other curve shows the simulated force profile for a circular loop in a field with no misalignments. Although there are significant differences between the curves, the simulation results capture the basic behavior of the measured results. The magnetic field strength over a much larger range is shown in Fig. 4. The experiment uses only the central 70 mm of the field, that has an almost constant radial component and almost no vertical component.

The simulations show that a small tilt of the suspended coil or of the superconducting magnet about the horizontal axis causes the horizontal force on the coil given by

$$F_x = \tfrac{1}{2}(\phi_{cy} + \phi_{my})F_{z0} \tag{6}$$

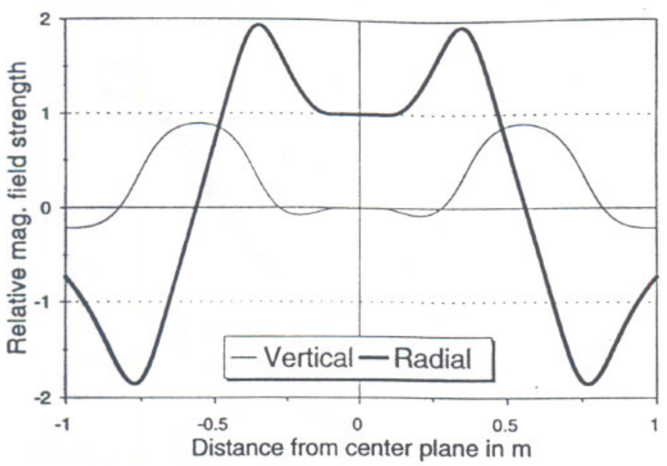

Fig. 4. Simulated vertical and radial magnetic field at radius of suspended coil over large distance above and below center plane.

and

$$F_y = \tfrac{1}{2}(\phi_{cx} + \phi_{mx})F_{z0} \tag{7}$$

where F_{z0} is the vertical magnetically induced force on the coil at $c_z = 0$. Although the force varies for $c_z \neq 0$, this causes only a very small change in the horizontal forces. Similarly, the simulations show that a small displacement of the center of the suspended coil in the horizontal plane from the axis of the magnet, $\mathbf{c}_0$, causes a torque, $\boldsymbol{\gamma}$, on the coil proportional to the displacement as

$$\boldsymbol{\gamma} = \tfrac{1}{2}\,\mathbf{c}_0 \times \mathbf{F}_z. \tag{8}$$

These results are in agreement with [6] for $c_z = 0$. However, the simulations showed that for c_z away from the center plane of the superconducting magnet, the torques have a z dependence that is a function of the gradient of the magnetic field. Fig. 5 shows the z dependence of γ_x, the torque about the x axis, for a rotation of 0.01 rad of the suspended coil about the x axis. This response is proportional to the gradient of the magnetic field in the vertical direction, i.e., the derivative of the force curve in Fig. 3. To first order in the field derivative, the horizontal torques are given by

$$\gamma_x = -\frac{1}{2}\left[c_y + \phi_{mx}c_z + \frac{|\mathbf{r}_c^2|}{F_{z0}} \cdot (\phi_{mx} - \phi_{cx})\,\frac{dF_z(c_z)}{dz}\right]F_{z0} \tag{9}$$

and

$$\gamma_y = \frac{1}{2}\left[c_x - \phi_{my}c_z - \frac{|\mathbf{r}_c^2|}{F_{z0}} \cdot (\phi_{my} - \phi_{cy})\,\frac{dF_z(c_z)}{dz}\right]F_{z0}. \tag{10}$$

For the NIST apparatus the field gradient term can be approximated as a linear and cubic z dependence over the region of interest as

$$\frac{|\mathbf{r}_c^2|}{F_{z0}}\,\frac{dF_z(c_z)}{dz} \approx c_z\left[0.143 - 13.3\left(\frac{c_z}{|\mathbf{r}_c|}\right)^2\right]. \tag{11}$$

The torque γ_z on the suspended coil about the vertical axis is not caused by any one misalignment, but requires two or more misalignments. By symmetry, there is no torque on the suspended coil unless it is tilted with respect to the vertical. Fig. 6 is a contour plot of simulated γ_z for a ϕ_{cx} of 0.01 rad and over ±40 mm of c_z and 0–3 mm of c_x. The figure shows that there is almost no z dependence of γ_z and that it increases linearly with c_x. For a combination of ϕ_{cx} of 0.01 rad and ϕ_{my} of 0.01 rad, γ_z does have a z dependence, as shown in Fig. 7. As with the horizontal torques, γ_z is sensitive to the field gradient. The deviation in this dependence from linear is

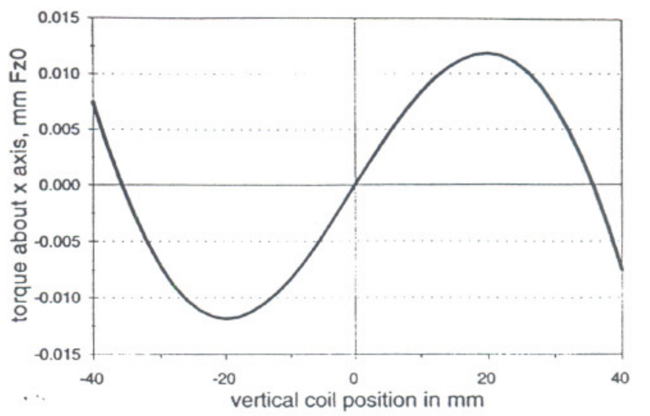

Fig. 5. The z dependence of γ_x, the torque about the x axis, for 0.01 rad in ϕ_{cx}, the tilt in the suspended coil about the x axis.

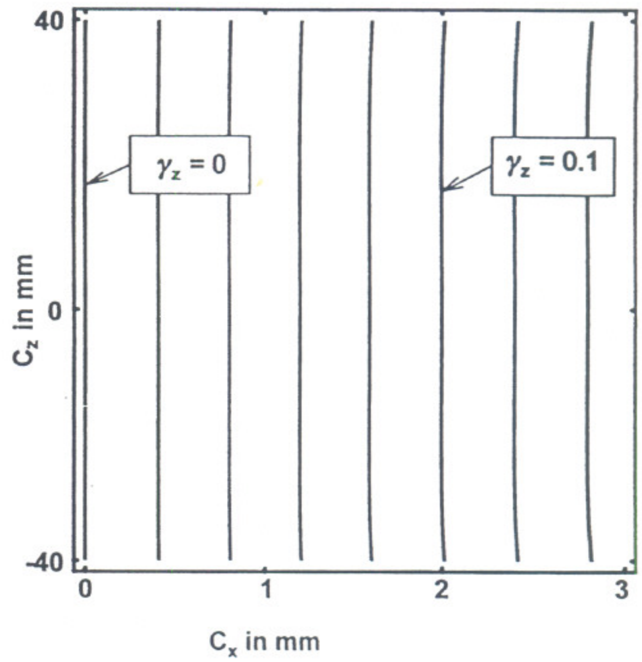

Fig. 6. Contour plot of γ_z, the torque about the z axis, with a tilt ϕ_{cx} of the suspended coil about the x axis of 0.01 rad. Results are shown for movement of the suspended coil over ±40 mm of vertical travel, c_z, shown vertically, and over 0–3 mm of travel in the x direction, c_x, shown horizontally.

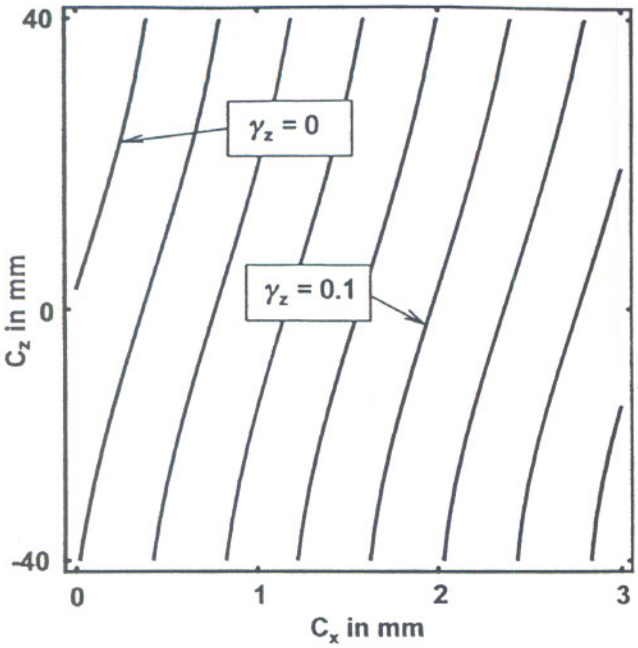

Fig. 7. Contour plot of γ_z, the torque about the z axis, with a tilt ϕ_{cx} of the suspended coil about the x axis and a tilt ϕ_{my} of the superconducting magnet about the y axis of 0.01 rad each. Results are shown for movement of the suspended coil over ± 40 mm of vertical travel, c_z, shown vertically, and over 0–3 mm of travel in the x direction, c_x, shown horizontally.

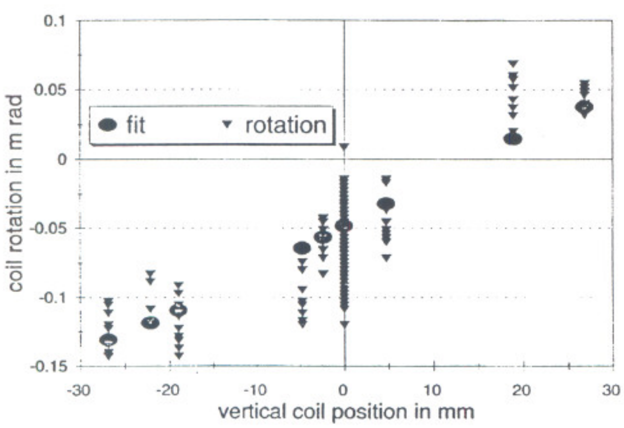

Fig. 8. Measurements of the rotation of the suspended coil about the vertical axis resulting from current reversals in the coil for several vertical positions, and a fit to the data based on the formula for the torque on the coil about the z axis.

20% at c_z of 40 mm. The equation for γ_z, to first order in the field derivative, is given by

$$\gamma_z = \frac{\phi_{cx}}{2}\left[c_x - \phi_{my}c_z - \frac{|\mathbf{r}_c^2|}{F_{z0}}\phi_{my}\frac{dF_z(c_z)}{dz}\right]F_{z0} + \frac{\phi_{cy}}{2}\left[c_y + \phi_{mx}c_z + \frac{|\mathbf{r}_c^2|}{F_{z0}}\phi_{mx}\frac{dF_z(c_z)}{dz}\right]F_{z0}. \quad (12)$$

Fig. 8 shows some recent measurements of rotation differences of the suspended coil resulting from reversal of current in the coil. The data is in m rad of rotation of the coil. A fit to the data, based on (12), is also shown.

V. Conclusions

The simulation program models the magnetic field of the NIST watt-balance apparatus. It gives results that match measurements and allows the examination of the effects of many misalignment parameters on the behavior of the apparatus. In particular, the results show that the torque that rotates the suspended coil about the vertical axis is the result of at least two misalignment parameters. The misalignments are tilt of the suspended coil about the horizontal axis, and translation of the coil center away from the axis of the magnet and along the axis of tilt. The simulation shows that the torques on the suspended coil have a dependence on the field gradient that can be observed even though the variation in field strength is less than 400 ppm.

The program is useful in developing an alignment procedure for the apparatus and for analyzing the error contributions of the residual misalignment factors.

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