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# Dynamics of overdamped Josephson junctions driven by a square-wave pulse 

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The periodic solutions of an overdamped Josephson junction driven by a square-wave pulse were found. Unlike those driven by a sinusoidal ac current. the integer step widths of a Josephson junction driven by a square-wave pulse train are reduced with increasing ac current frequency. The maximum value of any non-zero step width approaches $2 I_{c}$, where $I_{c}$ is the critical current, by decreasing either the pulse repetition rate or the pulse duration time. The characteristic features of any integer step are almost identical and we give an explanation of this. © 1998 American Institute of Phisics. [S0021-8979(98)06306-3]

## I. INTRODUCTION

If a Josephson junction ${ }^{1}$ is biased by an external ac current with frequency $f$. its dc $l-V$ curve shows voltage plateaus at the multiples of $h f / 2 e$. i.e..

$$
\begin{equation*}
V_{n}=n \frac{h f}{2 e}, \quad(n=0 . \pm 1 . \pm 2 \ldots), \tag{1.1}
\end{equation*}
$$

which are called the Shapiro steps. ${ }^{2}$ Here $h / 2 e$ is the flux quantum. The existence of these voltage plateaus is strong evidence of the existence of the phase-dependent supercurrent suggested by Josephson. ${ }^{1}$ If we measure the voltage difference between successive integer steps, we can determine the value of the fundamental constant $h / 2 e$. In reverse, if we know the value of $h / 2 e$, we can determine the output voltage of a Josephson junction, which is the basic concept of the Josephson voltage standard.

After several decades of effort, a dc Josephson voltage standard ${ }^{3}$ (DC-JVS) using Josephson junctions with highly hysteretic $I-V$ characteristics was made successfully and is now in use. It has been shown that this DC-JVS is very successful in calibrating solid-state voltage standards. Due to its hysteretic behavior and slow step selection procedure. however, it is not useful for an ac JVS (AC-JVS) which requires high speed step selection. To avoid this, an AC-JVS using Josephson junctions with non-hysteretic $I-V$ characteristics is proposed. ${ }^{4.5}$ Both DC-JVS and AC-JVS require large numbers of Josephson junctions connected in series to reach high enough voltage. So the variation of junction parameters, like critical current and normal resistance, is inevitable. These kinds of disorder always reduce the step width of an AC-JVS. For an AC-JVS to be applicable, the step width must be large enough to cover the uncertainty of an

[^0]external dc bias current. Because of this concern on the step width, several groups have tried to find a way to increase the step width.

The first important finding was made by Monaco ${ }^{6}$ who showed that a Josephson junction driven by a biharmonic voltage bias source gives larger non-zero steps than that driven by a single harmonic voltage source. The width of any non-zero step reaches its maximum value, $2 I_{c}$. if the voltage source becomes a periodic delta function which contains all the harmonics of equal amplitude. Maggi ${ }^{7}$ has shown that enhancement of the width of the non-zero step is true even for the resistively shunted-junction model and is significant if the pulse duration time, $\tau$, is smaller than the pulse period, $T$, and the pulse current, $I_{p}$, is greater than the critical current, $I_{c}$. Benz and Hamilton ${ }^{8}$ have found that the maximum width of any non-zero step of a pulse-driven Josephson junction does not depend on the pulse repetition rate, $f$, if $f$ is smaller than the characteristic frequency, $f_{c}$. This characteristic feature of non-integer step width in a pulse-driven system is quite different from that of a continuous wave (CW)driven system which exhibits maximum step width that increases with the external ac frequency. ${ }^{9}$ All these results suggest that using a pulse instead of a CW as an ac bias current is more favorable to get large non-zero steps. The experimental results performed by the same group. however, did not fit well to the results of the numerical calculation. ${ }^{8}$ The width of the $n=1$ step in a pulse-driven system decreased with the pulse repetition rate, like that in a CWdriven system.

There are several methods by which to find step boundaries as a function of $I_{\mathrm{ac}}$ and $f$. One of them is to integrate the equation of motion numerically using a computer or an analog simulator to get the $I-V$ curve for a given $I_{\mathrm{ac}}$ and $f$. Then the same procedure must be done repeatedly for differ
ent values of $I_{\mathrm{ac}}$ and $f$ to get the step boundaries as a function of $I_{\mathrm{ac}}$ and $f$. Another way of finding step boundaries is to use the harmonic balance equation. ${ }^{10}$ This method is simple and easy, giving a resuit that is in good agreement with the numerical simulation if the harmonic content is small. But in the case of a Josephson junction driven by a pulse which contains a lot of harmonic terms of high order, this is not a good approximation. Finally, if the external ac current is a square-wave pulse, another method is known. ${ }^{11.12}$ Starting from the general solution of the equation of motion and applying the matching condition at the boundary of two different bias regions and using the periodic boundary condition ( PBC ) requirement, one can get a set of equations that any periodic solution must satisfy. Even though this is a very powerful method by which to find the step boundaries, its applications so far were limited to the case of high dc-bias current ${ }^{11}$ or to the delta-function pulse limit. ${ }^{12}$ In this article, we generalize this method to include all the possible cases regarding the pulse amplitude and the dc-bias current. We first built a set of equations that any periodic solutions must satisfy, then solved this set of equations numerically, finding the step boundaries of a pulse-driven Josephson junction.

## II. ANALYTIC SOLUTIONS

In the resistively-shunted junction (RSJ) model, ${ }^{13.14}$ the equation of motion for the phase difference between the two superconductors in a Josephson junction, $\phi$, is given by

$$
\begin{equation*}
\frac{h}{4 \pi e R} \frac{d \phi}{d t}+I_{c} \sin \phi=I_{\mathrm{dc}}+I_{\mathrm{ac}}(t) \tag{2.1}
\end{equation*}
$$

for a junction with a negligible capacitance and a small resistance. Here $I_{c}$ is the critical current and $R$ is the normal resistance. We normalize the phase variable, $\phi$, and the time. $t$, by $\phi \rightarrow \phi / 2 \pi . t \rightarrow t / t_{c}$ where $t_{c}$ is the characteristic time defined by $t_{c}=1 / f_{c}=h / 2 e I_{c} R$. Then Eq. (2.1) takes the form

$$
\begin{equation*}
\frac{d \phi}{d t}+\sin (2 \pi \phi)=i_{\mathrm{dc}}+i_{\mathrm{ac}}(t) . \tag{2.2}
\end{equation*}
$$

where $i_{\mathrm{dc}}=I_{\mathrm{dc}} / I_{c}$, and $i_{\mathrm{ac}}=I_{\mathrm{ac}} / I_{c}$
Consider the case of no ac current. Integrating Eq. (2.2) for a given $i=i_{\mathrm{dc}}$, we get the following form of solution:

$$
\begin{align*}
& t-t_{0}=\frac{1}{\pi \sqrt{i^{2}-1}}\left[\tan ^{-1}\left(\frac{i \tan (\pi \phi)-1}{\sqrt{i^{2}-1}}\right) \pm m \pi\right] \\
& \text { for }|i|>1,  \tag{2.3}\\
& =\frac{1}{2 \pi_{\sqrt{ }} \overline{1-i^{2}}} \ln \left|\frac{i \tan (\pi \phi)-1-\sqrt{1-i^{2}}}{i \tan (\pi \phi)-1+\sqrt{1-i^{2}}}\right| \text { for }|i|<1 .
\end{align*}
$$

where $t_{0}$ is an integration constant and $m$ is an integer number. If we define $2 \pi \phi_{c}=\sin ^{-1}(i)\left(\left|\phi_{c}\right|<1 / 4\right)$, then the above equation can be written as

$$
\begin{aligned}
t-t_{0}= & \frac{1}{\pi \sqrt{i^{2}-1}}\left[\tan ^{-1}\left(\frac{i \tan (\pi \phi)-1}{\sqrt{i^{2}-1}}\right) \pm m \pi\right] \\
& \text { for }|i|>1, \\
= & \frac{1}{2 \pi \sqrt{1-i^{2}}} \ln \left|\frac{i \tan (\pi \phi)-\cot \left(\pi \phi_{c}\right)}{i \tan (\pi \phi)-\tan \left(\pi \phi_{c}\right)}\right| \text { for }|i|<1 .
\end{aligned}
$$

For $|i|<1$, in the limit of $t \rightarrow \infty$, $\phi(t)$ goes to $\phi_{c}$ regardless of the value of $t_{0}$. This means that, as time goes. all the solutions approach the fixed point. $\phi_{c}$, regardless of the initial starting point. This limiting value, $\phi_{\mathrm{s}} \equiv \phi_{C}$, is a pole of the right-hand side of the lower equation in Eq. (2.4) and is called the stable fixed point. There is another solution of the equation $\sin (2 \pi \phi)=i$ for $1 / 4<|\phi|<1 / 2$, satisfying $\phi_{u}$ $= \pm 1 / 2-\phi_{c}$. Like the stable solution $\phi(t)=\phi_{s}, \phi(t)=\phi_{u}$ is a solution of Eq. (2.2), but it is unstable: even an infinitesimal change in the initial value from $\phi_{u}$ will eventually lead the phase to the stable fixed point $\phi_{s}$. These two fixed points are branch points dividing the phase space $|\phi|<1 / 2$ into two separate regions and no solution can cross these branch points. If the initial value $\phi_{0}$ falls into $\phi_{s}<\phi_{0}<\phi_{u}$. the phase decreases in time approaching the limiting value $\phi_{c}$. Otherwise, the phase increases in time approaching the same limiting value $\phi_{c}$. Since no solution can cross the branch points, there are upper and lower bounds of the phase change given by

$$
\begin{align*}
& \max [\phi(t=x)-\phi(0)]=2 \phi_{c}+\frac{1}{2},  \tag{2.5}\\
& \min [\phi(t=x)-\phi(0)]=2 \phi_{c}-\frac{1}{2} .
\end{align*}
$$

For $|i|>1$, there is no branch point and no restriction in phase change. The phase increases or decreases in time with a mean slope given by

$$
\begin{equation*}
\left\langle\frac{d \phi}{d t}\right\rangle= \pm \sqrt{i^{2}-1} . \tag{2.6}
\end{equation*}
$$

This type of solution is called a running solution
We now consider the square-wave pulse with peak current, $i_{p}$, duration time, $\tau$, and repetition time. $T$. Throughout this article, we normalized all the current variables by the critical current. $i_{c}$. and all the time variables by the characteristic time, $t_{c}$. We define ac current as

$$
\begin{align*}
i_{\mathrm{ac}} & =i_{p} \text { for } 0<\bmod (t, T)<\tau,  \tag{2.7}\\
& =0 \text { otherwise. }
\end{align*}
$$

and the total current $i_{\text {tot }}$ is given by

$$
\begin{align*}
i_{\text {tot }} & =i_{p}+i_{\mathrm{dc}} \equiv i_{1} \text { for } 0<\bmod (t, T)<\tau .  \tag{2.8}\\
& =i_{\mathrm{dc}} \equiv i_{2} \text { otherwise. }
\end{align*}
$$

In order to get a complete picture of the step boundaries. one must consider four different cases according to the values of $\left|i_{1}\right|$ and $\left|i_{2}\right|$.

## A. Case I: $\left|i_{1}\right|<1$ and $\left|i_{2}\right|<1$

In this case, both $\left|i_{1}\right|$ and $\left|i_{2}\right|$ are smaller than 1 . Since no running solution is permitted, the dc voltage is always zero.

## B. Case II: $\left|i_{1}\right|>1$ and $\left|i_{2}\right|<1$

For $0<t<\tau$, the magnitude of the total current $\left|i_{\text {tor }}\right|=\left|i_{\mathrm{dc}}+i_{\mathrm{ac}}\right|=\left|i_{1}\right|$ is greater than 1. In this high-current region. the solution $\phi_{1}(t)$ is given by

$$
\begin{equation*}
\tan \left[\omega_{1}\left(t-t_{1}\right)\right]=\frac{\pi}{\omega_{1}}\left[i_{1} \tan \left(\pi \phi_{1}\right)-1\right] . \tag{2.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{1}=\pi, \overline{i^{2}-1}, \tag{2.10}
\end{equation*}
$$

and $t_{1}$ is an integration constant. If we define $\phi_{0}$ as the initial value at time $t=0$. then $t_{1}$ is given by

$$
\begin{equation*}
\tan \left(\omega_{1} t_{1}\right)=\frac{\pi}{\omega_{1}}\left[1-i_{1} \tan \left(\pi \phi_{0}\right)\right] . \tag{2.11}
\end{equation*}
$$

For $\tau<t<T$, the magnitude of the total current $\left|i_{\text {tor }}\right|=\left|i_{\text {de }}\right|=\left|i_{2}\right|$ is smaller than 1. In this low-current region. the solution $\phi_{2}(t)$ is given by

$$
\begin{equation*}
\exp \left[2 \omega_{2}\left(t-t_{2}\right)\right]=\left|\frac{\tan \left(\pi \phi_{2}\right)-\cot \left(\pi \phi_{c}\right)}{\tan \left(\pi \phi_{2}\right)-\tan \left(\pi \phi_{c}\right)}\right| . \tag{2.12}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{2}=\pi \sqrt{1-i_{2}^{2}}, 2 \pi \phi_{c}=\sin ^{-1}\left(i_{2}\right) \tag{2.13}
\end{equation*}
$$

and $t_{2}$ is another integration constant. The solution $\phi_{1}(t)$ and $\phi_{2}(t)$ must be identical at time $\tau$, i.e., $\phi_{1}(\tau)=\phi_{2}(\tau) \equiv \phi_{\tau}$. Substituting $\phi=\phi_{\tau}$ in Eqs. (2.9) and (2.12), we get the following equations.

$$
\begin{align*}
& \tan \left[\omega_{1}\left(\tau-t_{1}\right)\right]=\frac{\pi}{\omega_{1}}\left[i_{1} \tan \left(\pi \phi_{\tau}\right)-1\right],  \tag{2.14}\\
& \exp \left[2 \omega_{2}\left(\tau-t_{2}\right)\right]=\left|\frac{\tan \left(\pi \phi_{\tau}\right)-\cot \left(\pi \phi_{c}\right)}{\tan \left(\pi \phi_{\tau}\right)-\tan \left(\pi \phi_{c}\right)}\right|
\end{align*}
$$

The requirement of the PBC, $\phi_{0}=\phi_{2}(T)+n$ where $n$ is the step number, gives the following equation.

$$
\begin{equation*}
\exp \left[2 \omega_{2}\left(T-t_{2}\right)\right]=\left|\frac{\tan \left(\pi \phi_{0}\right)-\cot \left(\pi \phi_{c}\right)}{\tan \left(\pi \phi_{0}\right)-\tan \left(\pi \phi_{c}\right)}\right| . \tag{2.15}
\end{equation*}
$$

Combining Eqs. (2.14) and (2.15), we finally get the set of equations any periodic solution must satisfy.

$$
\begin{align*}
\exp & {\left[2 \omega_{2}(T-\tau)\right] } \\
& =\left|\frac{\left[\tan \left(\pi \phi_{0}\right)-\cot \left(\pi \phi_{c}\right)\right]\left[\tan \left(\pi \phi_{\tau}\right)-\tan \left(\pi \phi_{c}\right)\right]}{\left[\tan \left(\pi \phi_{0}\right)-\tan \left(\pi \phi_{c}\right)\right]\left[\tan \left(\pi \phi_{\tau}\right)-\cot \left(\pi \phi_{c}\right)\right]}\right| \tag{2.16}
\end{align*}
$$

with

$$
\begin{aligned}
& \tan \left(\pi \phi_{\tau}\right)=\frac{1}{i_{1}}\left[1+\frac{\omega_{1}}{\pi} \tan \left\{\omega_{1}\left(\tau-i_{1}\right)\right\}\right] . \\
& \tan \left(\omega_{1} t_{1}\right)=\frac{\pi}{\omega_{1}}\left[1-i_{1} \tan \left(\pi \phi_{0}\right)\right] .
\end{aligned}
$$



FIG. 1. The stable (solid circle) and unstable (cross) periodic solutions, of Eq. (2.15) with varying $i_{\mathrm{dc}}$ for $T=(\mathrm{a}) 0.5$ and (b) 1.0 . The values of $i_{\mathrm{ac}}$ and $\tau$ are 5 and 0.1 , respectively. The solid and dotted lines are the pole lines given by $\phi_{0}=\phi_{c}$ and $\phi_{T}= \pm 0.5-\phi_{c}$.

However, not all $\phi_{0}$ satisfying this set of equations give a periodic solution since there is a restriction imposed by the branch points. Due to the presence of the branch points in the low-current region, the phase value at time $\tau, \phi_{\tau}$, and at time $T, \phi_{2}(T)$, must be in the same branch. This requirement can be met if we take the positive argument of the right-hand side of Eq. (2.16) which then becomes
$\exp \left[2 \omega_{2}(T-\tau)\right]$

$$
\begin{equation*}
=\frac{\left[\tan \left(\pi \phi_{0}\right)-\cot \left(\pi \phi_{c}\right)\right]\left[\tan \left(\pi \phi_{\tau}\right)-\tan \left(\pi \phi_{c}\right)\right]}{\left[\tan \left(\pi \phi_{0}\right)-\tan \left(\pi \phi_{c}\right)\right]\left[\tan \left(\pi \phi_{\tau}\right)-\cot \left(\pi \phi_{c}\right)\right]} \tag{2.18}
\end{equation*}
$$

The problem of solving Eq. (2.2) to find a periodic solution reduces a situation of finding a $\phi_{0}$ to satisfy the above equation for a given set of parameters $\left\{i_{1}, i_{2}, \tau, T\right\}$. Note that the right-hand side of the above equation is independent of $T$ and has two poles and two zeros. With increasing $T$, the left-hand side of the above equation grows exponentially and in order to satisfy the equation. $\phi_{0}$ must be close to the poles, given by

$$
\begin{equation*}
\phi_{0}=\phi_{s} . \quad \phi_{\tau}=\phi_{k} . \tag{2.19}
\end{equation*}
$$

Figure 1 shows the solutions of Eq. (2.18) with varying $i_{\mathrm{dc}}$ for a given value of $i_{p}, T$ and $\tau$. As was expected, all the


FIG. 2. The evolution of phases starting from different initial values for $i_{\mathrm{dc}}$ $=(a)-0.7$. (b) 0.1 . and (c) 0.8 . The vertical dashed lines represent the boundary of high- and low-current regions and the horizontal dotted lines are branch cuts. The bold lines in (a) and (c) are solutions satisfying the PBC. The values of $i, T$. and $;$ are $5,0.5$, and 0.1 . respectively.
solutions appear near the pole lines, $\phi_{0}=\phi_{s}$ and $\phi_{\tau}=\phi_{u}$. denoted by solid and dotted lines. respectively. The larger $T$ becomes. the closer $\phi_{0}$ comes to the poles. making the nonstep region narrower. We will show later that the non-step region vanishes exponentially with increasing $T$.

A closer look at the phase evolution is helpful to understand the dynamical behavior of the pulse-driven Josephson junction. Figures 2(a)-2(c) show the evolution of phases starting from different initial values for $i_{\mathrm{dc}}=-0.7,0.1$, and 0.8 , respectively. The values of $i_{\mathrm{ac}}, T$ and $\tau$ are given by 10 , 0.5 , and 0.05 , respectively. For $i_{\mathrm{dc}}=-0.7$, the total external current in the high-current region is $i_{\text {tot }}=i_{1}=10-0.7=9.3$ w.ich is much greater than 1 . Since the magnitude of the Supercurrent is always smaller than 1 , most of the external current in this high-current region is carried by the normal current. The effect of the supercurrent, which can be estimated by the averaged supercurrent $\langle\sin (2 \pi \phi)\rangle$,

$$
\begin{equation*}
\langle\sin (2 \pi \phi)\rangle=i_{i}-\sqrt{i_{1}^{2}-1} \sim O\left(\frac{1}{i_{1}}\right) \tag{2.20}
\end{equation*}
$$

is negligible. Neglecting the supercurrent term in Eq. (2.2), we get an approximate solution.

$$
\begin{equation*}
\phi(t) \approx \phi_{0}+i_{1} t \tag{2.21}
\end{equation*}
$$

As is shown in Fig. 2, the phase increases almost linearly in time and the total phase change in this high-current region is given by

$$
\begin{equation*}
\phi_{T}-\phi_{0} \approx i_{1} \tau \tag{2.22}
\end{equation*}
$$

For $i_{1}=9.3$ and $\tau=0.05, i_{1} \tau$ is 0.47 , which means that the phase rotates about a 0.47 turn during $\tau$. Then it begins to decrease or increase depending on what branch the phase at time $\tau$ belongs. In the low-current region, the total external current is $i_{\text {tot }}=i_{\mathrm{dc}}=i_{2}=-0.7$ and the phase change is bounded by $-0.75<\Delta \phi<0.25$ [refer to Eq. (2.5)]. For any solution to satisfy the PBC, the total phase change must be an integer number, which can be satisfied only when the phase change at the low-current region is close to -0.47 , canceling the phase change made in the high-current region. So, any periodic solution, if it exists, must have $n=0$. In Fig. 2(a) are two periodic solutions, denoted by bold lines, with a net phase change of zero.

With increasing $i_{\mathrm{dc}}$, the two periodic solutions come closer to each other, fusing into one as shown in Fig 1(a). For $i_{\mathrm{dc}}=0.1$, there is no stable periodic solution as shown in Fig. 2(b). With a further increased $i_{\text {dc }}$, two periodic solutions begin to appear; their time evolutions are shown in Fig. 2(c), where $i_{\mathrm{dc}}=0.8$. This is different from the case of $i_{\mathrm{dc}}=-0.7$, especially in the low-current region. For $i_{\mathrm{dc}}=0.8$, the phase change in the high-current region is about 0.54 and the phase change in the low-current region is bound by -0.20 $<\Delta \phi<0.80$. Therefore, any periodic solution for $i_{\mathrm{dc}}=0.8$ must be an increasing function of time in the low-current region, giving a phase change of about 0.46 , which is required to make the total phase change to 1 .

Since the phase change in the low-current region is always smaller than 1 , the step number or the rotation number is mostly determined in the high-current region where phase change is unlimited. The magnitude of the total current in the high-current region. $\left|i_{1}\right|$, is dominated by the peak pulse current which can be much greater than the dc current. So. with increasing peak pulse current. we can expect steps with nonzero step numbers to occur, regardless of the value of the dc current. This is the reason why non-zero steps occur along the zero dc-bias line, i.e., $I_{\mathrm{dc}}=0 .^{7.8}$ We can determine at what value of $i_{p}$ the step width becomes maximum. For simplicity, we consider the limit $\tau \rightarrow 0$ while maintaining $\tau i_{p}$ constant. In this delta-function pulse limit, Eq. (2.22) become exact and for a $\tau i_{p}$ given by

$$
\begin{equation*}
\tau i_{p}=\tau i_{p, n} \equiv n \tag{2.23}
\end{equation*}
$$

the phase change in the high-current region becomes identical to $n$, which means that the phase rotates $n$ complete turns in the high-current region. Any solution starting from the initial value, $\phi_{0}$, identical to the pole or zeros of Eq. (2.18)


FIG. 3. The step boundaries in the region $\left|i_{\mathrm{dc}}\right|<1$ for $T=$ (a) 0.5 . (b) 1 . and (c) 2 . The value of pulse duration time, $\tau$, is 0.1 . Notice that with increasing $T$, the quasiperiodic region shrinks rapidly and the $n$th step width shows its maximum for $i_{p} \approx i_{p, n} \equiv n / \tau$.
then automatically satisfies the PBC. So, the two periodic solutions in this delta-function pulse limit are given by

$$
\begin{equation*}
\phi_{0}=\phi_{s} . \quad \phi_{0}=\phi_{u} . \tag{2.24}
\end{equation*}
$$

Since these solutions always exist for $\left|i_{\mathrm{dc}}\right|<1$. the step width defined by the maximum extent to which a stable periodic solutions exists becomes identical to 2 . This result looks similar to that obtained by Monaco ${ }^{6}$ for a voltage-driven system which is a good approximation of the RSJ model in the Gigh-current limit.
One can notice that $\tau i_{p, n}$ is independent of $T$. implying that the pulse current required to exhibit the maximum step width is independent of the pulse repetition rate. In contrast, for a CW-driven Josephson junction. the higher the frequency, the larger $i_{\mathrm{ac}}$ must be in order to get the maximum non-zero step width. ${ }^{9}$ Figure 3 shows the step boundaries for different values of $T$. The $n$th step width becomes maximum


FIG. 4. The log-log plot of $2-\Delta i_{\mathrm{dc}}(n=1)$ vs $T$ for several values of $\tau$. The peak pulse current is determined by $\tau i_{p}=1$. Note that for $T<1$. $2-\Delta i_{\mathrm{dc}}(n=1$, fits well to a line with a slope of 1 . The inset shows the same graph with a different $x$ axis. $\pi / T$. The slope of the line is 1 .
for $\tau i_{p} \approx \tau i_{p . n}=n$, proving that Eq. (2.23) is a good approximation even for a pulse with a relatively long duration time. With increasing $T$, the step regions expand and the non-step (quasiperiodic) region shrinks. We have calculated the $n=1$ step width as a function of $T$ for several values of $\tau$. We have chosen the pulse power $i_{n}=1 / \tau$ to get the step width close to its maximum value. As shown in Fig. 4, the step width increases with $T$ approaching 2 . The smaller the duration time, $\tau$. the faster the step width approaches 2 . We found that the maximum first step width fits well to the formula

$$
\begin{equation*}
\max \left[\Delta i_{\mathrm{dc}}(n=1)\right] \approx 2\left(1-\frac{\tau}{T}\right) \tag{2.25}
\end{equation*}
$$

The inset of Fig. 4 shows that most of data fits the above equation regardless of the value of $\tau$. The smaller the $\tau$, the wider the range of $T$ over which the data fit well to the above equation. Here we draw an important conclusion of our work: in order to get the maximum step width close to 2 one can either increase the repetition time. $T$, or decrease the duration time, $\tau$. Note that the functional form of the maximum step width for a CW-driven Josephson junction can be approximated by

$$
\begin{equation*}
\max \left[\Delta i_{\mathrm{dc}}(n=1)\right] \approx \frac{C}{T} \tag{2.26}
\end{equation*}
$$

for $T \geqslant 1$, where $C$ is a constant with a positive value. The maximum step width decreases with $T$ for a CW-driven system, while it increases with $T$ for a pulse-driven system. We also have found that the quasiperiodic region shrinks exponentially with increasing $T$. Let us define the width of quasiperiodic region $\Delta i_{q p}\left(n, i_{p}\right)$ by

$$
\begin{equation*}
\Delta i_{q p}\left(n, i_{p}\right)=i_{\mathrm{dc}}^{-}\left(n+1, i_{p}\right)-i_{\mathrm{dc}}^{+}\left(n, i_{p}\right), \tag{2.27}
\end{equation*}
$$

where $i_{\mathrm{dc}}\left[n, i_{p}\right)\left(i_{\mathrm{dc}}^{-}\left(n, i_{p}\right)\right]$ is the upper (lower) boundary of the $n$th step for a given $i_{p}$. In Fig. 5 we have shown the width of the quasiperiodic region for $n=0$ and $\pi i_{p}=1 / 2$ as a


FIG. 5. The width of the quasiperiodic region, $\Delta i_{q p}(n=0)$ as a function of $T$ for $\tau=0.1,0.05$, and 0.01 . The peak pulse current has been changed with $\tau$ to satisfy $\tau i_{p}=1 / 2$. The three lines are almost indistinguishable, implying that $\tau$ has little effect on the width of the quasiperiodic region. For $T>1$. $\ln \left[\Delta i_{q p}(n=0)\right]$ fits well to a line. showing the exponential dependency of $\Delta i_{q p}(n=0)$ _on $T$. The inset shows the same graph in the log-log plot, showing the power-law dependence of $\Delta i_{q p}(n=0)$ on $T$. The slope is close to -1 .
function of $T$. Although it is not apparent, there are three curves with different values of $\tau$. implying that the width of the quasiperiodic region is nearly independent of $\tau$ for $\tau i_{p}=1 / 2$. The logarithm of $\Delta_{q p}$ fits well to a line with a slope 3.14 for $T>0.2$ which corresponds to $f<5 f_{c}$. In other words, the quasiperiodic region vanishes exponentially with decreasing $f$ for $f<5 f_{c}$. Due to this exponential dependence on $T$, the width of the quasiperiodic region shrinks fast with increasing $T$. becoming less than $5 \%$ of the width of the step region for $T>1.2\left(f<0.8 f_{c}\right)$. For a small $T$, as shown in the inset of the Fig. 5(b), the width of quasiperiodic region fits well to $\Delta i_{q p} \sim 1 / T$. The transition from the power-law dependency to the exponential dependency to $T$ happens close to $2 \pi T=1$. So, we get the following functional form of the width of quasiperiodic region:

$$
\begin{align*}
\Delta i_{q p} & \approx A \exp (-B T) \text { for } 2 \pi T>1,  \tag{2.28}\\
& \approx D / T \text { for } 2 \pi T<1 .
\end{align*}
$$

where $A . B$ and $D$ are parameters weakly dependent on $i_{\mathrm{dc}}$, $n$, and $\tau$.

## C. Case III: $\left|i_{1}\right|<1$ and $\left|i_{2}\right|>1$

In this case, we define the pulse differently from case II.

$$
\begin{align*}
i_{\mathrm{ac}} & =0 \text { for } 0<\bmod (t, T)<T-\tau,  \tag{2.29}\\
& =i_{p} \text { otherwise. }
\end{align*}
$$

Then the total current $i_{\text {tot }}$ is given by

$$
\begin{align*}
i_{\text {tot }} & =i_{\mathrm{dc}} \equiv i_{1} \text { for } 0<\bmod (t, T)<T-\tau,  \tag{2.30}\\
& =i_{\mathrm{dc}}+i_{p} \equiv i_{2} \text { otherwise. }
\end{align*}
$$



FIG. 6. The step boundaries in the $\left(i_{p}, i_{\mathrm{sc}}\right)$ plane for $T=(\mathrm{a}) 0.5$ and (b) 1 . The value of the pulse duration time. $\tau$. is 0.1 .

Following the same procedure as in case II. we get the following set of equations:
$\exp \left(2 \omega_{2} \tau\right)$

$$
=\frac{\left[\tan \left(\pi \phi_{0}\right)-\cot \left(\pi \phi_{c}\right)\right]\left[\tan \left(\pi \phi_{T-\tau}\right)-\tan \left(\pi \phi_{c}\right)\right]}{\left[\tan \left(\pi \phi_{0}\right)-\tan \left(\pi \phi_{c}\right)\right]\left[\tan \left(\pi \phi_{T-\tau}\right)-\cot \left(\pi \phi_{c}\right)\right]} .
$$

$\tan \left(\pi \phi_{T-\tau}\right)=\frac{1}{i_{1}!}\left[1+\frac{\omega_{1}}{\pi} \tan \left\{\omega_{1}\left(T-\tau-t_{1}\right)\right\}\right]$,
$\tan \left(\omega_{1} t_{1}\right)=\frac{\pi}{\omega_{1}}\left[1-i_{1} \tan \left(\pi \phi_{0}\right)\right]$.
where $\phi_{T-\tau}$ is the phase value at time $T-\tau$. In the previous case, the phase is subjected to a low bias current for time $T-\tau$ which is considered to be much greater than $\tau$. The larger the $T-\tau$, the more the phase can change in the lowcurrent region. The more the phase can change in the lowcurrent region, the greater the possibility of existing periodic solutions for a given value of $i_{\mathrm{dc}}$. As a result, the larger the $T-\tau$, the wider the step region. In the present case, the phase is subjected to a low bias current for a very short time, resulting in narrow steps in the $\left(i_{p}, i_{\text {dc }}\right)$ plane.

## D. Case IV: $\left|i_{1}\right|>1$ and $\left|i_{2}\right|>1$

In this case both $\left|i_{1}\right|$ and $\left|i_{1}\right|$ are greater than 1 , implying that there is no pole or branch point. There still can exist steps of finite width. even though their width is much smaller than in case II. Following the same procedure as in case II. we get the following set of equations:
$\tan \left(\pi \phi_{0}\right)=\frac{1}{i_{2}}\left[1+\frac{\omega_{2}^{\prime}}{\pi} \tan \left\{\omega_{2}^{\prime}\left(T-t_{2}\right)\right\}\right]$.

$$
\begin{align*}
& \frac{1}{i_{2}^{\prime}} 1-\frac{\omega_{2}^{\prime}}{\pi} \tan \left\{\omega_{2}^{\prime}\left(\tau-t_{2}\right)\right\} \\
& \left.\quad=\frac{1}{i_{1}} 1-\frac{\omega_{1}}{\pi} \tan \left\{\omega_{1}\left(\tau-t_{1}\right)\right\}\right], \tag{2.32}
\end{align*}
$$

$\tan \left(\omega_{1} t_{1}\right)=\frac{\pi}{\omega_{1}}\left[1-i_{1} \tan \left(\pi \phi_{0}\right)\right]$,
where
$\omega=\pi \sqrt{i_{2}-1}$.

## iii. RESULTS

We have solved Eqs. (2.18), (2.31), and (2.32) numerically to get the step boundaries in the ( $i_{p}, i_{\mathrm{dc}}$ ) plane. Figures tria and 6 (b) show, respectively. the overall shape of the step soundaries for $T=1$ and 0.5 . Compared to those of a CWdriven Josephson junction, the step boundaries of a pulsedriven Josephson junction show quite different characteristic features. First of all. the step boundaries are not symmetric with respect to the zero dc-bias line, $i_{\mathrm{dc}}=0$. All the non-zero steps cross the zero dc-bias line which. in the CW-drive system. can be crossed only by the $n=0$ step. This characteristic feature is due to the asymmetry of the ac current: the square-wave pulse we have considered is not a true ac current in that it has a nonvanishing dc component given by

$$
\begin{equation*}
\left\langle i_{\mathrm{ac}}\right\rangle=\frac{\tau}{T} i_{p} \tag{3.1}
\end{equation*}
$$

In case of a CW bias. one needs to apply a large enough dc current to see high-order steps. For a pulse bias, however. there is no need of applying a large dc current to see a highorder step since the step number is primarily determined by the pulse which contains a non-vanishing dc component in itself. If the step number is determined only by the pulse current. what is the difference between $n=0$ step and $n \neq 0$ steps? In fact. the shape of $n$th step boundary is almost identical to that of the zero step, except that it is displaced in the $i_{p}$ axis by $i_{p, n}=n / \tau$. This point becomes clear if we define $i_{\mathrm{dc}}^{\prime}$ by

$$
\begin{equation*}
i_{\mathrm{dc}}=i_{\mathrm{dk}}+\frac{\tau}{T} i_{p} \tag{3.2}
\end{equation*}
$$

and redraw Fig. 6 using this newly defined dc current axis, as shown in Fig 7. This newly defined dc current includes all the dc component in the total external current. In the ( $i_{p}, i_{\mathrm{dc}}^{\prime}$ )


FIG. 7. The step boundaries in the ( $i_{p}, i_{\mathrm{dc}}^{\prime}$ ) plane where $i_{\mathrm{dc}}^{\prime}$ is defined in the text. No integer step other than $n=0$ step crosses the newly defined zerobias line. $i_{\mathrm{d}}=0$. The value of the pulse duration time. $\tau$. is 0.1 .
plane, the shapes of the step boundaries look more like those of a CW-driven system and no non-zero step can cross the newly defined zero dc-bias line, i.e., $i_{\mathrm{dc}}^{\prime}=0$.

## IV. DISCUSSION

In sum, using a pulsed ac current instead of a CW has a great advantage regarding application of the Shapiro steps to a voltage standard. Its enhanced step width will increase the step stability and will reduce the voltage uncertainty originating from the uncertainty in the bias current. In order to get a large integer step, a pulse train with short duration time, low repetition rate and high power is favorable. Low repetition rate. however. implies a small voltage difference between successive integer steps and more junctions are required to get a certain output voltage. The disorder in junction parameter, which reduces step stability, increases with the number of junctions. Also, a transmitting highpower pulse train of several tens of GHz without distorting the pulse form is, in general. more difficult than transmitting sinusoidal ac current. So. for a practical application, there must be optimization of pulse repetition rate, duration time. and pulse power.

There are still many questions as yet unanswered. For instance, if the pulse becomes symmetric, how does the dynamical behavior of the system change? Is this system robust to a small disorder in pulse shape or pulse period? Another
unresolved issue is the discrepancies between the results of numerical calculations. mostly based on the RSJ model. and the experimental results of Benz and Hamilton. ${ }^{8}$ especially the frequency dependency of the step width for a pulsedriven system. If we use other models. like the RSJN (resistively-shunted junction with nonlinear resistance) model, ${ }^{15}$ what will the result be? More extensive studies are required to resolve these issues.

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