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Allan variance of time series models for measurement data

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Abstract

The uncertainty of the mean of autocorrelated measurements from a stationary process has been discussed in the literature. However, when the measurements are from a non-stationary process, how to assess their uncertainty remains unresolved. Allan variance or two-sample variance has been used in time and frequency metrology for more than three decades as a substitute for the classical variance to characterize the stability of clocks or frequency standards when the underlying process is a $1/f$ noise process. However, its applications are related only to the noise models characterized by the power law of the spectral density. In this paper, from the viewpoint of the time domain, we provide a statistical underpinning of the Allan variance for discrete stationary processes, random walk and long-memory processes such as the fractional difference processes including the noise models usually considered in time and frequency metrology. Results show that the Allan variance is a better measure of the process variation than the classical variance of the random walk and the non-stationary fractional difference processes including the $1/f$ noise.

1. Introduction

In metrology, it is a common practice that the dispersion or standard deviation of the average of repeated measurements is calculated by the sample standard deviation of the measurements divided by the square root of the sample size. When the measurements are autocorrelated, this calculation is not appropriate as pointed out by the *Guide to the Expression of Uncertainty in Measurement (GUM)* [1, 4.2.7]. Thus, appropriate approaches are needed to calculate the corresponding uncertainty. Recently, a practical approach was proposed in [2] to calculate the uncertainty of the mean of autocorrelated measurements when the data are from a stationary process. Related to [2], reference [3] discussed the use of autocorrelation function to characterize time series of voltage measurements for stationary processes. However, as stated in [2], if measurements are from a non-stationary process, using the average value and the corresponding variance to characterize the measurement standard may be misleading.

In time and frequency metrology, the power spectral density has been proposed to measure frequency stability in the frequency domain. As discussed in [4–11], it has been found empirically that random fluctuations in standards can be modelled or at least usefully classified by a power law of the power spectral density given by

$$f(\omega) = \sum_{\alpha=-2}^{2} h_\alpha \omega^\alpha, \tag{1}$$

when ω is small. In (1), $f(\omega)$ is the spectral density at the Fourier frequency ω and the h_αs are intensity coefficients. Among several common noise types classified based on (1) and encounters in practice, a process which has the property that $f(\omega) \sim 1/\omega$ when $\omega \to 0$ is called $1/f$ noise or flicker frequency noise. For such processes, it was concluded that the process variance is infinite while its Allan variance is finite. Recently, [12] and [13] found that in electrical metrology the measurements of Zener-diode voltage standards retain some 'memory' of its previous values and can be modelled as $1/f$ noise. In addition to the spectral density, Allan variance or two-sample variance has been used widely in time and frequency metrology as a substitute for the classical variance to characterize the stability of clocks or frequency standards in the time domain. In [1, 4.2.7], it states that specialized methods such as the Allan variance should be used to treat autocorrelated measurements of frequency standards. However, the stochastic processes studied in time and frequency metrology and in [12] and [13] are characterized by the power law of frequencies as in (1). The Allan variance often gives an impression of being an ad hoc technique. For that we think it is important to provide a stronger statistical underpinning of the Allan variance technique. In this paper, we study the properties of the Allan variance for a wide range of time series including various non-stationary processes. Therefore, some basic concepts of time series models, in particular the stationary processes, are needed.

In the time and frequency literature, discrete observations are implicitly treated as being made on continuous time series. In most cases, the time indices are expressed in units of time such as seconds. In this paper, we consider discrete time series with equally spaced (time) intervals. The time indices here are counts and thus are unitless. Specifically, we consider a discrete weakly stationary process $\{X(t), t = 1, 2, ...\}$. By stationarity, we mean $E[X(t)] = \mu$ (constant) and the covariance between $X(t)$ and $X(t+\tau)$ $(\tau = 0, 1, \ldots)$ is finite and depends only on τ, i.e.

$$r(\tau) = \mathrm{Cov}[X(t), X(t+\tau)].$$

Here τ is the time lag. Again t and τ are unitless. In particular, the process variance is $\sigma_X^2 = r(0)$. The autocorrelation function of $\{X(t)\}$ is $\rho(\tau) = r(\tau)/r(0)$. Obviously, $\rho(0) = 1$. The simplest stationary process is white noise, which has a mean of zero and $\rho(\tau) = 0$ for all $\tau \neq 0$. We define the process $\{Y_n(T), T = 1, 2, \ldots\}$, $n \geqslant 2$ of arithmetic means of n consecutive $X(t)$s $(n > 1)$ or moving averages as

$$Y_n(1) = \frac{X(1) + \cdots + X(n)}{n}$$

$$\cdots\cdots\cdots\cdots$$

$$Y_n(T) = \frac{X((T-1)n+1) + \cdots + X(Tn)}{n}. \quad (2)$$

It is obvious that

$$E[Y_n(T)] = \mu. \quad (3)$$

The autocovariance function of $\{Y_n(T)\}$ can be expressed in terms of the autocovariance of $\{X(t)\}$ and n. When lag $m > 0$, the autocovariance function of $\{Y_n(T)\}$ can be expressed by

$$R_n(m) = \mathrm{Cov}[Y_n(T), Y_n(T+m)]$$
$$= \frac{n \cdot r(mn) + \sum_{i=1}^{n-1} i \cdot [r(mn-n+i) + r(mn+n-i)]}{n^2}. \quad (4)$$

From (3) and (4), it is clear that for any fixed n, $\{Y_n(T), T = 1, 2, \ldots\}$ is also a stationary process. When $m = 0$ and $n \geqslant 2$, the variance of $\{Y_n(T)\}$ is given by

$$\mathrm{Var}[Y_n(T)] = R_n(0)$$
$$= \frac{n \cdot r(0) + 2\sum_{i=1}^{n-1} i \cdot r(n-i)}{n^2}. \quad (5)$$

As shown in [14, p 319] and [15], the variance can also be expressed as

$$\mathrm{Var}[Y_n(T)] = \frac{\sigma_X^2}{n}\left[1 + \frac{2\sum_{i=1}^{n-1}(n-i)\rho(i)}{n}\right] = \mathrm{Var}[\bar{X}], \quad (6)$$

where $\bar{X}$ is the sample mean and $n \geqslant 2$. In particular, when $\{X(t)\}$ is a stationary and uncorrelated process or an independently identically distributed (i.i.d.) sequence, $\mathrm{Var}[Y_n(T)] = \sigma_X^2/n$. This fact has been used in metrology to reduce the standard deviation or uncertainty of the $Y_n(T)$ or $\bar{X}$, using a large sample size n. When $\{X(t)\}$ is autocorrelated, the variance of $Y_n(T)$ or $\bar{X}$ can be calculated from (6) and is used to calculate the uncertainty of the mean of autocorrelated measurements when they are from a stationary process. That is, the uncertainty of $Y_n(T)$ can be calculated by substituting σ_X^2 by the sample variance S_X^2 and substituting $\rho(i)$ by the corresponding sample autocorrelation $\hat{\rho}(i)$ and thus given by

$$u_{Y_n(T)}^2 = \frac{S_X^2}{n}\left[1 + \frac{2\sum_{i=1}^{n_t}(n-i)\hat{\rho}(i)}{n}\right],$$

where n_t is a cut-off lag estimated from the data. See [2]. For example, when $\{X(t)\}$ is a first order autoregressive (AR(1)) process,

$$X(t) - \mu - \phi_1[X(t-1) - \mu] = a(t).$$

The above expression can be rewritten as

$$(1 - \phi_1 B)[X(t) - \mu] = a(t),$$

where B is the back shift operator, i.e. $B[X(t)] = X(t-1)$ and $\{a(t)\}$ is white noise with a variance of σ_a^2. Without loss of generality, we assume that the process means for AR(1) and other stationary processes in this paper are zero. From (6) when $\{X(t)\}$ is a stationary AR(1) process with $|\phi_1| < 1$, in [2] it is shown that

$$\mathrm{Var}[Y_n(T)] = \frac{n - 2\phi_1 - n\phi_1^2 + 2\phi_1^{n+1}}{n^2(1-\phi_1)^2}\sigma_X^2. \quad (7)$$

In this case, the variance of $Y_n(T)$ or $\bar{X}$ still decreases with a rate of $1/n$ when the sample size n increases.

However, for some processes which are not stationary, the variance of $Y_n(T)$ or $\bar{X}$ may not decrease with a rate of $1/n$ or even may not always decrease when n increases. For illustration, we show the behaviour of a time series of a Zener voltage standard measured against a Josephson voltage standard with a nominal value of 10 V. Figure 1 shows time series of the differences of the 8192 voltage measurements in units of microvolt (µV). The length of the time interval between successive measurements is 0.06 s. Figure 2 demonstrates the autocorrelations of the time series for the first 500 lags with an approximate 95% confidence band centred at zero and with limits of $\pm 1.96/\sqrt{n}$ assuming the process is white noise with $n = 8192$. Obviously, the data are autocorrelated and the autocorrelation persists even for lags larger than 250. In figure 3, sample variance of $Y_n(T)$ is plotted against n. Note that when n increases, the sample variance of $Y_n(T)$ decreases quickly when $n < 100$. But the decrease becomes slow when $n > 200$. This means that in this case continued averaging of repeated measurements does not reduce the uncertainty and improve the quality of measurements, as when the measurements are statistically independent. This concern was the main reason for the use

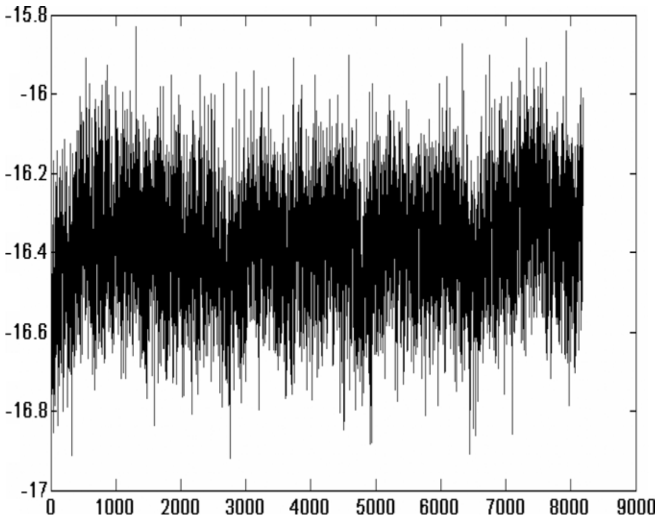

Figure 1. The differences between the measurements from a Zener voltage standard and a Josephson voltage standard in the unit of microvolt.

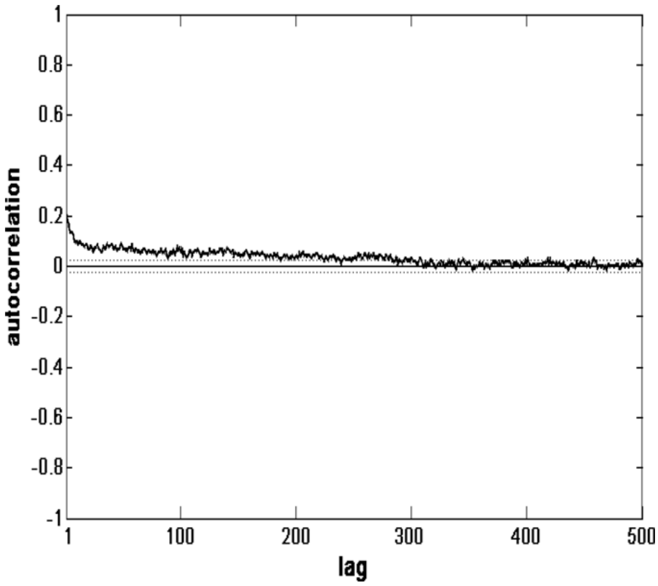

Figure 2. Sample autocorrelation function of the time series of the voltage differences.

of Allan variance in [6] and [16]. In section 2, Allan variance is introduced for a weakly stationary process in general and its properties are studied for autoregressive-moving average (ARMA) processes. Sections 3, 4 and 5 will discuss the Allan variance for random walk, ARIMA(0,1,1) processes and the fractional difference ARFIMA$(0, d, 0)$ processes, including the cases where $d = -0.5$ and 0.5, respectively. The results are extended to ARFIMA (p, d, q) processes in section 5 followed by the summary and conclusions.

2. Allan variance for a weakly stationary process

In [5], [17], [10] and [11], the two-sample variance or Allan variance of $\{X(t)\}$ for the average size of $n \geqslant 2$ is defined as

$$\mathrm{AVar}_n[X(t)] = \frac{E[[Y_n(T) - Y_n(T-1)]^2]}{2}. \quad (8)$$

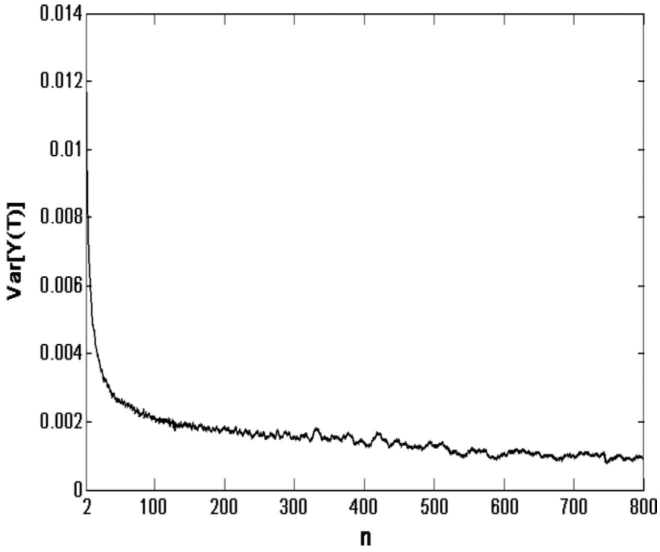

Figure 3. Variance of the moving averages of the time series of voltage differences in the unit of μV^2 as a function of the average size.

When $\{X(t)\}$ is stationary, the Allan variance can be expressed as

$$\begin{aligned}\mathrm{AVar}_n[X(t)] &= \frac{\mathrm{Var}[Y_n(T) - Y_n(T-1)]}{2} \\ &= \mathrm{Var}[Y_n(T)] - R_n(1).\end{aligned} \quad (9)$$

From (4),

$$\begin{aligned}R_n(1) &= \frac{n \cdot r(n) + \sum_{i=1}^{n-1} i \cdot [r(i) + r(2n-i)]}{n^2} \\ &= \frac{\sigma_X^2}{n^2}\left[n \cdot \rho(n) + \sum_{i=1}^{n-1} i \cdot [\rho(i) + \rho(2n-i)]\right].\end{aligned} \quad (10)$$

From (5), (9) and (10),

$$\mathrm{AVar}_n[X(t)] = \frac{n[1-\rho(n)] + \sum_{i=1}^{n-1} i[2\rho(n-i) - \rho(i) - \rho(2n-i)]}{n^2}\sigma_X^2. \quad (11)$$

From (6), (9) and (11), it is clear that both the variance of moving averages and the Allan variance of a stationary process are functions of the autocorrelation function, the size of the average and the variance of the process. From (8), for a given data set of $\{X(1), \ldots, X(N)\}$, the Allan variance for the average size of n is estimated by

$$\widehat{\mathrm{AVar}}_n[X(t)] = \frac{\sum_{i=2}^{m}[Y_n(i) - Y_n(i-1)]^2}{2(m-1)}, \quad (12)$$

where $m = [N/n]$. See [6, 8]. Obviously, this is an unbiased estimator of $\mathrm{AVar}_n[X(t)]$. Reference [18] studies the uncertainty of $\widehat{\mathrm{AVar}}_n[X(t)]$. Denote $Z_n(T) = Y_n(T) - Y_n(T-1)$ for $T = 2, \ldots, m$. Since $\{X(t)\}$ is stationary, $E[Z_n(T)] = 0$. Thus,

$$\widehat{\mathrm{AVar}}_n[X(t)]\frac{S^2_{Z_n(T)}}{2} \quad (13)$$

for large n, where $S^2_{Z_n(T)}$ is the sample variance of $\{Z_n(T)\}$. For the asymptotic distribution of $\widehat{\mathrm{AVar}}_n[X(t)]$, consider a general linear process

$$X(t) = \mu + \sum_{k=-\infty}^{\infty} \gamma_k a(t-k),$$

where $\{a(t)\}$ consists of i.i.d. random variables with $E[a(t)] = 0$ and finite variance. Obviously, $\{Z_n(T)\}$ is a general linear process. When some regularity conditions are met, by a theorem in [19, p 478] it can be shown that for any fixed n the limiting distribution of $\sqrt{N}\{\widehat{\mathrm{AVar}}_n[X(t)] - \mathrm{AVar}_n[X(t)]\}$ when $N \to \infty$ is normal with zero mean and a certain variance.

We now investigate the properties of Allan variance for several stationary time series models in the following subsections.

2.1. i.i.d. sequences

When $\{X(t)\}$ is an i.i.d. sequence or a stationary and uncorrelated process, $\rho(i) = 0$ when $i \neq 0$ and $\rho(0) = 1$. The spectral density $f(\omega) = \sigma_X^2/2\pi$, which is a constant for $-\pi \leqslant \omega \leqslant \pi$. From (10), it is obvious that when $n > 1$, $R_n(1) = 0$. Therefore, from (9),

$$\mathrm{AVar}_n[X(t)] = \mathrm{Var}[Y_n(T)] = \frac{\sigma_X^2}{n}. \tag{14}$$

That is, in the case of an i.i.d. sequence, the Allan variance equals the variance of the sample average.

2.2. MA(q) processes

When $\{X(t)\}$ is an MA(q) process,

$$X(t) = a(t) - \theta_1 a(t-1) - \cdots - \theta_q a(t-q), \tag{15}$$

where $\{a(t)\}$ is white noise with a variance of σ_a^2. Using the backshift operator, the above expression can be rewritten as

$$X(t) = [1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q]a(t).$$

In this paper we assume that the roots of the characteristic equation $\theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q = 0$ lie out of the unit circle. Thus, the MA(q) process is invertible (see [20, pp 86–7]). An MA(q) process has the property that its autocovariance and autocorrelation are zero after lag q. That is, the autocorrelation function at lag m is

$$\rho(m) = \frac{-[\theta_m - \theta_1\theta_{m+1} - \cdots - \theta_{q-m}\theta_q]}{1 + \theta_1^2 + \cdots + \theta_q^2}, \tag{16}$$

when $m = 1, 2, \ldots, (q-1)$ and $\rho(q) = -\theta_q/[1+\theta_1^2+\cdots+\theta_q^2]$. See [21, p 68]. When $m > q$, $\rho(m) = 0$. From (15), it is obvious that

$$\mathrm{Var}[X(t)] = \left(1 + \sum_{i=1}^{q}\theta_i^2\right)\sigma_a^2. \tag{17}$$

Assuming $n \geqslant q + 1$, from (6)

$$\mathrm{Var}[Y_n(T)] = \frac{\sigma_X^2}{n}\left[1 + \frac{2\sum_{i=1}^{q}(n-i)\rho(i)}{n}\right]. \tag{18}$$

In particular, when $q = 1$ from (16),

$$\mathrm{Var}[Y_n(T)] = \frac{\sigma_X^2}{n}\left[1 - \frac{2(n-1)\dfrac{\theta_1}{1+\theta_1^2}}{n}\right]. \tag{19}$$

In general, when $\theta_1 \neq 1$, $\mathrm{Var}[Y_n(T)]$ decreases with a rate of $1/n$ when n increases. When $\theta_1 = 1$, $\mathrm{Var}[Y_n(T)]$ decreases with a rate of $1/n^2$ when n increases.

Now we consider the behaviour of Allan variance when n increases. Under the condition of $n \geqslant q + 1$, from (9) it can be shown that

$$\mathrm{AVar}_n[X(t)] = \frac{n \cdot r(0) + \sum_{i=1}^{q}(2n - 3i)\cdot r(i)}{n^2}. \tag{20}$$

Hence, in general the Allan variance of an MA(q) process decreases to zero when $n \to \infty$ with a rate of $1/n$. In particular, when $q = 1$, i.e. for an MA(1) process,

$$\begin{aligned}\mathrm{AVar}_n[X(t)] &= \frac{n \cdot r(0) + (2n-3)\cdot r(1)}{n^2} \\ &= \frac{n + (2n-3)\rho(1)}{n^2}\sigma_X^2,\end{aligned} \tag{21}$$

which can also be expressed as

$$\mathrm{AVar}_n[X(t)] = (1+\theta_1^2)\frac{n - (2n-3)\dfrac{\theta_1}{1+\theta_1^2}}{n^2}\sigma_a^2. \tag{22}$$

The spectral density of an MA(1) process is given by (see [21, p 69])

$$f(\omega) = \frac{(1 - 2\theta_1\cos\omega + \theta_1^2)\sigma_a^2}{2\pi} \tag{23}$$

for $-\pi \leqslant \omega \leqslant \pi$. In particular, an MA(1) process with $\theta_1 = 1$ is not invertible and is sometimes called white phase noise (see table 1 in [22]). In this case, from (19)

$$\mathrm{Var}[Y_n(T)] = \frac{2\sigma_a^2}{n^2}. \tag{24}$$

From (21),

$$\mathrm{AVar}_n[X(t)] = \frac{3\sigma_a^2}{n^2}. \tag{25}$$

Thus, the variance of the average of a white phase noise process and its Allan variance decrease with a rate of $1/n^2$, which is faster than those for white noise. The corresponding spectral density is

$$f(\omega) = \frac{2\sin^2\dfrac{\omega}{2}}{\pi}\sigma_a^2 \tag{26}$$

for $-\pi \leqslant \omega \leqslant \pi$. Obviously, when $\omega \to 0$, $f(\omega) \sim \omega^2$.

2.3. AR(1) processes

When $\{X(t)\}$ is an AR(1) process with the parameter of ϕ_1,

$$X(t) - \phi_1 X(t-1) = a(t). \quad (27)$$

We assume that $|\phi_1| < 1$, which guarantees the stationarity of the process. The process $\{a(t)\}$ is white noise with a variance of σ_a^2. In this case, $\rho(i) = \phi_1^i$ for $i = 1, 2, \ldots$ and the spectral density from [20, p 123], is given by

$$f(\omega) = \frac{\sigma_a^2}{2\pi(1 - 2\phi_1 \cdot \cos\omega + \phi_1^2)} \quad (28)$$

when $-\pi \leqslant \omega \leqslant \pi$. When $\omega \to 0$,

$$f(\omega) \to \frac{\sigma_a^2}{2\pi(1-\phi_1)^2}.$$

From (10),

$$R_n(1) = \frac{\phi_1^{2n+1} - 2\phi_1^{n+1} + \phi_1}{n^2(1-\phi_1)^2}\sigma_X^2. \quad (29)$$

From (7) and (9), the Allan variance for the AR(1) process is

$$\begin{aligned} \mathrm{AVar}_n[X(t)] &= \frac{n - 3\phi_1 - n\phi_1^2 + 4\phi_1^{n+1} - \phi_1^{2n+1}}{n^2(1-\phi_1)^2}\sigma_X^2 \\ &= \frac{n - 3\phi_1 - n\phi_1^2 + 4\phi_1^{n+1} - \phi_1^{2n+1}}{n^2(1-\phi_1)^2(1-\phi_1^2)}\sigma_a^2 \end{aligned} \quad (30)$$

since $\sigma_X^2 = \sigma_a^2/(1-\phi_1^2)$. Note that when $\phi_1 = 0$, $X(t) = a(t)$, which is a white noise process, then (30) becomes (14). We now consider the case of the values of the Allan variance when n increases. From (30), when $n \to \infty$,

$$\mathrm{AVar}_n[X(t)] \sim \frac{\sigma_a^2}{n(1-\phi_1)^2}. \quad (31)$$

Namely, for a fixed $|\phi_1| < 1$, the Allan variance approaches zeros with a rate of $1/n$ when $n \to \infty$.

2.4. AR(p) processes

We now consider an AR(p) process,

$$X(t) - \phi_1 X(t-1) - \cdots - \phi_p X(t-p) = a(t). \quad (32)$$

We assume that the process is stationary, which means that all the roots of its characteristic equation, i.e. $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0$, are out of the unit circle. Without loss of generality, we assume that the roots are real and distinct. The autocorrelation function can be expressed as

$$\rho(k) = c_1 v_1^{-k} + c_2 v_2^{-k} + \cdots + c_p v_p^{-k}, \quad (33)$$

where $v_i (i = 1, \ldots, p)$ are the roots of characteristic equation and $c_i (i = 1, \ldots, p)$ are constants (see [20, p 94]). $|v_i| > 1$ for $i = 1, \ldots, p$. Similar to the case of AR(1), from (6)

$$\mathrm{Var}[Y_n(T)] = \frac{\sigma_X^2}{n}\left[1 + 2\sum_{j=1}^{p} c_j v_j^{-1} \frac{n - 1 - n v_j^{-1} + v_j^{-n}}{n(1-v_j^{-1})^2}\right]$$

while from (10)

$$\begin{aligned} R_n(1) = \frac{\sigma_X^2}{n^2}\Bigg[& n\sum_{j=1}^{p} c_j v_j^{-n} \\ & + \sum_{j=1}^{p} c_j v_j^{-1} \frac{1 - n v_j^{-(n-1)} + (n-1)v_j^{-n}}{(1-v_j^{-1})^2} \\ & + \sum_{j=1}^{p} c_j v_j^{1-2n} \frac{1 - n v_j^{n-1} + (n-1)v_j^{n}}{(1-v_j)^2}\Bigg]. \end{aligned}$$

From (9), similar to AR(1) the Allan variance of a stationary AR(p) process approaches zero with a rate of $1/n$ when n increases.

2.5. ARMA(p, q) processes

An ARMA(p, q) process is a combination of AR and MA processes. If $\{a(t)\}$ is white noise, the ARMA(p, q) process is defined by

$$\begin{aligned} X(t) &- \phi_1 X(t-1) - \cdots - \phi_p X(t-p) \\ &= a(t) - \theta_1(t-1) - \cdots - \theta_q(t-q) \end{aligned} \quad (34)$$

assuming the process mean is zero. We assume that the process is stationary and the roots of the characteristic equation of the process are distinct and also assume that the process is invertible. Then, for the autocorrelation of $\{X(t)\}$ when lag $k \geqslant \max(p, q+1) - p$, (33) holds (see [20, pp 91–3]). Thus, when n is large enough, a result similar to AR(p) processes holds. Namely, the Allan variance of a stationary and invertible ARMA(p, q) process approaches zero with a rate of $1/n$ when n increases. In particular, for an ARMA(1,1) process,

$$X(t) - \phi_1 X(t-1) = a(t) - \theta_1(t-1)$$

from [21, pp 76–7]

$$\rho(1) = \frac{(1-\phi_1\theta_1)(\phi_1 - \theta_1)}{1 + \theta_1^2 - 2\phi_1\theta_1} \qquad \text{and}$$

$$\rho(k) = \rho(1)\phi_1^{k-1} \quad (35)$$

for $k \geqslant 1$ and

$$\mathrm{Var}[X(t)] = \frac{1 - 2\theta_1\phi_1 + \theta_1^2}{1 - \phi_1^2}\sigma_a^2. \quad (36)$$

The spectral density is given by

$$f(\omega) = \frac{1 - 2\theta_1\cos\omega + \theta_1^2}{1 - 2\phi_1\cos\omega + \phi_1^2}\frac{\sigma_a^2}{2\pi}. \quad (37)$$

By (11), the Allan variance for an ARMA(1,1) process can be expressed as

$$\begin{aligned} \mathrm{AVar}_n[X(t)] = &\left[n(1-\phi_1)^2(1 - 2\phi_1\theta_1 + \theta_1^2)\right. \\ &\left. + (\phi_1 - \theta_1)(1 - \phi_1\theta_1)(2n - 3 - 2n\phi_1 + 4\phi_1^n - \phi_1^{2n})\right] \\ &\times \left[n^2(1-\phi_1)^2(1-\phi_1^2)\right]^{-1}\sigma_a^2. \end{aligned} \quad (38)$$

From (38), it is obvious that the Allan variance of an ARMA(1,1) process approaches zero with a rate of $1/n$ when n increases.

In this section, we show that for stationary and invertible ARMA processes, the variance of the sample average and the Allan variance have the same convergent rate of $1/n$ when $n \to \infty$. Therefore, they are similar uncertainty measures for measurements from a stationary ARMA process.

3. Random walk and ARIMA(0,1,1) processes

3.1. Random walk

A random walk process $\{X(t), t = 0, 1, \ldots\}$ is defined as follows (see [20, p 10]): $X(0) = 0$ and

$$X(t) = X(t-1) + a(t) \tag{39}$$

for $t > 1$, where $\{a(t)\}$ is white noise with variance of σ_a^2. From [2],

$$\mathrm{Var}[X(t)] = t \cdot \sigma_a^2. \tag{40}$$

Since

$$Y_n(T) = \frac{X((T-1)n+1) + \cdots + X(Tn)}{n},$$

from (39)

$$\mathrm{Var}[Y_n(T)] = \frac{\sigma_a^2}{n} \frac{2n^2(3T-2) + 3n + 1}{6}. \tag{41}$$

Thus, the variance of $Y_n(T)$ depends on T and approaches infinity with a rate of n when n increases. Although a random walk is not stationary, we can treat it as a limiting case of an AR(1) process (with $X(0) = 0$) when $\phi \to 1$. From (28), the limit of spectral density is

$$f(\omega) \to \frac{\sigma_a^2}{4\pi(1-\cos\omega)} = \frac{\sigma_a^2}{8\pi \cdot \sin^2 \dfrac{\omega}{2}} \tag{42}$$

for $-\pi \leqslant \omega \leqslant \pi$ although it is in the sense of non-Wienran spectral theory as indicated by [23]. It is clearly that when $\omega \to 0$, $f(\omega)$ increases and approaches infinity with a rate of

$$f(\omega) \sim \frac{1}{\omega^2}, \tag{43}$$

which is consistent with the results in the frequency domain [7]. The Allan variance of a random walk $\{X(t)\}$ is given by

$$\mathrm{AVar}_n[X(t)] = \frac{2n^2+1}{6n}\sigma_a^2. \tag{44}$$

The derivation is in appendix A. Note that the Allan variance of a random walk is independent of the time index T and depends on n only. In that sense, it is a better measure than the variance of the moving averages. Obviously, the Allan variance of random walk increases and approaches infinity with a rate of n when $n \to \infty$. Alternatively, (44) can also be obtained as a limit from (30). For an AR(1) process with $\phi_1 = 1$ or an ARIMA(0,1,0) and $X(0) = 0$, it is a random walk. In (30), the Allan variance for an AR(1) process for fixed n has a limit when $\phi_1 \to 1$. The limit is obtained by using L'Hospital Rule three times, i.e. when $\{X(t)\}$ is an AR(1) process,

$$\lim_{\phi\to 1} \mathrm{AVar}_n[X(t)] = \frac{2n^2+1}{6n}\sigma_a^2. \tag{45}$$

3.2. ARIMA(0,1,1) processes

An ARIMA (0,1,1) process, also called an integrated moving average (IMA(1,1)) process, is a non-stationary process:

$$X(t) - X(t-1) = a(t) - \theta_1 a(t-1), \tag{46}$$

where $\{a(t)\}$ is white noise with a variance of σ_a^2. Treating it as a limiting case of an ARMA(1,1) process when $\phi_1 \to 1$, from (36), the process has an infinity variance. From (37), when $\phi_1 \to 1$ the limit of spectral density is

$$f(\omega) \to \frac{(1 - 2\theta_1\cos\omega + \theta_1^2)\sigma_a^2}{8\pi\sin^2\left(\dfrac{\omega}{2}\right)} \tag{47}$$

for $-\pi \leqslant \omega \leqslant \pi$, which is similar to that for a random walk shown in (43). That is, when $\omega \to 0$, $f(\omega) \sim 1/\omega^2$. The Allan variance of $\{X(t)\}$ can be calculated as a limit of that for an ARMA(1,1) process when $\phi_1 \to 1$ similar to (45) and is given by

$$\mathrm{AVar}_n[X(t)] = \frac{(2n^2+1)(1+\theta_1^2) - 4\theta_1(n^2-1)}{6n}\sigma_a^2. \tag{48}$$

Thus, the Allan variance of an ARIMA(0,1,1) approaches infinity with the same rate of a random walk when $n \to \infty$.

4. Fractional difference ARIMA (ARFIMA) (0, *d*, 0) processes

4.1. Stationary ARFIMA (0, d, 0) process

The fractional difference model was proposed by [24] and [25]. In particular, the ARFIMA(0,d,0) is defined as

$$(1-B)^d X(t) = a(t), \tag{49}$$

where $\{a(t)\}$ is white noise and d can be any real number. By a binomial series the fractional difference $(1-B)^d$ is defined as

$$(1-B)^d = 1 - \mathrm{d}B - \frac{d(1-d)}{2}B^2 - \frac{d(1-d)(2-d)}{6}B^3 - \cdots.$$

When $d = 1$ and $X(0) = 0$, it is a random walk. When d is not an integer, $\{X(t)\}$ is called a fractional difference process. When $d < 0.5$, the process is stationary and when $d > -0.5$ it is invertible [25]. When $d < 0.5$, reference [25] shows that the spectral density of a stationary $\{X(t)\}$ can be written as

$$f(\omega) = \frac{\sigma_a^2}{2\pi\left(2\sin\dfrac{\omega}{2}\right)^{2d}} \tag{50}$$

for $-\pi \leqslant \omega \leqslant \pi$. When $\omega \to 0$

$$f(\omega) \sim \omega^{-2d}.$$

Thus, it is a long-memory process. Reference [25] shows that the autocovariance function at lag k of $\{X(t)\}$ can be expressed by gamma functions as

$$r(k) = \frac{(-1)^k \Gamma(1-2d)}{\Gamma(k-d+1)\cdot\Gamma(1-k-d)}\sigma_a^2 \qquad (51)$$

for $k = 1, 2, \ldots$. In particular, the variance of $\{X(t)\}$ is

$$\mathrm{Var}[X(t)] = \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\sigma_a^2. \qquad (52)$$

Obviously, when $d \to 0.5$ $\mathrm{Var}[X(t)] = \sigma_X^2 \to \infty$. The autocorrelation function of $\{X(t)\}$ is given by

$$\rho(k) = \frac{d\cdot(1+d)\ldots(k-1+d)}{(1-d)\cdot(2-d)\ldots(k-d)} = \frac{\Gamma(1-d)\Gamma(k+d)}{\Gamma(d)\Gamma(k+1-d)} \qquad (53)$$

for $k = 1, 2, \ldots$. As shown in [25], when $-0.5 < d < 0.5$ and $k \to \infty$, the autocorrelation of $\{X(t)\}$ has the following property:

$$\rho(k) \sim \frac{\Gamma(1-d)}{\Gamma(d)}k^{2d-1}. \qquad (54)$$

From (6),

$$\mathrm{Var}[Y_n(T)] = \left[1 + \frac{2\sum_{i=1}^{n-1}(n-i)\rho(i)}{n}\right]\frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\frac{\sigma_a^2}{n}. \qquad (55)$$

When $n \geqslant 2$, from (11)

$$\mathrm{AVar}_n[X(t)] = \frac{n[(r(0)-r(n)] + \sum_{i=1}^{n-1} i[2r(n-i) - r(i) - r(2n-i)]}{n^2}. \qquad (56)$$

By (52) and (56), the Allan variance can be expressed as

$$\mathrm{AVar}_n[X(t)] = \frac{n[1-\rho(n)] + \sum_{i=1}^{n-1} i[2\rho(n-i) - \rho(i) - \rho(2n-i)]}{n^2} \times \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\sigma_a^2. \qquad (57)$$

Thus, by (53) and (57) for given d and n, the Allan variance for a stationary fractional difference process can be calculated.

4.2. ARFIMA$(0, d, 0)$ processes with $d = -0.5$

As shown in [25], when $d > -0.5$, an ARFIMA$(0, d, 0)$ is invertible. In particular, when $d = -0.5$, this is called flicker phase noise (see table 1 in [22]). The process is stationary but not invertible as described in [25]. From (50),

$$f(\omega) = \frac{\sin\dfrac{\omega}{2}}{\pi}\sigma_a^2 \qquad (58)$$

for $-\pi \leqslant \omega \leqslant \pi$. Obviously, when $\omega \to 0$, $f(\omega) \sim \omega$. From (53)

$$\rho(k) = -\frac{0.25}{k^2 - 0.25}. \qquad (59)$$

Thus, $\rho(k) < 0$ when $k > 1$ and $\rho(k) \sim k^{-2}$ when $k \to \infty$. It is shown in appendix B that when $n \to \infty$

$$\mathrm{Var}[Y_n(T)] \sim \frac{\ln n}{n^2}, \qquad (60)$$

which is faster than the rate of $1/n$.

Similar to the argument for the rate of convergence for the variance of $Y_n(T)$, it is also shown in appendix B that when $d = -0.5$ and n approaches infinity, the Allan variance will go to zero with a rate of $\ln n/n^2$, i.e.

$$\mathrm{AVar}_n[X(T)] \sim \frac{\ln n}{n^2}. \qquad (61)$$

4.3. ARFIMA$(0, d, 0)$ processes with $d = 0.5$

We now consider the fractional difference process when $d = 0.5$. First we know that in this case, the process has long-memory and is not stationary. From (50), as a limit, the spectral density of the process is given by

$$f(\omega) = \frac{\sigma_a^2}{4\pi \sin\dfrac{\omega}{2}} \sim \omega^{-1} \qquad (62)$$

as $\omega \to 0$. Because of the limiting behaviour of the spectral density this process is a flicker frequency process (see table 1 in [22]) or a discrete $1/f$ noise as indicated in [25]. We re-express (6) as

$$\mathrm{Var}[Y_n(T)] = \frac{1}{n}\left[1 + \frac{2\sum_{i=1}^{n-1}(n-i)\rho(i)}{n}\right]\sigma_X^2 \triangleq b(n,d)\sigma_X^2, \qquad (63)$$

where

$$b(n,d) = \frac{1}{n}\left[1 + \frac{2\sum_{i=1}^{n-1}(n-i)\rho(i)}{n}\right] \qquad (64)$$

and σ_X^2 is given by (52). As shown in [24] when $d = 0.5$, σ_X^2 is infinite, and this is the case in (52) as $d \to 0.5$. For a fixed n, when $d \to 0.5$, $\rho(k) \to 1$ by (53). Hence,

$$\lim_{d\to 0.5} b(n,d) = \frac{1}{n}\left[1 + \frac{2\sum_{i=1}^{n-1}(n-i)}{n}\right] = 1. \qquad (65)$$

Thus, when $d \to 0.5$, $\mathrm{Var}[Y_n(T)]/\mathrm{Var}[X(t)] \to 1$ for any fixed n. That is, when $d \to 0.5$, $\mathrm{Var}[Y_n(T)]$ and $\mathrm{Var}[X(t)]$ go to infinity with a same rate. Hence, $\mathrm{Var}[Y_n(T)]$ is not a good measure of uncertainty of ARFIMA(0,0.5,0). From (52), (57) can be written as

$$\mathrm{AVar}_n[X(t)] = C(n,d)\cdot \mathrm{Var}[X(t)],$$

Table 1. Allan variance for ARFIMA(0, d, 0) in the units of σ_a^2.

	$n = 2$	40	100	200	400	800
$d = 0.49$	0.5078	0.4151	0.4072	0.4016	0.3960	0.3906
0.499	0.5091	0.4389	0.4378	0.4371	0.4365	0.4359
0.499 9	0.5093	0.4414	0.4410	0.4409	0.4408	0.4407
0.499 99	0.5093	0.4416	0.4413	0.4413	0.4412	0.4412
0.499 999	0.5093	0.4417	0.4413	0.4413	0.4413	0.4413

where

$$C(n,d) = \frac{n[1-\rho(n)] + \sum_{i=1}^{n-1} i[2\rho(n-i) - \rho(i) - \rho(2n-i)]}{n^2}.$$

Note that all $\rho(i)$s in $C(n, d)$ are functions of d. Thus, the Allan variance is a function of d as well as n. For a fixed n, the Allan variance for an ARFIMA(0,0.5,0) may be obtained by letting $d \to 0.5$ in (56). Note that in (57) when $d \to 0.5$, $\Gamma(1-2d) \to \infty$. In addition, for a fixed n, when $d \to 0.5$ the numerator in the first term on the right-hand side of (57)

$$n[1-\rho(n)] + \sum_{i=1}^{n-1} i[2\rho(n-i) - \rho(i) - \rho(2n-i)] \to 0$$

since $\rho(k) \to 1$ when $d \to 0.5$ for $k = 1, 2, \ldots$ from (53). For a fixed n, for example, $n = 2$, analytically, it seems intractable to find the limit of Allan variance in (57) when $d \to 0.5$. Numerically, for various n the Allan variance of ARFIMA(0, d, 0) can be calculated from (57) for those ds which are very close to 0.5. Table 1 lists the Allan variance in the unit of σ_a^2 for various d close to 0.5 and n. For example, when $n = 2$, the limit seems equal to $0.5093 \cdot \sigma_a^2$ when $d \to 0.5$. In general, we found that for a fixed n, the Allan variance increases when $d \to 0.5$. On the other hand, for a fixed d, the Allan variance decreases when $n \to \infty$, which is reasonable. The computation results show that when $d \geqslant 0.499\,99$ and $n \geqslant 100$, the Allan variance stabilized at the value of $0.4413 \cdot \sigma_a^2$, which is close to $(2\ln 2/\pi)\sigma_a^2$. In summary, when $n \geqslant 100$ and $d \to 0.5$, $\text{AVar}_n[X(T)]$ approaches $(2\ln 2/\pi)\sigma_a^2$. This property characterizes the ARFIMA(0,0.5,0) process and to a great extent is consistent with the Allan variance for the $1/f$ noise in the frequent domain and listed in table 1 in [11] and the observations by [13], where the property of a stabilized Allan variance was exploited to test whether the measurements of the Zener-diode standards is $1/f$ noise. As pointed out above that the variance of a moving average of ARFIMA(0,d,0) will approach infinity as $d \to 0.5$, it is clear that using Allan variance has an advantage over using the variance of the moving average. Figure 4 shows the behaviours of Allan variance for AR(1) with $\phi_1 = 0.75$ and 0.9, ARFIMA(0,0.499,0), ARFIMA(0,0.25,0), ARFIMA(0,−0.5,0) and MA(1) with $\theta_1 = 1$ as functions of n. For these processes, we set the white noise variance $\sigma_a^2 = 1$. The Allan variances are calculated by (25), (30) and (57), respectively. The figure shows that the Allan variance of a stationary AR(1) increases first when n increases then decreases and approaches zero. It also shows that the Allan variance of ARFIMA(0,0.499,0) with $\sigma_a^2 = 1$, which is close

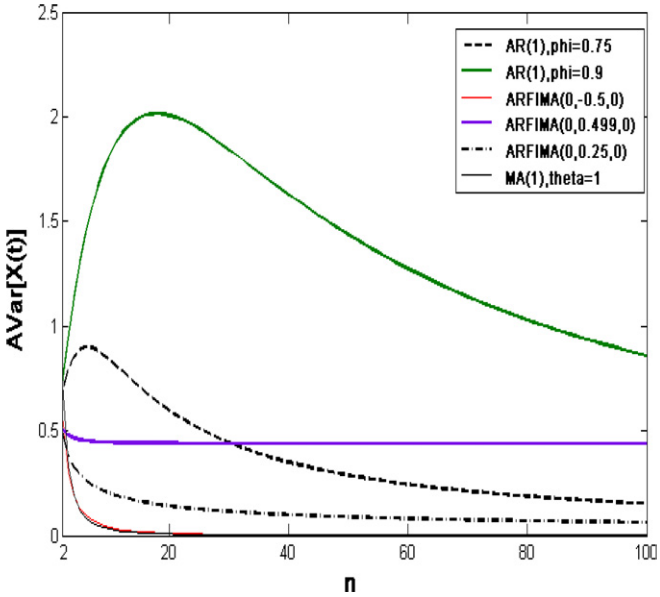

Figure 4. Allan variance as a function of the average size for various processes.
(This figure is in colour only in the electronic version)

to the $1/f$ noise, is almost at the level of 0.44 when $n > 20$. As noted in [11], for $1/f$ noise the increase in n will not reduce the uncertainty. The figure demonstrates that the convergent rates of Allan variance for stationary fractional difference processes are slower than that of stationary AR processes when $0 \leqslant d < 0.5$ and vice versa when $-0.5 \leqslant d < 0$. Figure 4 also shows that the convergent rate of Allan variance for the ARFIMA(0,−0.5,0) (i.e. flicker phase noise) is slower than that of the MA(1) process with $\theta_1 = 1$ (i.e. white phase noise). This is certainly true from (25) and (61). However, when n becomes larger, the differences become smaller. In time and frequency metrology, the relatively poor discrimination against white and flicker phase noises prompted the development of the modified Allan variance. See [6, 26].

5. ARFIMA (p, d, q) processes

The fractional difference process ARFIMA(0,d,0) was extended to ARFIMA(p, d, q) by [25]. $\{X(t), t = 1, 2, \ldots\}$ is said to be an ARFIMA(p, d, q) process if $\{X(t)\}$ satisfies the following difference equation,

$$\phi(B)(1-B)^d X(t) = \theta(B)a(t), \qquad (66)$$

where $\phi(B)$ is for the AR part and $\theta(B)$ is for the MA part and $\{a(t)\}$ is white noise with a variance of σ_a^2. The properties

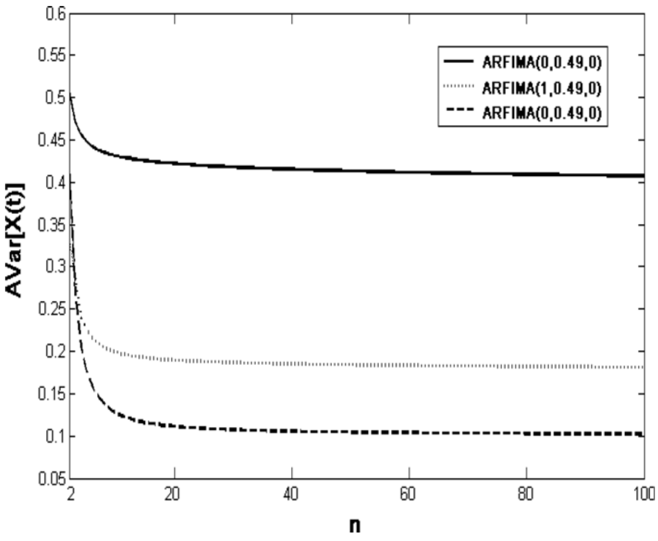

Figure 5. Allan variance for ARFIMA(0,0.49,0), ARFIMA(1,0.49,0) and ARFIMA(0,0.49,1).

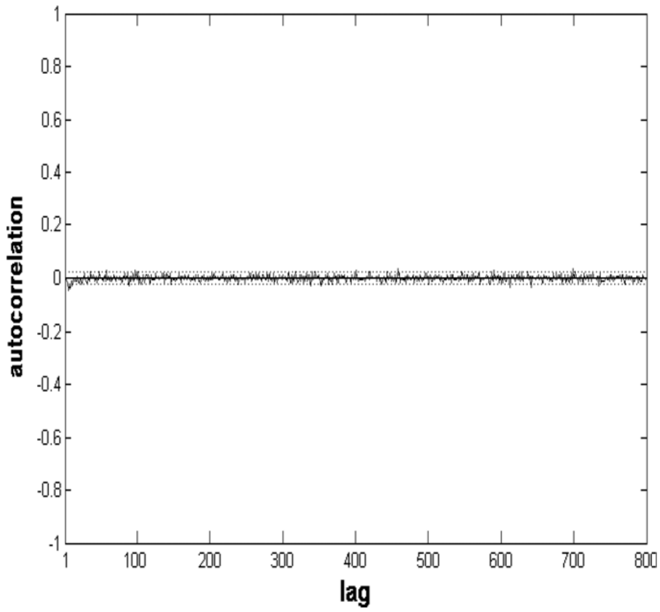

Figure 6. Sample autocorrelation of the residuals of the time series of voltage differences after a fit of ARFIMA(5,0.497,0).

of an ARFIMA(p, d, q) process are similar to those of an ARFIMA(0, d, 0) process. Theorem 2 in [25] showed that when $|d| < 0.5$ and all the roots of the characteristic equations $\phi(z) = 0$ and $\theta(z) = 0$ lie outside the unit circle, $\{X(t)\}$ is stationary and invertible. It also shows that the process is a long-memory process since the autocorrelation and spectral density satisfy

(a) $\lim k^{1-2d}\rho(k)$ exists as $k \to \infty$ and is finite,
(b) $\lim \omega^{2d} f(\omega)$ exists as $\omega \to 0$ and is finite.

(a) in the above gives the convergent rate of $\rho(k)$ for $\{X(t)\}$, which is the same for ARFIMA(0,d,0) given in (54). Let $W(t) = \{\theta(B)\}^{-1}\phi(B)X(t)$ leading to $(1-B)^d W(t) = a(t)$. Thus $\{W(t)\} \sim$ ARFIMA$(0, d, 0)$. Therefore, an ARFIMA(p, d, q) process can be treated as an ARMA(p, q) process with a noise of an ARFIMA(0,d,0) process. In that theorem, the relationship between the autocovariances of $\{X(t)\}$ and $\{W(t)\}$ was established, which were demonstrated in lemmas 1 and 2 in [25] for ARFIMA$(1, d, 0)$ and ARFIMA$(0, d, 1)$. When $\{X(t)\} \sim$ ARFIMA$(1, d, 0)$, $|d| < 0.5$ and $|\phi_1| < 1$,

$$\rho_X(k) = \rho_W(k) \times \frac{F(1, d+k; 1-d+k; \phi_1) + F(1, d-k; 1-d-k; \phi_1) - 1}{(1-\phi_1)F(1, 1+d; 1-d; \phi_1)} \tag{67}$$

and

$$\mathrm{Var}[X(t)] = \frac{F(1, 1+d; 1-d; \phi_1)}{1+\phi_1} \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\sigma_a^2, \tag{68}$$

where F is the hypergeometric function. See [27, p 361]. When$\{X(t)\}$ is an ARFIMA$(0, d, 1)$, similarly,

$$\rho_X(k) = \rho_W(k)\frac{ak^2 - (1-d)^2}{k^2 - (1-d)^2} \tag{69}$$

and

$$\mathrm{Var}[X(t)] = \left[1 + \theta_1^2 - \frac{2\theta_1 d}{1-d}\right] \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\sigma_a^2, \tag{70}$$

where $a = (1-\theta_1)^2/\{1+\theta_1^2 - 2\theta_1 d/(1-d)\}$. Thus from (57), (67) and (70) for given d and n, the Allan variance for stationary ARFIMA$(1, d, 0)$ and ARFIMA$(0, d, 1)$ can be calculated. In figure 5, the Allan variances for ARFIMA(1,0.49,0) with $\phi_1 = -0.5$ and ARFIMA(0,0.49,1) with $\theta_1 = 0.5$ as well as the Allan variance for ARFIMA(0,0.49,0) for $\sigma_a^2 = 1$ and n from 2 to 100 are plotted. It is clear that when $n > 20$, the curves for the Allan variances of the three processes are almost parallel and flat, but at different levels. As demonstrated in figure 5, for ARFIMA (1,0.49,0) with $\phi_1 = -0.5$, its Allan variance is stabilized at the level of 0.18, which is different from 0.40 for ARFIMA (0,0.49,0) while for ARFIMA(0,0.49,1) with $\theta_1 = 0.5$, its Allan variance is stabilized at the level of 0.10. Not shown in the figure, for ARFIMA(1,0.49,0) with $\phi_1 = 0.5$, its Allan variance is stabilized at the level of 1.64 while for ARFIMA(0,0.49,1) with $\theta_1 = -0.5$, its Allan variance is stabilized at the level of 0.92. These stabilized levels are similar to the $1/f$ noise floors for the Zener voltage measurements as discussed in [12, 13].

In addition, when $d \to 0.5$, it can be shown that for an ARFIMA$(1, d, 0)$ with $|\phi_1| < 1$ the ratio between $\rho_X(k)$ and $\rho_W(k)$ approaches 1. Namely,

$$\frac{\rho_X(k)}{\rho_W(k)} \to 1. \tag{71}$$

In addition, when $d \to 0.5$

$$\frac{\mathrm{Var}[Y_n(T)]}{\mathrm{Var}[X(t)]} \to 1 \tag{72}$$

and

$$\frac{\mathrm{AVar}_n[X(t)]}{\mathrm{AVar}_n[W(t)]} \to \frac{1}{(1-\phi_1)^2}, \tag{73}$$

where $W(t) = X(t) - \phi_1 X(t-1)$ is the corresponding ARFIMA$(0, d, 0)$. Thus, from section 4.3, when n is large and $d \to 0.5$, we believe

$$\mathrm{AVar}_n[X(t)] \to \frac{(2\ln 2/\pi)}{(1-\phi_1)^2}\sigma_a^2.$$

For an ARFIMA(0, d, 1) process with $|\theta_1| < 1$, corresponding to (73), when $d \to 0.5$, (71) and (72) hold and

$$\frac{\mathrm{Var}[X(t)]}{\mathrm{Var}[W(t)]} \to (1-\theta_1)^2, \quad (74)$$

where $\{W(t)\}$ is the corresponding ARFIMA(0, d, 0) process. Similarly, it has been shown that for ARFIMA(0, d, 1),

$$\frac{\mathrm{AVar}_n[X(t)]}{\mathrm{AVar}_n[W(t)]} \to (1-\theta_1)^2. \quad (75)$$

Thus, from section 4.3, when n is large and $d \to 0.5$, we believe

$$\mathrm{AVar}_n[X(t)] \to (2\ln 2/\pi)(1-\theta_1^2)\sigma_a^2.$$

The derivations of (71)–(75) can be found in appendix C.

For a general ARFIMA(p, d, q) process, it is expected that the properties for ARFIMA(1, d, 0) and ARFIMA(0, d, 1) will hold for the general case and the Allan variance will approach a constant when $d \to 0.5$.

We now return to the example of the time series of the differences in voltage measurements presented in the introduction. Using time series software [28], a model of ARFIMA(5,0.497,0),

$$(1-B)^{0.497}[X(t) + 0.35X(t-1) + 0.2X(t-2) + 0.1X(t-3) + 0.1X(t-4) + 0.06X(t-5) + 29.98] = a(t)$$

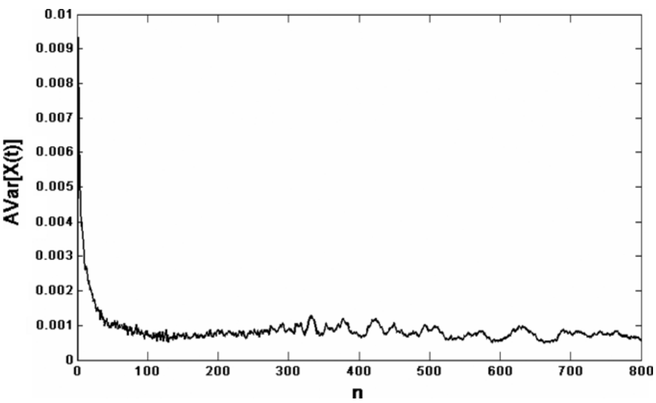

Figure 7. Allan variances in the unit of μV^2 for the time series of voltage differences plotted in figure 1.

with $\hat{\sigma}_a^2 = 0.02$, was found to be a good fit to this data set. Note that all ϕ_is ($i = 1, \ldots, 5$) are negative. Figure 6 demonstrates the sample autocorrelation of the residuals indicating a good fit from the ARFIMA(5,0.497,0) model. By (12) the sample Allan variance was calculated based on the data. Thus, this data set can be treated from an ARFIMA(5,0.5,0) or an AR(5) process with an ARFIMA(0,0.5,0) noise, which is a $1/f$ noise process. Figure 7 shows the Allan variance against the average size. It is clear that the Allan variance decreases slowly and becomes stabilized at the level of 0.0006 μV^2 when $n > 100$, which can be treated as a $1/f$ noise floor.

6. Summary and conclusions

As a summary, table 2 lists the limiting behaviours of spectral densities and Allan variances for various time series models discussed in this paper. The table includes the five common types of noise in time and frequency metrology as listed in [4, 6]. It is clear that for these time series the results are consistent with those in [4, 6].

We have shown that the variance of moving averages and the Allan variance of a stationary ARMA process and a stationary fractional difference ARMA process are closely related. They decrease with a same rate when the size of the average increases. For a random walk process, which is a non-stationary process, or in general an ARIMA(0,1,1) process the variance of the moving averages of it will go to infinity when the size of the averages increases. While the Allan variance of random walk also approaches infinity with the same rate for the variance of the moving averages when the size of the averages increases, it is independent of the time index. For a non-stationary fractional ARFIMA(0, d, 0) process as well as an ARFIMA(1, d, 0) or an ARFIMA(0, d, 1) process when $d \to 0.5$, which is a $1/f$ noise process, we have demonstrated that their Allan variances are stabilized at certain levels when the size of the average is large enough while the variances of the moving averages will approach infinity. We expect that this property holds for a general ARFIMA(p, 0.5, q) when the corresponding AR part is stationary and the MA part is invertible.

We conclude that the Allan variance is a measure of uncertainty similar to the variance of moving averages for the measurements from stationary processes. However, for the measurements from a random walk and in general non-stationary ARFIMA(p, 0.5, q) processes the Allan variance is stabilized when the size of the average increases and it is thus

Table 2. Limiting behaviours of spectral densities and Allan variances for various time series.

Time series	$f(\omega)$	$\mathrm{AVar}_n[X(t)]$
White noise	$\sigma_X^2/2\pi$ for all ω	σ_X^2/n
Stationary ARMA(1,1)	$[(1-\theta_1)^2/(1-\phi_1)^2]\sigma_a^2/2\pi$ when $\omega \to 0$	$\sim 1/n$ when $n \to \infty$
MA(1) with $\theta_1 = 1$ (white phase noise)	$\sim\omega^2$ when $\omega \to 0$	$\sim 1/n^2$ when $n \to \infty$
Random walk and ARIMA(0,1,1)	$\sim 1/\omega^2$ when $\omega \to 0$	$\sim n^2$ when $n \to \infty$
ARFIMA(0,0.5,0) ($1/f$ noise)	$\sim 1/\omega$ when $\omega \to 0$	$\to (2\ln 2/\pi)\sigma_a^2$ when n is large and $d \to 0.5$
ARFIMA(0,−0.5,0) (flicker phase noise)	$\sim\omega$ when $\omega \to 0$	$\sim \ln n/n^2$ when $n \to \infty$

a better uncertainty measure than the variance of the moving averages.

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Appendix A. Derivation of equation (44)

To calculate the Allan variance of a random walk process $\{X(t)\}$, by (39), the difference in $Y_n(T) - Y_n(T-1)$ can be expressed as

$$\begin{aligned}
Y_n(T) - Y_n(T-1) &= \big\{[X((T-1)n+1) - X((T-2)n+1)] + \cdots \\
&\quad + [X(Tn) - X((T-1)n)]\big\}\{n\}^{-1} \\
&= \left[\sum_{j=1}^{n} j \cdot a((T-2)n + j + 1) \right. \\
&\quad \left. + \sum_{j=1}^{n-1} j \cdot a(Tn - j + 1)\right][n]^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var}[Y_n(T) - Y_n(T-1)] &= \frac{\sigma_a^2}{n^2}\left[\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n-1} j^2\right] \\
&= \frac{\sigma_a^2}{n^2}\left[2\sum_{j=1}^{n-1} j^2 + n^2\right] \\
&= \frac{2n^2+1}{3n}\sigma_a^2.
\end{aligned}$$

Hence, from (9) the Allan variance of random walk is given by

$$\mathrm{AVar}_n[X(t)] = \frac{2n^2+1}{6n}\sigma_a^2.$$

Appendix B. Derivation of equations (60) and (61)

For $\mathrm{Var}[Y_n(T)]$ when $d = -0.5$, first we inspect $\sum_{i=1}^{n-1} (n-i)\rho(i)$. When $d = -0.5$, from (59)

$$\begin{aligned}
\sum_{i=1}^{n-1} \rho(i) &= (-0.25)\sum_{i=1}^{n-1} \frac{1}{i^2 - 0.25} \\
&= (-0.25)\left(2 - \frac{1}{n-0.5}\right).
\end{aligned} \tag{B.1}$$

We also have

$$\begin{aligned}
\sum_{i=1}^{n-1} i\rho(i) &= (-0.25)\sum_{i=1}^{n-1} \frac{i}{i^2-0.25} \\
&= (-0.25)\left[\sum_{i=1}^{n-1}\frac{1}{i+0.5} + \sum_{i=1}^{n-1}\frac{0.5}{i^2-0.25}\right] \\
&= (-0.25)\sum_{i=1}^{n-1}\frac{1}{i+0.5} - 0.125\left(2 - \frac{1}{n-0.5}\right).
\end{aligned}$$

From P. 3, 0.131 of [29] or P. 14, (70) from [30],

$$\sum_{i=1}^{n}\frac{1}{i} \sim \ln n$$

when $n \to \infty$. In addition,

$$\sum_{i=1}^{n-1}\frac{1}{i+1} < \sum_{i=1}^{n-1}\frac{1}{i+0.5} < \sum_{i=1}^{n-1}\frac{1}{i}.$$

Both sides in the above inequalities $\sim \ln n$ when $n \to \infty$. Thus, when $n \to \infty$

$$\sum_{i=1}^{n-1} i\rho(i) \sim \ln n. \tag{B.2}$$

From (B.1) and (B.2), when $n \to \infty$

$$\sum_{i=1}^{n-1}(n-i)\rho(i) \sim -0.5n + \ln n. \tag{B.3}$$

From (6), when $n \to \infty$

$$\mathrm{Var}[Y_n(T)] \sim \frac{\ln n}{n^2}.$$

For the proof of (61),

$$\sum_{i=1}^{n-1} i\rho(n-i) = \sum_{i=1}^{n-1}(n-i)\rho(i).$$

Thus,

$$\sum_{i=1}^{n-1} i\rho(n-i) \sim -0.5n + \ln n. \tag{B.4}$$

Similarly, it can be shown that

$$\sum_{i=1}^{n-1} i\rho(2n-i) \sim O(1). \tag{B.5}$$

From (57), (59), (B.2), (B.4) and (B.5), (61) holds.

Appendix C. The proofs of equations (71)–(75) regarding the limiting behaviours of Allan variances for ARFIMA(1,*d*,0) and ARFIMA(0,*d*,1) processes

(1) *For ARFIMA(1,d,0).* From (67), first we show that when $d \to 0.5$ the ratio between $\rho_X(k)$ and $\rho_W(k)$ approaches 1.

$$\frac{\rho_X(k)}{\rho_W(k)} = \big[F(1, d+k; 1-d+k; \phi_1) + F(1, d-k; 1-d-k; \phi_1) - 1\big]\big[(1-\phi_1) \times F(1, 1+d; 1-d; \phi_1)\big]^{-1} \triangleq \frac{R_n}{R_d}. \tag{C.1}$$

From [25],

$$F(a, b; c; z) = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1) \cdot 1 \cdot 2} z^2 + \cdots. \tag{C.2}$$

From [27], $F(a, b; c; z)$ converges when $|z| < 1$. By (C.2) when $d \to 0.5$,

$$R_n \to 2(1 + \phi_1 + \phi_1^2 + \cdots) - 1 = 1 + 2(\phi_1 + \phi_1^2 + \cdots).$$

Similarly, when $d \to 0.5$

$$R_d \to (1-\phi_1)(1 + 3\phi_1 + 5\phi_1^2 + 7\phi_1^3 + \cdots) = 1 + 2\phi_1 + 2\phi_1^2 + 2\phi_1^3 + \cdots.$$

Thus, from (C.1) when $d \to 0.5$,

$$\frac{\rho_X(k)}{\rho_W(k)} \to 1 \tag{C.3}$$

uniformly for all k. Similar to ARFIMA(0,*d*,0) and (63)–(65), for any fixed n, when $\{X(t)\}$ is an ARFIMA(1,*d*,0) and $d \to 0.5$, $\mathrm{Var}[Y_n(T)]/\mathrm{Var}[X(t)] \to 1$.

Now we show that when $\{X(t)\}$ is an ARFIMA(1,*d*,0) and $d \to 0.5$, $\mathrm{Var}[X(t)] \to \infty$. In this case, $W(t) = X(t) - \phi_1 X(t-1)$ with $|\phi_1| < 1$. Since $\{X(t)\}$ is stationary,

$$\mathrm{Var}[W(t)] = [1 + \phi_1^2 - 2\phi_1 \rho_X(1)]\mathrm{Var}[X(t)],$$

where $\rho_X(1)$ is used to denote the autocorrelation of $\{X(t)\}$ at lag 1. When $d \to 0.5$,

$$\frac{\mathrm{Var}[X(t)]}{\mathrm{Var}[W(t)]} = \frac{1}{1 + \phi_1^2 - 2\phi_1 \rho_X(1)} \sim \frac{1}{1 + \phi_1^2 - 2\phi_1 \rho_W(1)} \to \frac{1}{(1-\phi_1)^2} \tag{C.4}$$

since $\{W(t)\}$ is an ARFIMA(0,*d*,0) process and $\rho_W(1) \to 1$. Since $\mathrm{Var}[W(t)] \to \infty$ when $d \to 0.5$, $\mathrm{Var}[X(t)] \to \infty$. Therefore, when $d \to 0.5$, $\mathrm{Var}[Y_n(T)] \to \infty$.

From (11), (C.3) and (C.4),

$$\frac{\mathrm{AVar}_n[X(t)]}{\mathrm{AVar}_n[W(t)]} = \left\{ n[1 - \rho_X(n)] + \sum_{i=1}^{n-1} i[2\rho_X(n-i) - \rho_X(i) - \rho_X(2n-i)]\mathrm{Var}[X(t)] \right\} \times \left\{ n[1 - \rho_W(n)] + \sum_{i=1}^{n-1} i[2\rho_W(n-i) - \rho_W(i) - \rho_W(2n-i)]\mathrm{Var}[W(t)] \right\}^{-1} \to \frac{1}{(1-\phi_1)^2}. \tag{C.5}$$

(2) *For ARFIMA(0,d,1).* Similar to the case of ARFIMA (1,*d*,0), from (69),

$$\frac{\rho_X(k)}{\rho_W(k)} = \frac{ak^2 - (1-d)^2}{k^2 - (1-d)^2},$$

where $a = (1-\theta_1)^2/\{1 + \theta_1^2 - 2\theta_1 d/(1-d)\}$ with $|\theta_1| < 1$. Obviously, when $d \to 0.5$, $a \to 1$ and thus $\rho_X(k)/\rho_W(k) \to 1$ uniformly for all k. Similar to the case of ARFIMA(1,*d*,0), when $\{X(t)\}$ is an ARFIMA(0,*d*,1), $W(t) - \theta_1 W(t-1) = X(t)$ with $|\theta_1| < 1$. Since $\{W(t)\}$ is stationary,

$$[1 + \theta_1^2 - 2\theta_1 \rho_W(1)]\mathrm{Var}[W(t)] = \mathrm{Var}[X(t)],$$

where $\rho_W(1)$ is used to denote the autocorrelation of $\{W(t)\}$ at lag 1. When $d \to 0.5$,

$$\frac{\mathrm{Var}[X(t)]}{\mathrm{Var}[W(t)]} = 1 + \theta_1^2 - 2\theta_1 \rho_W(1) \to (1-\theta_1)^2 \tag{C.6}$$

since $\{W(t)\}$ is an ARFIMA(0,*d*,0) process and $\rho_W(1) \to 1$. Similar to the case of ARFIMA(1,*d*,0), when $\{X(t)\}$ is an ARFIMA(0,*d*,1), the corresponding $\mathrm{Var}[Y_n(T)] \to \infty$ when $d \to 0.5$. Similarly to (C.5), from (11), (C.3) and (C.6),

$$\frac{\mathrm{AVar}_n[X(t)]}{\mathrm{AVar}_n[W(t)]} \to (1-\theta_1)^2.$$

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