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Augmented Lagrangian Homotopy Method for the Regularization of Total Variation Denoising Problems

L.A. Melara · A.J. Kearsley · R.A. Tapia

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Abstract This paper presents a homotopy procedure which improves the solvability of mathematical programming problems arising from total variational methods for image denoising. The homotopy on the regularization parameter involves solving a sequence of equality-constrained optimization problems where the positive regularization parameter in each optimization problem is initially large and is reduced to zero. Newton's method is used to solve the optimization problems and numerical results are presented.

Keywords Constrained optimization · Newton's method · Regularization · Homotopy method

1 Introduction

The construction and transmission of images through electronic machinery is a common process in information technology. Invariably, this process contaminates these images with noise. As a result, there is a need to develop methods for denoising images. Various tools in computational and applied mathematics are used to address this problem. Many existing techniques do not work well with block-like images that

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L.A. Melara (✉)
Department of Mathematics, Colorado College, Colorado Springs, CO, USA
e-mail: lmelara@ColoradoCollege.edu

A.J. Kearsley
Mathematical and Computational Sciences Division, National Institute of Standards and Technology, Gaithersburg, MD, USA

R.A. Tapia
Department of Computational and Applied Mathematics, Rice University, Houston, TX, USA

contain sharp edges or piecewise smooth segments. For these cases, total variational techniques are effective, see Refs. [1–8].

Let $z=(x,y)\in\Omega$, where Ω is a convex, bounded region of $\mathbb{R}^2$ and consider $\bar{u}:\Omega\to\mathbb{R}$ where values of $\bar{u}(z)$ represent the intensities of a given image. Assume that the image has been corrupted by some error noise $\eta(z)\in H^1(\Omega)$ and results in an observed image $u_0(z)$:

$$u_0(z)=\bar{u}(z)+\eta(z). \tag{1}$$

For brevity, function arguments will be suppressed when they are clear from context; for example, we write η for $\eta(z)$. For the purposes of this paper, it suffices to consider only noise functions arising from noise distributions. We will assume further that the $L^2(\Omega)$-norm of the noise distribution η is known. The standard deviation of the noise σ, see Ref. [4], is defined to be

$$\sigma=\left(\int_\Omega |\bar{u}-u_0|^2 dz\right)^{1/2}, \tag{2}$$

where $\bar{u}$ is the true image and $\sigma>0$. We will assume that σ is known, if σ is not known, a different approach is taken. This case will be addressed later. The notation $\|\cdot\|_2$ refers to the 2-norm in $\mathbb{R}^2$,

$$\|z\|_2=\sqrt{x^2+y^2},$$

for $z\in\mathbb{R}^2$; $\|\cdot\|_{L^2(\Omega)}$ denotes the norm on the Hilbert space $L^2(\Omega)$ defined by

$$\|f\|_{L^2(\Omega)}=\left(\int_\Omega |f(z)|^2 dz\right)^{1/2},$$

for $f\in L^2(\Omega)$. Similarly, the norm in the Hilbert space $H^1(\Omega)$ is defined by

$$\|f\|_{H^1(\Omega)}=\left(\int_\Omega |f(z)|^2 dz+\int_\Omega |\partial f(z)/\partial x|^2 dz+\int_\Omega |\partial f(z)/\partial y|^2 dz\right)^{1/2},$$

for $f\in H^1(\Omega)$.

Many mathematical formulations of the problem have been proposed. This paper focuses on an optimization-based method for restoring a corrupted image. We consider an equality constrained optimization problem where the objective function is a regularized bounded variational seminorm and the equality constraint is constructed using (2). Hence, we have

$$\min_{u\in H^1(\Omega)} \int_\Omega \|\nabla u\|_2 dz, \tag{3a}$$

$$\text{s.t.}\quad 1/2(\|u-u_0\|^2_{L^2(\Omega)}-\sigma^2)=0, \tag{3b}$$

where σ was defined in (2). The denoised image is a solution of (3).

The Lagrangian functional $\ell(u,\lambda)$ for this constrained problem is given by

$$\ell(u,\lambda)=\int_\Omega \|\nabla u\|_2 dz+\lambda/2(\|u-u_0\|^2_{L^2(\Omega)}-\sigma^2), \tag{4}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated with the constraint. Since our objective function is a seminorm, we expect many solutions; therefore, we look for a solution which satisfies the boundary condition $\partial u/\partial n = 0$, $z \in \Gamma$, where Γ is the boundary of Ω. This is a common practice. As a result, we obtain conveniently a representation of the gradient of $\ell(u, \lambda)$ with respect to u in the L^2-inner product given by

$$\begin{aligned}
\ell'(u,\lambda)m &= \frac{d}{dt}\ell(u+tm,\lambda)\bigg|_{t=0} \\
&= \frac{d}{dt}\int_\Omega \|\nabla(u+tm)\|_2 dz + \lambda/2(\|u+tm-u_0\|^2_{L^2(\Omega)} - \sigma^2)\bigg|_{t=0} \\
&= \int_\Omega \frac{\nabla u \cdot \nabla m}{\|\nabla u\|_2} dz + \lambda \int_\Omega (u-u_0) m dz \\
&= -\int_\Omega \nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|_2}\right) m dz + \int_\Gamma \left(\frac{m}{\|\nabla u\|_2}\right)\left(\frac{\partial u}{\partial n}\right) d\Gamma \\
&\quad + \lambda \int_\Omega (u-u_0) m dz \\
&= \left\langle -\nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|_2}\right) + \lambda(u-u_0), m \right\rangle.
\end{aligned} \tag{5}$$

We use (5) and write the first-order necessary conditions, which require that we solve for $u^* \in H^1(\Omega)$ and $\lambda^* \in \mathbb{R}$ such that

$$-\nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|_2}\right) + \lambda(u-u_0) = 0, \quad z \in \Omega,$$

$$1/2(\|u-u_0\|^2_{L^2(\Omega)} - \sigma^2) = 0, \quad z \in \Omega;$$

with

$$\frac{\partial u}{\partial n} = 0, \quad z \in \Gamma.$$

The gradient of (4) with respect to u is given by

$$\mathcal{M}(u,\lambda) = -\nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|_2}\right) + \lambda(u-u_0), \quad z \in \Omega; \tag{6}$$

for $z \in \Omega$; for fixed $\lambda \in \mathbb{R}$, the root u^* of $\mathcal{M}(u,\lambda)$ will be our approximation to the restored image. The Lagrange multiplier λ is obtained using a least squares formula, see for example Ref. [9]. The algorithm will be described in detail in Sect. 2.

In Ref. [5], the authors use a parabolic equation with the time s as an evolution parameter to find the roots u^* of (6). This method is equivalent to a gradient descent method, Ref. [5]. Essentially, they solve

$$\frac{\partial u}{\partial s} = -\nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|_2}\right) + \lambda(u-u_0), \tag{7}$$

for $s > 0$, $z \in \Omega$ and with $u(x, y, 0)$ given and $\partial u/\partial n = 0$ for $z \in \Gamma$. As s increases, a denoised version of the image is approached. If a steady state solution is reached, the left side of (7) will vanish, i.e. $\partial u/\partial s = 0$. At this point, the multiplier λ is obtained by multiplying both sides of (7) by $u - u_0$ and integrating by parts in Ω. This results in λ

$$\lambda = -\frac{1}{2\sigma^2}\int_\Omega \|\nabla u\|_2 - \frac{\nabla u_0 \cdot \nabla u}{\|\nabla u\|_2} dz.$$

If u is a constant function, then $\mathcal{M}(u, \lambda)$ in (6) is undefined. To address this, the objective function in (3) is regularized, resulting in the following term:

$$\int_\Omega \sqrt{\|\nabla u\|_2^2 + \epsilon^2} dz. \tag{8}$$

In Refs. [6, 7], the authors solve the following penalized least squares problem which incorporates the regularized functional (8),

$$\min_{u\in H^1(\Omega)} \frac{1}{2}\|u - u_0\|^2_{L^2(\Omega)} + \gamma \int_\Omega \sqrt{\|\nabla u\|_2^2 + \epsilon^2}, \tag{9}$$

and where $\gamma \in \mathbb{R}$ is the penalization parameter. Unlike the optimization problem in (3), the problem in (9) corresponds to the case when σ is not known. The chosen value for the penalized parameter γ is relatively small, particularly when the data contain small amounts of error. In Refs. [6, 7], the authors analyze the discrete approximations of a Newton type method. In Ref. [4], Majava experimented also with various computations of the penalization parameter γ. In one instance, the value of γ was chosen for the result which looked best; in another instance, γ minimized the reconstruction error.

In this work, we consider the following optimization problem:

$$\min_{u\in H^1(\Omega)} \int_\Omega \sqrt{\|\nabla u\|_2^2 + \epsilon^2} dz, \tag{10a}$$

$$\text{s.t.} \quad 1/2(\|u - u_0\|^2_{L^2(\Omega)} - \sigma^2) = 0. \tag{10b}$$

Letting

$$f(u) = \int_\Omega \sqrt{\|\nabla u\|_2^2 + \epsilon^2} dz, \tag{11}$$

$$g(u) = 1/2(\|u - u_0\|^2_{L^2(\Omega)} - \sigma^2), \tag{12}$$

we construct the augmented Lagrangian

$$\mathcal{L}(u, \lambda, \rho) = f(u) + \lambda g(u) + \rho g(u)^2, \tag{13}$$

where λ is the Lagrange multiplier associated to the equality constraint and ρ is the penalization parameter. We use Newton's method to find the roots $u^* \in H^1(\Omega)$ of the gradient of (13) with respect to u. This gradient is computed similarly to (5). In particular, the representation for the gradient of the objective function in the L^2-inner product is now given by

$$f'(u)m = \left\langle -\nabla \cdot \left(\frac{\nabla u}{\sqrt{\|\nabla u\|_2^2 + \epsilon^2}} \right), m \right\rangle, \tag{14}$$

for $z \in \Omega$, which is well-defined for constant u. Further, we require also that

$$\frac{\partial u}{\partial n} = 0,$$

for $z \in \Gamma$, under the same justifications.

For fixed $\lambda \in \mathbb{R}$, Newton's method is used to solve for the roots $u^* \in H^1(\Omega)$ of the gradient of the augmented Lagrangian (13) with respect to u. The application of Newton's method to the solution of general Euler–Lagrange type equations and equality constrained optimization problems was also previously studied by Tapia in Refs. [10, 11].

It appears that the methods presented in Refs. [6, 7] are very efficient for well-chosen values of ϵ, otherwise, these methods may be slowly convergent or even fail to converge. The underlying idea for the method presented in this work is based on the observation that when ϵ is large, the gradient is well-behaved but the optimization problem (10) differs significantly from the original problem shown in (3), see Ref. [4]. For small ϵ, on the other hand, Newton's method does not perform satisfactorily, see Ref. [4]. A small value of ϵ is desired for the regularization parameter, in order to obtain a good approximation to a solution.

The framework of the homotopy technique consists of solving a sequence of equality constrained optimization problems with decreasing values of the regularization parameter $\epsilon > 0$. This permits the procedure to solve a problem with Newton's method behaving well initially and hence allowing for a larger pool of acceptable initial iterates $u^{(0)}$. For each incremental reduction of ϵ, the solution to the previous equality constrained optimization problem serves as the initial iterate for the subsequent optimization problem.

Our numerical intuition and results show that the solutions obtained when implementing this homotopy technique contain less noise than the solutions obtained when the homotopy technique is not used. Thus, in this paper, we describe a technique whereby the regularization parameter ϵ is employed as a homotopy parameter to approximate solutions to the original equality constrained optimization problem presented in (3).

The paper is organized as follows. In Sect. 2, we present the numerical implementation. In Sect. 3, we report on numerical results. We end with the conclusion in Sect. 4.

2 Numerical Implementation

2.1 Objective Function and Equality Constraint Approximation

Let the partitioning of the domain $\Omega = (0, 1) \times (0, 1)$ be

$$\Omega = \bigcup_{k=1}^{N(Q)} Q_k,$$

where $N(Q)$ is the total number of partitions in Ω and the partitions have the form

$$Q_k = ((i-1)h, ih) \times ((j-1)h, jh),$$

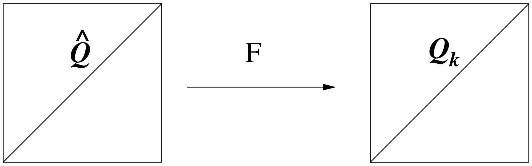

Fig. 1 Visual description of map $F(\hat{Q}) = Q_k$

for $k = 1, 2, \ldots, N(Q)$, with appropriate values of i and j; $h = 1/N$ is the mesh size for N partitions along both axes. We approximate the objective function in (10) using an affine map with a $\mathcal{P}_1$ finite element method, Ref. [12]. The affine map $F : \mathbb{R}^2 \to \mathbb{R}^2$ projects from the parent element $\hat{Q} = (0, 1) \times (0, 1)$ to the basic cell $Q_k = T_1 \cup T_2 \subset \Omega$, where T_1 and T_2 are the triangular elements, Ref. [12]. This is depicted in Fig. 1. The map F is given by

$$F(\hat{z}) = D\hat{z} + d = z,$$

with

$$D = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \qquad d = \begin{pmatrix} ih \\ jh \end{pmatrix}, \qquad i, j = 1, \ldots, N \text{ and } h = 1/N,$$

$\hat{z} \in \hat{Q}$ and $z \in Q_k$. The inverse of the affine function F is composed with the function $\hat{u} : \hat{Q} \to \mathbb{R}$ to evaluate the function $u : Q_k \subset \Omega \to \mathbb{R}$, for $k = 1, \ldots, N(Q)$. Therefore, we have

$$u(z) = (\hat{u} \circ F^{-1})(z) = \hat{u}(F^{-1}(z)); \tag{15}$$

consequently, by the chain rule,

$$\nabla u(z) = \nabla F^{-1} \cdot \hat{\nabla}\hat{u}(F^{-1}(z)),$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ and $\hat{\nabla} = (\partial/\partial \hat{x}, \partial/\partial \hat{y})$. Hence, the composition (15) facilitates the evaluation of the objective function and equality constraint in (10).

The function u is approximated by

$$u = \sum_{k=1}^{N(h)} v_k \varphi_k,$$

where $v_k = u(ih, jh)$, for some i, j; $N(h)$ is the total number of nodes in Ω and φ_k is the kth basis function from the polynomial space

$$\mathcal{P}_1 = \text{span}\{1, x, y\}.$$

To approximate the objective function, the integral over Ω is separated into the sum of integrals over the basic cells Q_k,

$$\int_\Omega \sqrt{\|\nabla u\|_2^2 + \epsilon^2} dz = \sum_{k=1}^{N(Q)} \int_{Q_k} \sqrt{\|\nabla u\|_2^2 + \epsilon^2} dz. \tag{16}$$

Then, using the inverse affine map F^{-1}, each term on the right-hand side of (16) can be approximated by

$$\int_{Q_k} \sqrt{\|\nabla u\|_2^2 + \epsilon^2}dz = \int_{\hat{Q}} \sqrt{\|\nabla F^{-1}\hat{\nabla}\hat{u}\|_2^2 + \epsilon^2} \det \hat{\nabla} F d\hat{z}.$$

The equality constraint in (10b) is computed similarly using the relationship

$$\int_{Q_k} |u|^2 dz = \int_{\hat{Q}} |\hat{u}|^2 \det \hat{\nabla} F d\hat{z}.$$

2.2 Optimization Method

To solve the optimization problem (10) numerically, we use the augmented Lagrangian $\mathcal{L}(u, \lambda, \rho)$ presented in (13). In our numerical implementation, we use Newton's method to find roots u^* of the augmented Lagrangian (13); λ is updated using a least squares formula and a sufficiently large penalization parameter ρ is chosen and kept fixed for all iterates.

The discrete approximation of $f(u)$ is denoted by $f(v)$, similarly $g(u)$ is denoted by $g(v)$; for (13), we simplify, $\mathcal{L}(v) = \mathcal{L}(v, \lambda, \rho)$, where $v \in \mathbb{R}^{N(h)}$. In particular, we emphasize the dependence on the vector v, since the penalization parameter ρ is fixed and an update formula for the Lagrange multiplier λ is used. The method requires the gradient vector of the augmented Lagrangian (13), denoted by $\nabla\mathcal{L}(v) \in \mathbb{R}^{N(h)}$. The ith component of $\nabla\mathcal{L}(v)$ corresponds to the ith partial derivative of (13) and is given by

$$\begin{aligned}(\nabla\mathcal{L}(v))_i &= (\nabla f(v))_i + \lambda(\nabla g(v))_i + 2\rho g(v)(\nabla g(v))_i \\ &= \partial f(v)/\partial v_i + \lambda \partial g(v)/\partial v_i + 2\rho g(v)\partial g(v)/\partial v_i. \end{aligned} \tag{17}$$

In our numerical implementation, we solve $\nabla\mathcal{L}(v) = 0$.

The Hessian matrix $H \in \mathbb{R}^{N(h)\times N(h)}$ of the augmented Lagrangian in (13) has the ijth component

$$H_{ij} = \frac{\partial^2 \mathcal{L}(v)}{\partial v_i \partial v_j}$$

and is symmetric positive definite for sufficiently large ρ, see Ref. [13]. The gradient vector $\nabla\mathcal{L}(v)$, Hessian matrix H and vector v are assembled to minimize, for fixed λ, the augmented Lagrangian (13) via Newton's method. Given $v^{(0)}$, the sequence $\{v^{(n)}\}$ generated by

$$v^{(n+1)} = v^{(n)} + \alpha s^{(n)}, \tag{18}$$

is called the Newton sequence for $v^{(0)}$ with respect to $\mathcal{L}(v)$. The vector $s^{(n)}$ is obtained by solving the linear system

$$H^{(n)} s^{(n)} = -\nabla\mathcal{L}(v^{(n)}).$$

The Newton step $s^{(n)}$ is a descent direction on the gradient of the augmented Lagrangian because

$$\begin{aligned}(s^{(n)})^T \nabla \mathcal{L}(v^{(n)}) &= -\nabla \mathcal{L}(v^{(n)})^T (H^{(n)})^{-T} \nabla \mathcal{L}(v^{(n)}) \\ &= -\nabla \mathcal{L}(v^{(n)})^T (H^{(n)})^{-1} \nabla \mathcal{L}(v^{(n)}) < 0,\end{aligned}$$

since $(H^{(n)})^{-1}$ is also symmetric positive definite. The linesearch parameter $\alpha \in \mathbb{R}$ is initially set to $\alpha := 1$. The value of α is determined based on a sufficient decrease criterion. If the sufficient decrease criterion,

$$\mathcal{L}(v^{(n)} + \alpha s^{(n)}) < \mathcal{L}(v^{(n)}) + 10^{-4} \alpha \nabla \mathcal{L}(v^{(n)}) \cdot s^{(n)}, \tag{19}$$

is not met, then we reduce the steplength by $1/2$,

$$\alpha := \alpha/2,$$

until (19) is satisfied (Ref. [14]). Then, we compute the $(n+1)$st iterate in (18) and reset $\alpha := 1$. The Lagrange multiplier λ is updated using the least squares update formula (see Ref. [11])

$$\lambda = -(\nabla g(v) \cdot \nabla f(v))/(\nabla g(v) \cdot \nabla g(v)). \tag{20}$$

We implemented the algorithm in Matlab. After solving the optimization problem (10) by finding a root of the gradient of the augmented Lagrangian (13), the initial ϵ is reduced and then we solve again for a root of the gradient of the augmented Lagrangian (13) with the updated and reduced value of ϵ. Our numerical results show that, as the ϵ values decrease, the solution v^*, obtained by solving $\nabla \mathcal{L}(v) = 0$ using Newton's method, contains far less noise than the observed image v_0. The algorithm is described below:

```
Step 0. Given ϵ̄. Initialize λ, ρ, ϵ.
While ϵ ≥ ϵ̄, execute the steps below:
Step 1. Use Newton's method to solve ∇L(v) = 0.
Step 2. Compute λ via the least squares update (20).
Step 3. Reduce ϵ. Return to Step 0.
```

For Newton's method, the stopping criterion is either $\|\nabla \mathcal{L}(v)\|_2 < \delta$, for some given δ, or when the maximum number of iterations is reached. The regularization parameter ϵ is reduced by a factor of 10 until the lower bound $\bar{\epsilon}$ is reached. This algorithm can be viewed as a large-scale analog of the diagonalized multiplier method (see Tapia, Ref. [11]) with the Newton method used to solve for the primal variable and a least-squares approximation to the dual variables. Section 3 presents a simple example implementing the procedure.

3 Numerical Results

In these numerical examples, we set $\delta = 10^{-5}$, $\bar{\epsilon} = 10^{-4}$, and the maximum number of iterations for Newton's method is 100. The penalization parameter ρ is fixed at

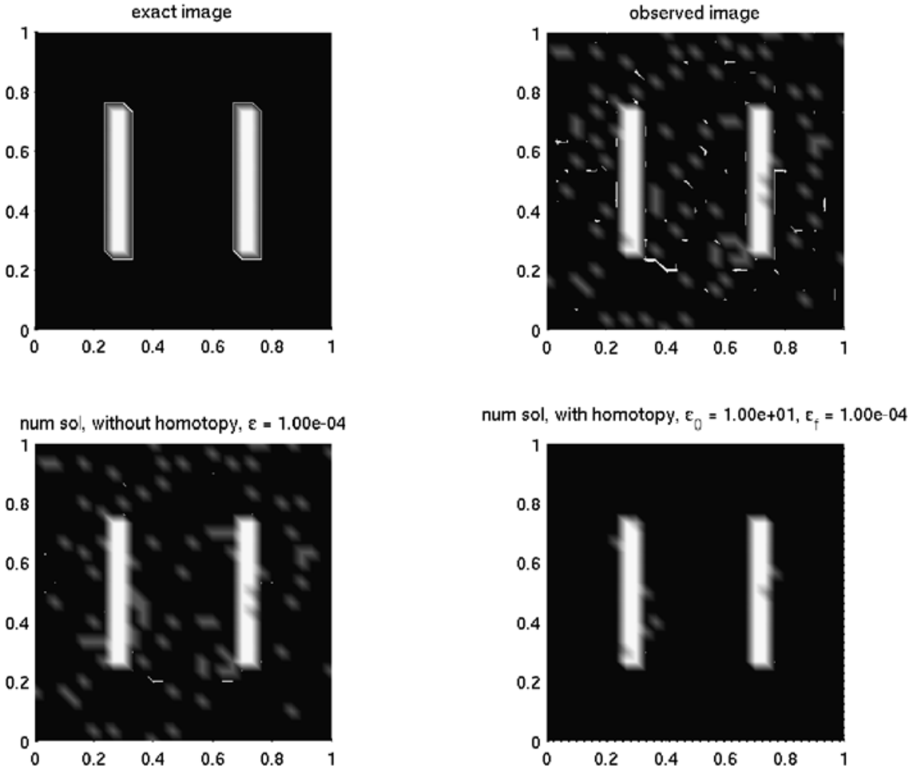

Fig. 2 *Top left*: Exact image. *Top right*: Observed image. *Bottom left*: Numerical solution obtained using method without homotopy technique. The value of regularization parameter is fixed at $\epsilon = 10^{-4}$. *Bottom right*: Numerical solution obtained using method with homotopy technique. The initial value of regularization parameter is $\epsilon = 10$ and the final value is $\epsilon = 10^{-4}$

$\rho = 300$ and the initial value of the Lagrange multiplier is $\lambda = 10$. The regularization parameter ϵ is initially set to $\epsilon = 10$ and is reduced to a final value of $\epsilon = 10^{-4}$.

We construct two functions $u : \Omega \to \mathbb{R}$, where $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$. The functions display the following: two vertical bars as shown in the top left of Fig. 2 and the word '*NIST*' as shown in the top left of Fig. 3. These functions are the true images, corresponding to $\bar{u}$. The observed image u_0 is constructed as described in (1). This image is stored in the vector v_0 with ith component given by

$$(v_0)_i = (\bar{v})_i + \text{rand}(1) * 3.80e - 1,$$

where rand.m is a built-in Matlab function which produces uniformly distributed random numbers on the interval (0.00, 1.00) and $\bar{v}$ is the vector storing the exact image produced by the function $\bar{u}$. The observed image $v_0 \in \mathbb{R}^{N(h)}$ is used as the initial iterate $v^{(0)} \in \mathbb{R}^{N(h)}$. In both figures, the images are on a 30×30 grid. The number of elements in Ω is $N(Q) = 900$. Thus, the total number of nodes in Ω is $N(h) = 31^2 = 961$.

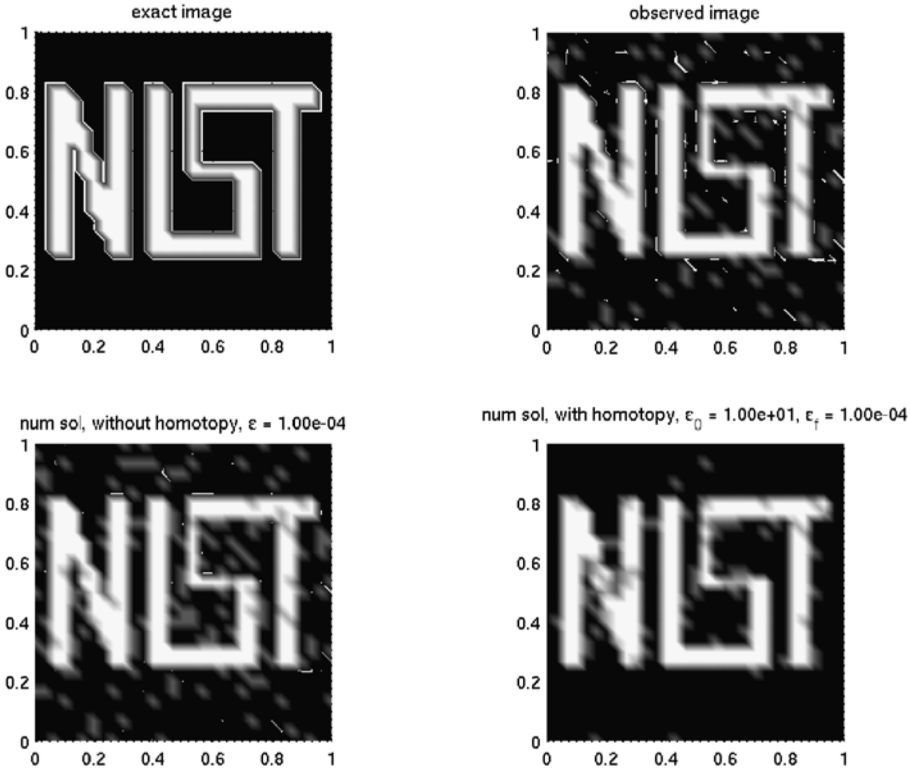

Fig. 3 *Top left*: Exact image. *Top right*: Observed image. *Bottom left*: Numerical solution obtained using the method without homotopy technique. The value of the regularization parameter is fixed at $\epsilon = 10^{-4}$. *Bottom right*: Numerical solution obtained using the method with homotopy technique. The initial value of the regularization parameter is $\epsilon = 10$ and the final value is $\epsilon = 10^{-4}$

We compare our results to those obtained by a method which does not employ the regularization parameter ϵ as a homotopy parameter. In the method without the homotopy procedure, the regularization parameter ϵ is kept fixed at $\epsilon = 10^{-4}$. Also, for the method without the homotopy technique, we use the observed image v_0 as the initial iterate $v^{(0)}$ and we seek to denoise the same observed image v_0. The results obtained for both methods are presented in Figs. 2 and 3.

The plot of the exact image $\bar{v}$ is on the top left frame in Figs. 2 and 3. The top right frame in both figures shows the contaminated image v_0. In each of the figures, the bottom left frame contains the graph of the numerical solution obtained without the homotopy technique and the bottom right frame displays the numerical solution computed using the technique described in this paper. Desingularization will often lead to highly nonlinear optimization problems for which obvious starting points are not sufficiently close to the optimal solution but ensures desirable convergence properties of Newton's method. By reducing the regularization parameter ϵ, the solution to the optimization problem (10) better approximates a solution to the optimization problem (3).

4 Conclusions

This paper presents a practical way for approximating solutions to regularized problems encountered in total variational methods for image denoising by employing the regularization parameter ϵ as a homotopy parameter. The numerical solutions computed using this technique show less noise, as opposed to solutions obtained when the method is not implemented. In the homotopy method, we solve a sequence of equality constrained optimization problems as the regularization parameter $\epsilon \to 0$. Our numerical results suggest that the pool of acceptable initial iterates for Newton's method may have enlarged. In Refs. [15, 16], the authors consider a sequence of equality constrained optimization problems as the log barrier parameter $\mu \to 0$ and compare the behavior of the radius of convergence of Newton's method when applied to two equivalent systems. Therefore, we are motivated to investigate the relationship between the radius of the Kantorovich ball for Newton's method and the regularization/homotopy parameter ϵ as used in this paper. This is a topic of further research (Ref. [17]).

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