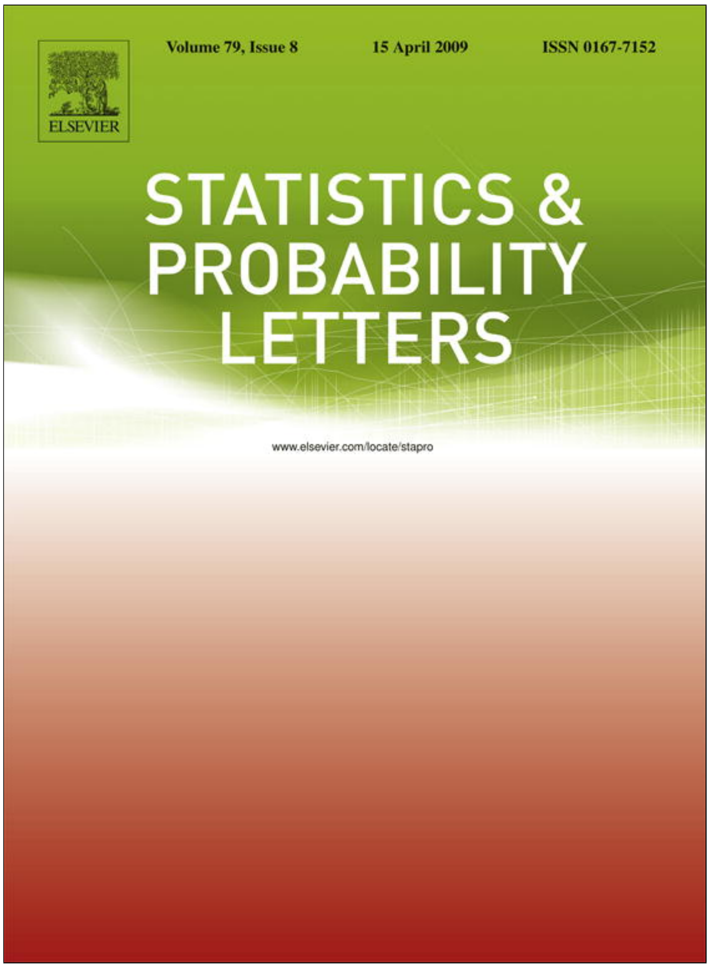
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Identities for negative moments of quadratic forms in normal variables

Andrew L. Rukhin *

National Institute of Standards and Technology, University of Maryland at Baltimore County, USA

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ABSTRACT

Two formulas for the central inverse moments of a quadratic form in normal variables and of the ratio of such forms are established. They relate the quadratic form determined by a positive definite matrix to that defined by the inverse matrix.

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1. Introduction

Quadratic forms in normal variables appear in many different areas of statistics: time series, decision theory, hypothesis testing, particularly, in the context of the general linear model, etc. By now there is a substantial literature dedicated to evaluation of their moments as well as the moments of ratios of quadratic forms. Numerous references are given by Mathai and Provost (1992) who gave a compendium of formulas for inverse moments of quadratic forms in normal variables in terms of Lauricella's function. There is also an extensive bibliography (76 items) on the moments of quadratic forms and of their ratios in Meng (2005).

Out of these references we mention that of Smith (1989) who expressed the expectation of the ratio via an infinite series involving invariant (zonal) polynomials in three matrix arguments, and Lieberman (1994) who derived a saddlepoint approximation. Meng (2005) provided rigorous derivation of many earlier results along with some generalizations and gave an excellent review of their applications.

Assuming normality, we provide here some formulas relating negative central moments of the quadratic form defined by a positive definite matrix to those determined by the inverse matrix. The ratios of quadratic forms exhibit a similar relationship.

2. Main results

We start with a formula relating inverse central moments of the quadratic form determined by a positive definite matrix Λ and those determined by Λ^{-1}.

Theorem 2.1. *Let $Z = (Z_1, \ldots, Z_r)^{\mathrm{T}}$ be a random vector composed by independent standard normal variables, and assume that Λ is an $r \times r$ positive definite matrix. Then if $0 < q < r/2$,*

$$E(Z^{\mathrm{T}}\Lambda Z)^{-q} = \frac{\Gamma(r/2-q)}{2^{2q-r/2}\Gamma(q)\det(\Lambda)^{1/2}} E(Z^{\mathrm{T}}\Lambda^{-1}Z)^{q-r/2}. \tag{1}$$

* Corresponding address: Statistical Engineering Division, NIST, 100 Bureau Drive, 20899-8980 Gaithersburg, MD, USA. Tel.: +1 301 975 2951; fax: +1 301 975 3144.

E-mail address: andrew.rukhin@nist.gov.

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Proof. Without loss of generality Λ can be taken to be a diagonal matrix, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$. Then

$$\begin{aligned}
2^q\Gamma(q)E(Z^{\mathrm{T}}\Lambda Z)^{-q} &= \int_0^\infty t^{q-1} E\mathrm{e}^{-tZ^{\mathrm{T}}\Lambda Z/2}\,\mathrm{d}t = \int_0^\infty t^{q-1}\prod_i E\mathrm{e}^{-t\lambda_i Z_i^2/2}\,\mathrm{d}t \\
&= \int_0^\infty \frac{t^{q-1}\,\mathrm{d}t}{\prod_i\sqrt{1+\lambda_i t}} = \int_0^\infty \frac{t^{q-r/2-1}\,\mathrm{d}t}{\prod_i\sqrt{t^{-1}+\lambda_i}} = \int_0^\infty \frac{u^{r/2-q-1}\,\mathrm{d}u}{\prod_i\sqrt{\lambda_i+u}} \\
&= (\lambda_1\ldots\lambda_r)^{-1/2}\int_0^\infty \frac{u^{r/2-q-1}\,\mathrm{d}u}{\prod_i\sqrt{1+u/\lambda_i}} \\
&= \det(\Lambda)^{-1/2}\int_0^\infty u^{r/2-q-1}E\mathrm{e}^{-uZ^{\mathrm{T}}\Lambda^{-1}Z/2}\,\mathrm{d}u \\
&= \frac{\Gamma(r/2-q)}{2^{q-r/2}\det(\Lambda)^{1/2}}E(Z^{\mathrm{T}}\Lambda^{-1}Z)^{q-r/2},
\end{aligned}$$

so that indeed (1) holds. ∎

Theorem 2.1 is valid if Λ is a non-negative definite matrix, i.e. if some of the λ's vanish. In this case r denotes the rank of Λ, Λ^{-1} is to be replaced by the Moore–Penrose generalized inverse of Λ (Rao and Mitra, 1971), and the product of r positive λ's substitutes for $\det(\Lambda)$. Indeed the proof of Theorem 2.1 holds when $\prod_i$ is taken over i such that $\lambda_i > 0$.

Under the notation,

$$\tilde{\Lambda} = \Lambda/\det(\Lambda)^{1/r}, \tag{2}$$

it is suggestive to rewrite (1) as

$$\frac{E(Z^{\mathrm{T}}\tilde{\Lambda}Z)^{-q}}{E(Z^{\mathrm{T}}Z)^{-q}} = \frac{E(Z^{\mathrm{T}}\tilde{\Lambda}^{-1}Z)^{q-r/2}}{E(Z^{\mathrm{T}}Z)^{q-r/2}}.$$

Indeed $\tilde{\Lambda}^{-1} = \Lambda^{-1}/\det(\Lambda)^{-1/r}$, $\det(\tilde{\Lambda}) = 1$, and

$$E(Z^{\mathrm{T}}Z)^{-q} = \frac{\Gamma(r/2-q)}{2^q\Gamma(r/2)}.$$

In particular,

$$\det(\Lambda)^{1/2}E(Z^{\mathrm{T}}\Lambda Z)^{-r/4} = E(Z^{\mathrm{T}}\Lambda^{-1}Z)^{-r/4}. \tag{3}$$

Formula (1) gives the limit when $q \to r/2$,

$$\frac{E(Z^{\mathrm{T}}\Lambda Z)^{-q}}{E(Z^{\mathrm{T}}Z)^{-q}} \to \det(\Lambda)^{-1/2}.$$

The condition $0 < q < r/2$ guarantees the existence of moments in both sides of (1) so that interchange of integration with differentiation in the proof of Theorem 2.1 is legitimate. The same comment holds for the next result which extends Theorem 2.1 to the expectation of the ratio of powers of two quadratic forms.

Theorem 2.2. *Under the notation of Theorem 2.1, assume that $0 < q < p + r/2$ with $p \geq 0$, Λ a positive definite matrix, and Ψ is a symmetric matrix. If p is a non-negative integer or if Ψ is non-negative definite,*

$$E\frac{(Z^{\mathrm{T}}\Psi Z)^p}{(Z^{\mathrm{T}}\Lambda Z)^q} = \frac{\Gamma(r/2+p-q)}{2^{2q-p-r/2}\Gamma(q)\det(\Lambda)^{1/2}}E\frac{(Z^{\mathrm{T}}\Lambda^{-1/2}\Psi\Lambda^{-1/2}Z)^p}{(Z^{\mathrm{T}}\Lambda^{-1}Z)^{p-q+r/2}}. \tag{4}$$

Proof. As in the proof of Theorem 2.1, one has for a non-negative integer p,

$$\begin{aligned}
2^q\Gamma(q)E\frac{(Z^{\mathrm{T}}\Psi Z)^p}{(Z^{\mathrm{T}}\Lambda Z)^q} &= \int_0^\infty t^{q-1}E(Z^{\mathrm{T}}\Psi Z)^p\mathrm{e}^{-tZ^{\mathrm{T}}\Lambda Z/2}\,\mathrm{d}t \\
&= (-2)^p\int_0^\infty t^{q-1}\frac{\mathrm{d}^p}{\mathrm{d}s^p}E\mathrm{e}^{-Z^{\mathrm{T}}(s\Psi+t\Lambda)Z/2}\Big|_{s=0}\,\mathrm{d}t \\
&= (-2)^p\int_0^\infty t^{q-1}\frac{\mathrm{d}^p}{\mathrm{d}s^p}\frac{1}{\sqrt{\det(I+s\Psi+t\Lambda)}}\Big|_{s=0}\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-2)^p}{\det(\Lambda)^{1/2}} \int_0^\infty t^{q-r/2-1} \frac{\mathrm{d}^p}{\mathrm{d}s^p} \frac{1}{\sqrt{\det(I + st^{-1}\Lambda^{-1/2}\Psi\Lambda^{-1/2} + t^{-1}\Lambda^{-1})}}\Bigg|_{s=0} \mathrm{d}t \\
&= \frac{(-2)^p}{\det(\Lambda)^{1/2}} \int_0^\infty u^{r/2-q-1} \frac{\mathrm{d}^p}{\mathrm{d}s^p} \frac{1}{\sqrt{\det(I + su\Lambda^{-1/2}\Psi\Lambda^{-1/2} + u\Lambda^{-1})}}\Bigg|_{s=0} \mathrm{d}u \\
&= \frac{(-2)^p}{\det(\Lambda)^{1/2}} \int_0^\infty u^{r/2-q-1} \frac{\mathrm{d}^p}{\mathrm{d}s^p} E\mathrm{e}^{-Z^{\mathrm{T}}(u\Lambda^{-1}+su\Lambda^{-1/2}\Psi\Lambda^{-1/2})Z/2}\Big|_{s=0} \mathrm{d}u \\
&= \det(\Lambda)^{-1/2} \int_0^\infty u^{r/2+p-q-1} E(Z^{\mathrm{T}}\Lambda^{-1/2}\Psi\Lambda^{-1/2}Z)^p \mathrm{e}^{-uZ^{\mathrm{T}}\Lambda^{-1}Z/2}\, \mathrm{d}u \\
&= \frac{\Gamma(r/2+p-q)}{2^{q-p-r/2}\det(\Lambda)^{1/2}} E\frac{(Z^{\mathrm{T}}\Lambda^{-1/2}\Psi\Lambda^{-1/2}Z)^p}{(Z^{\mathrm{T}}\Lambda^{-1}Z)^{p-q+r/2}}.
\end{aligned}
$$

Justification of these formulas follows from Lemma 1 in Meng (2005), or the fact that for a positive definite matrix B, with $Q_p(\Psi)$ denoting a polynomial in elements of Ψ of degree p,

$$\frac{\mathrm{d}^p}{\mathrm{d}s^p}\frac{1}{\sqrt{\det(B+s\Psi)}}\Bigg|_{s=0} = \frac{Q_p(\Psi)}{\det(B)^{p+1/2}},$$

is well defined even when Ψ is not a a positive definite matrix.

If p is not an integer (but Ψ is a non-negative definite matrix, so that $Z^{\mathrm{T}}\Psi Z$ is positive with probability one), we use the fact that the pth moment is the pth fractional derivative of the moment generating function. Decompose $p = \lceil p \rceil + \langle p \rangle$, where $\lceil p \rceil$ is the largest integer not exceeding p, and $\langle p \rangle$, $0 < \langle p \rangle < 1$, denotes its fractional part. Then according to Lemma 2 in Meng (2005),

$$
\begin{aligned}
2^q\Gamma(q)\Gamma(\langle p \rangle)E\frac{(Z^{\mathrm{T}}\Psi Z)^p}{(Z^{\mathrm{T}}\Lambda Z)^q} &= \int_0^\infty\int_0^\infty s^{\langle p \rangle-1}t^{q-1}E(Z^{\mathrm{T}}\Psi Z)^{\lceil p \rceil}\mathrm{e}^{-sZ^{\mathrm{T}}\Psi Z/2-tZ^{\mathrm{T}}\Lambda Z/2}\,\mathrm{d}s\mathrm{d}t \\
&= (-2)^{\lceil p \rceil}\int_0^\infty\int_0^\infty s^{\langle p \rangle-1}t^{q-1}\frac{\mathrm{d}^{\lceil p \rceil}}{\mathrm{d}s^{\lceil p \rceil}}E\mathrm{e}^{-Z^{\mathrm{T}}(s\Psi+t\Lambda)Z/2}\,\mathrm{d}s\mathrm{d}t \\
&= (-2)^{\lceil p \rceil}\int_0^\infty\int_0^\infty s^{\langle p \rangle-1}t^{q-1}\frac{\mathrm{d}^{\lceil p \rceil}}{\mathrm{d}s^{\lceil p \rceil}}\frac{1}{\sqrt{\det(I+s\Psi+t\Lambda)}}\,\mathrm{d}s\mathrm{d}t \\
&= \frac{(-2)^{\lceil p \rceil}}{\det(\Lambda)^{1/2}}\int_0^\infty\int_0^\infty s^{\langle p \rangle-1}t^{q-1}\frac{\mathrm{d}^{\lceil p \rceil}}{\mathrm{d}s^{\lceil p \rceil}}\frac{1}{\sqrt{\det(I+st^{-1}\Lambda^{-1/2}\Psi\Lambda^{-1/2}+t^{-1}\Lambda^{-1})}}\,\mathrm{d}s\mathrm{d}t \\
&= \frac{(-2)^{\lceil p \rceil}}{\det(\Lambda)^{1/2}}\int_0^\infty\int_0^\infty s^{\langle p \rangle-1}u^{r/2-q-1}\frac{\mathrm{d}^{\lceil p \rceil}}{\mathrm{d}s^{\lceil p \rceil}}\frac{1}{\sqrt{\det(I+su\Lambda^{-1/2}\Psi\Lambda^{-1/2}+u\Lambda^{-1})}}\,\mathrm{d}s\mathrm{d}u \\
&= \frac{(-2)^{\lceil p \rceil}}{\det(\Lambda)^{1/2}}\int_0^\infty\int_0^\infty s^{\langle p \rangle-1}u^{r/2-q-1}\frac{\mathrm{d}^{\lceil p \rceil}}{\mathrm{d}s^{\lceil p \rceil}}E\mathrm{e}^{-Z^{\mathrm{T}}(u\Lambda^{-1}+su\Lambda^{-1/2}\Psi\Lambda^{-1/2})Z/2}\,\mathrm{d}s\mathrm{d}u \\
&= \frac{(-2)^{\lceil p \rceil}}{\det(\Lambda)^{1/2}}\int_0^\infty\int_0^\infty v^{\langle p \rangle-1}u^{r/2+\lceil p \rceil+\langle p \rangle-q-1}\frac{\mathrm{d}^{\lceil p \rceil}}{\mathrm{d}v^{\lceil p \rceil}}E\mathrm{e}^{-Z^{\mathrm{T}}(u\Lambda^{-1}+v\Lambda^{-1/2}\Psi\Lambda^{-1/2})Z/2}\,\mathrm{d}v\mathrm{d}u \\
&= \frac{\Gamma(\langle p \rangle)\Gamma(r/2+p-q)}{2^{q-p-r/2}\det(\Lambda)^{1/2}}E\frac{(Z^{\mathrm{T}}\Lambda^{-1/2}\Psi\Lambda^{-1/2}Z)^{\lceil p \rceil+\langle p \rangle}}{(Z^{\mathrm{T}}\Lambda^{-1}Z)^{\lceil p \rceil+\langle p \rangle-q+r/2}}. \qquad \blacksquare
\end{aligned}
$$

The formula derived from (4) when $q = r/4 + p/2$,

$$\det(\Lambda)^{1/2}E\frac{(Z^{\mathrm{T}}\Psi Z)^p}{(Z^{\mathrm{T}}\Lambda Z)^{r/4+p/2}} = E\frac{(Z^{\mathrm{T}}\Lambda^{-1/2}\Psi\Lambda^{-1/2}Z)^p}{(Z^{\mathrm{T}}\Lambda^{-1}Z)^{r/4+p/2}},$$

extends (3).

By using notation (2), identity (4) can be rewritten as

$$\frac{E\frac{(Z^{\mathrm{T}}\Psi Z)^p}{(Z^{\mathrm{T}}\tilde{\Lambda}Z)^q}}{E(Z^{\mathrm{T}}Z)^{p-q}} = \frac{E\frac{(Z^{\mathrm{T}}\tilde{\Lambda}^{-1/2}\Psi\tilde{\Lambda}^{-1/2}Z)^p}{(Z^{\mathrm{T}}\tilde{\Lambda}^{-1}Z)^{p-q+r/2}}}{E(Z^{\mathrm{T}}Z)^{q-r/2}}.$$

When $\Psi = I$, $0 < p = q < r/2$, (4) and (1) give

$$E\left(\frac{Z^{\mathrm{T}}Z}{Z^{\mathrm{T}}\Lambda Z}\right)^p = \frac{\Gamma(r/2)}{2^{p-r/2}\Gamma(p)\det(\Lambda)^{1/2}}E(Z^{\mathrm{T}}\Lambda^{-1}Z)^{p-r/2}$$

$$= \frac{2^p \Gamma(r/2)}{\Gamma(r/2-p)} E(Z^{\mathrm{T}} \Lambda Z)^{-p} = \frac{E(Z^{\mathrm{T}} \Lambda Z)^{-p}}{E(Z^{\mathrm{T}} Z)^{-p}}.$$

For $0 < q < r/2$ and positive p,

$$\frac{E \frac{(Z^{\mathrm{T}} Z)^p}{(Z^{\mathrm{T}} \Lambda Z)^q}}{E(Z^{\mathrm{T}} Z)^{p-q}} = \frac{E(Z^{\mathrm{T}} \Lambda^{-1} Z)^{q-r/2}}{\det(\Lambda)^{1/2} E(Z^{\mathrm{T}} Z)^{q-r/2}} = \frac{2^{2q-r/2} \Gamma(q) E(Z^{\mathrm{T}} \Lambda Z)^{-q}}{\Gamma(r/2-q) E(Z^{\mathrm{T}} Z)^{q-r/2}},$$

so that

$$E \frac{(Z^{\mathrm{T}} Z)^p}{(Z^{\mathrm{T}} \Lambda Z)^q} = \frac{2^p \Gamma(r/2-q+p)}{\Gamma(r/2-q)} E(Z^{\mathrm{T}} \Lambda Z)^{-q}.$$

If $m = r - 2(q-p)$ is a positive integer, then this formula takes the form

$$E \frac{(Z^{\mathrm{T}} Z)^p}{(Z^{\mathrm{T}} \Lambda Z)^q} = \frac{E(Z^{\mathrm{T}} \Lambda Z)^{-q}}{E(U^{\mathrm{T}} U)^{-p}},$$

where U is an m-dimensional random vector with independent standard normal coordinates.

Further formulas of this type showing the close relationship between moments of ratios of quadratic forms and their inverse moments can be obtained from (1) and (4). They could be used for checking the numerical accuracy of different algorithms for evaluation of these moments (Paolella, 2003).

Obviously all results can be reformulated in terms of zero mean normal vectors with an arbitrary covariance matrix Σ, e.g. in Theorem 2.1, Λ has to replaced by $\Sigma^{1/2} \Lambda \Sigma^{1/2}$, and additionally in Theorem 2.2, Ψ is to be taken as $\Sigma^{1/2} \Psi \Sigma^{1/2}$.

Jones (1987), as well as Mathai and Provost (1992), noticed that the function $E(Z^{\mathrm{T}} \Lambda Z)^{-q}$ of $\lambda_1, \ldots, \lambda_r$ can be expressed in terms of a multiple hypergeometric function (Carlson's R-function or Lauricella's function). Formula (1) can be obtained by application of the so-called Euler transformation (Carlson, 1977, Theorem 6.8-3). However the possibility of deriving (4) via this route is not clear.

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