10.1098/rspa.2003.1156

FirstCite® e-publishing

THE ROYAL SOCIETY

The dilogarithm function for complex argument

By Leonard C. Maximon

Department of Physics, The George Washington University, Washington, DC 20052, USA (max@gwu.edu)

Received 13 January 2003; accepted 12 March 2003

This paper summarizes the basic properties of the Euler dilogarithm function, often referred to as the Spence function. These include integral representations, series expansions, linear and quadratic transformations, functional relations, numerical values for special arguments and relations to the hypergeometric and generalized hypergeometric function. The basic properties of the two functions closely related to the dilogarithm (the inverse tangent integral and Clausen's integral) are also included. A brief summary of the defining equations and properties for the frequently used generalizations of the dilogarithm (polylogarithm, Nielsen's generalized polylogarithm, Jonquière's function, Lerch's function) is also given. A résumé of the earliest articles that consider the integral defining this function, from the late seventeenth century to the early nineteenth century, is presented. Critical references to details concerning these functions and their applications in physics and mathematics are listed.

Keywords: Euler dilogarithm; Spence function; Debye function; Jonquière's function; polylogarithms; Clausen's integral

1. Introduction

The dilogarithm function, also referred to as the Spence function, has a long history connected with some of the great names in the history of mathematics. The integral that defines it first appears in one of the letters from Leibniz to Johann Bernoulli in 1696, part of an extensive correspondence between Leibniz and the Bernoullis. However, the properties of this integral as a distinct function were first studied by Landen in 1760. Since then it has, along with its generalization, the polylogarithm, been studied by some of the great mathematicians of the past—Euler, Abel, Lobachevsky, Kummer and Ramanujan among others. It appears in a very wide range of fields—number theory, algebraic geometry, electric network and radiation problems, the statistical mechanics of ideal gases, and, in quantum electrodynamics, in any calculation of higher-order processes such as vacuum polarization and radiative corrections. Nonetheless, there does not seem to be a concise reference work summarizing the essential properties of the dilogarithm as a function of complex argument. With this paper we hope to provide such a reference.

2. Definition and notation

The Euler dilogarithm is defined for complex argument z by

$$\mathrm{L}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\,\mathrm{d}t. \tag{2.1}$$

It is also useful to write this integral in the equivalent form

$$\mathrm{L}_2(z) = -\int_0^1 \frac{\ln(1-zt)}{t}\,\mathrm{d}t. \tag{2.2}$$

We consider here the principal branch of the dilogarithm, defined by taking the principal branch of the logarithm, for which $\ln z$ has a cut along the negative real axis, with $|\arg z| < \pi$. This defines the principal branch of the dilogarithm as a single-valued function in the complex plane cut along the real axis from 1 to $+\infty$ ($0 < \arg(z-1) < 2\pi$). A survey of the notations and definitions adopted by different authors may be found in Lewin (1981, § 1.10, pp. 27–29). In particular, the function $\mathrm{L}_2(z)$ is denoted by $\mathcal{L}_2(z)$ in Gröbner & Hofreiter (1975), by $S_2(z)$ in Kölbig *et al.* (1970) and Kölbig (1986), by $\mathrm{Li}_2(z)$ in Lewin (1981) and Roskies *et al.* (1990) and by $\mathrm{Sp}(z)$ in 't Hooft & Veltman (1979).

3. Analytic continuation for dilogarithms

Using either of the representations (2.1) and (2.2), one may expand the logarithm in powers of z, obtaining the Taylor series expansion for the dilogarithm, valid for $|z| \leqslant 1$,

$$\mathrm{L}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}. \tag{3.1}$$

However, the principal branch of the dilogarithm is defined by the integrals in (2.1) and (2.2) as a single-valued analytic function in the entire z-plane, with the exception of the points on the cut along the real axis from 1 to $+\infty$. The integrals (2.1) and (2.2) may therefore be used to obtain analytic continuations of the dilogarithm for arguments outside the unit circle. These transformations are given below.

(*a*) *Transformation formulae*

(i) *Linear transformations*

The linear transformations of the dilogarithm are

$$\mathrm{L}_2\left(\frac{1}{z}\right) = -\mathrm{L}_2(z) - \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\ln^2(-z), \qquad z \notin [0,\infty), \tag{3.2}$$

$$\mathrm{L}_2(1-z) = -\mathrm{L}_2(z) + \tfrac{1}{6}\pi^2 - \ln(1-z)\ln z, \qquad z \notin (-\infty,0]\cup[1,\infty), \tag{3.3}$$

$$\mathrm{L}_2\left(\frac{z}{z-1}\right) = -\mathrm{L}_2(z) - \tfrac{1}{2}\ln^2(1-z), \qquad z \notin [1,\infty), \tag{3.4}$$

$$\mathrm{L}_2\left(\frac{z-1}{z}\right) = \mathrm{L}_2(z) - \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\ln^2 z + \ln(1-z)\ln z, \qquad z \notin (-\infty,0]\cup[1,\infty), \tag{3.5}$$

$$\begin{aligned}\mathrm{L}_2\left(\frac{1}{1-z}\right) &= \mathrm{L}_2(z) + \tfrac{1}{6}\pi^2 \\ &\quad + \ln(-z)\ln(1-z) - \tfrac{1}{2}\ln^2(1-z), \qquad z \notin [0,\infty).\end{aligned} \tag{3.6}$$

The transformation (3.3) provides directly the expansion of the dilogarithm about the point $z = 1$, namely,

$$\mathrm{L}_2(z) = -\sum_{k=1}^{\infty} \frac{(1-z)^k}{k^2} + \ln(1-z) \sum_{k=1}^{\infty} \frac{z^k}{k} + \tfrac{1}{6}\pi^2. \tag{3.7}$$

As noted in equations (3.2)–(3.6) and (3.11), each of these transformations is valid for z in the entire cut plane apart from real numbers that lie on the cut of either of the dilogarithm functions. (Note that this restriction also precludes the argument of any of the logarithms from being on the cut of the logarithm.) These transformations may be obtained as follows. Making the substitution of variables $t = 1 - s$ in (2.1) and then integrating by parts gives

$$\mathrm{L}_2(z) = -\ln(1-z)\ln z - \int_{1-z}^{1} \frac{\ln(1-s)}{s}\,\mathrm{d}s. \tag{3.8}$$

Equation (3.3) then follows using (6.2). Next, using (2.1) to write $\mathrm{L}_2(z/(z-1))$ and making the substitution of variables $t = s/(s-1)$, we have

$$\mathrm{L}_2\left(\frac{z}{z-1}\right) = \int_0^z \frac{\ln(1-s)}{s(1-s)}\,\mathrm{d}s. \tag{3.9}$$

Splitting the denominator in partial fractions and integrating then gives (3.4). Making the substitution $z \to 1 - z$ in (3.4), we have $z/(z-1) \to (z-1)/z$, giving

$$\mathrm{L}_2\left(\frac{z-1}{z}\right) = -\mathrm{L}_2(1-z) - \tfrac{1}{2}\ln^2 z. \tag{3.10}$$

This equation, together with (3.3), then gives (3.5). Next, adding the left- and right-hand sides of (3.4) and (3.5) and making the substitution $w = z/(z-1)$ then gives (3.2). Finally, setting $z = 1 - u$ in (3.2), together with (3.3), then gives (3.6).

(ii) *Quadratic transformations*

The quadratic transformation of the dilogarithm follows directly from (2.2),

$$\mathrm{L}_2(z) + \mathrm{L}_2(-z) = \tfrac{1}{2}\mathrm{L}_2(z^2), \quad z \notin (-\infty, -1] \cup [1, \infty). \tag{3.11}$$

More generally,

$$\sum_{k=0}^{m-1} \mathrm{L}_2(\omega^k z) = \frac{1}{m}\mathrm{L}_2(z^m), \quad \text{where } \omega = \mathrm{e}^{2\pi\mathrm{i}/m}, \quad m = 1, 2, 3, \ldots. \tag{3.12}$$

A simple proof of this generalization is given in Andrews *et al.* (1999).

(*b*) *Analytic continuation around the branch points*

We have thus far considered only the principal branch of the dilogarithm, which is a single-valued analytic function in the cut plane. If, however, we permit the variable of integration t to wander around the complex plane without restriction, we then create the general branch of the function $\mathrm{L}_2(z)$, and, in the integrand of (2.1),

$\ln(1-t)$ may no longer have its principal value at $t=0$, but instead equals $2k\pi\mathrm{i}$ with a non-zero value of the integer k. The dilogarithm defined by (2.1) is then a multivalued analytic function in the complex plane. Thus, if we begin with the principal branch of $\mathrm{L}_2(z)$ at any point z in the plane and integrate along a closed contour that goes in a continuous manner once around the branch point at $z=1$ in the positive rotational sense, then the value of the function on returning to z is $\mathrm{L}_2(z)-2\pi\mathrm{i}\ln z$. For this branch and, more generally, for all of the branches of the dilogarithm other than the principal branch, the point $z=0$ is a 'hidden' branch point, again of the logarithmic type. Thus, at a point z, the value of the dilogarithm on a general branch of the function is given in terms of its value on the principal branch, $\mathrm{L}_2^*(z)$, by

$$\mathrm{L}_2(z)=\mathrm{L}_2^*(z)+2m\pi\mathrm{i}\ln z+4k\pi^2, \tag{3.13}$$

in which $m=0,\pm1,\pm2,\dots$, $k=0,\pm1,\pm2,\dots$. It is to be noted here that the values of m and k depend critically on the path of integration. Specifically, they depend not only on how many times and in which direction each of the two branch points is encircled, which is usual, but also on the *order* in which the branch points are encircled, which is unusual. For further references, see Erdélyi (1953, § 1.11.1, pp. 31, 32) and Hölder (1928).

4. Series expansions for the dilogarithm

The Taylor series (3.1) converges for $|z|\leqslant 1$. Although this condition can always be obtained using the transformation (3.2) if $|z|>1$, this series is clearly very slowly convergent for $|z|$ near unity. A more satisfactory series has been given in 't Hooft & Veltman (1979), in which the dilogarithm is written in terms of the Debye function, $D(z)$ (see Abramowitz & Stegun 1972, § 27.1.1, p. 998), defined by

$$D(z)=\int_0^z \frac{u}{\mathrm{e}^u-1}\,\mathrm{d}u. \tag{4.1}$$

Substituting $t=1-\mathrm{e}^{-u}$ in the integrand in (2.1), we have

$$\mathrm{L}_2(z)=D(-\ln(1-z)). \tag{4.2}$$

Here, the integrand of the Debye function can be expanded in terms of the Bernoulli numbers, B_n (see Erdélyi 1953, § 1.13, pp. 35, 36), giving

$$\mathrm{L}_2(z)=\sum_{n=0}^{\infty} B_n\frac{(-\ln(1-z))^{n+1}}{(n+1)!}. \tag{4.3}$$

Since, for $n=1,2,3,\dots$,

$$B_{2n}=2(-1)^{n+1}(2n)!(2\pi)^{-2n}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}, \tag{4.4}$$

$$B_{2n+1}=0, \tag{4.5}$$

it follows that the series in (4.3) converges for $|-\ln(1-z)|<2\pi$.

5. Functional relations involving two variables

A number of relations between dilogarithms involving two variables have been studied extensively. One, given by Abel (1881), is

$$\begin{aligned}\mathrm{L}_2\left(\frac{x}{1-x}\frac{y}{1-y}\right) &= \mathrm{L}_2\left(\frac{x}{1-y}\right)+\mathrm{L}_2\left(\frac{y}{1-x}\right)\\ &\quad -\mathrm{L}_2(x)-\mathrm{L}_2(y)-\ln(1-x)\ln(1-y), \quad x,y,x+y<1. \end{aligned} \tag{5.1}$$

Making the substitutions $x/(1-y)\to x$, $y/(1-x)\to y$ in (5.1) and using (3.4) to transform the last two dilogarithms on the right-hand side, one obtains a similar five-term relation, due to Hill (1830, p. 9, eqn X),

$$\mathrm{L}_2(xy)=\mathrm{L}_2(x)+\mathrm{L}_2(y)+\mathrm{L}_2\left(\frac{xy-x}{1-x}\right)+\mathrm{L}_2\left(\frac{xy-y}{1-y}\right)+\tfrac{1}{2}\ln^2\left(\frac{1-x}{1-y}\right), \quad x,y,xy<1. \tag{5.2}$$

A number of other functional relations involving five dilogarithm functions are given in Lewin (1981, § 1.5, pp. 7–11) and Kirillov (1995, § 1.6, pp. 88, 89). As shown in Lewin (1981), any one of these five-term relations may be derived from any of the others by use of the transformations given above as well as redefining the variables in the arguments of the functions. Moreover, any number of single-variable relations may be obtained by taking y as some suitable function of x (satisfying the conditions given above in (5.1) and (5.2)). A number of such relations are given in Lewin (1981, § 1.5.4, pp. 10, 11), in Nielsen (1909) and in Kirillov (1995, § 1.2, pp. 70–74). Functional relations involving six dilogarithm functions are given in Lewin (1981, § 1.6, pp. 11–16) and relations involving nine dilogarithm functions are given in Kirillov (1995, § 1.3, p. 84, § 1.6, p. 89).

6. Numerical values for special arguments

For special arguments, the numerical value of the dilogarithm function may be expressed directly in terms of simpler functions, in closed form. The only known results (see Lewin 1991, ch. 1, 13) are

$$\mathrm{L}_2(0)=0, \tag{6.1}$$

$$\mathrm{L}_2(1)=\tfrac{1}{6}\pi^2, \tag{6.2}$$

$$\mathrm{L}_2(-1)=-\tfrac{1}{12}\pi^2, \tag{6.3}$$

$$\mathrm{L}_2(\tfrac{1}{2})=\tfrac{1}{12}\pi^2-\tfrac{1}{2}\ln^2 2, \tag{6.4}$$

$$\mathrm{L}_2(\tfrac{1}{2}(3-\sqrt{5}))=\tfrac{1}{15}\pi^2-\tfrac{1}{4}\ln^2(\tfrac{1}{2}(3-\sqrt{5})), \tag{6.5}$$

$$\mathrm{L}_2(\tfrac{1}{2}(\sqrt{5}-1))=\tfrac{1}{10}\pi^2-\ln^2(\tfrac{1}{2}(\sqrt{5}-1)), \tag{6.6}$$

$$\mathrm{L}_2(\tfrac{1}{2}(1-\sqrt{5}))=-\tfrac{1}{15}\pi^2+\tfrac{1}{2}\ln^2(\tfrac{1}{2}(\sqrt{5}-1)), \tag{6.7}$$

$$\mathrm{L}_2(-\tfrac{1}{2}(1+\sqrt{5}))=-\tfrac{1}{10}\pi^2+\tfrac{1}{2}\ln^2(\tfrac{1}{2}(\sqrt{5}+1)). \tag{6.8}$$

For similar equations containing the sum of dilogarithms of different arguments, see Lewin (1991, ch. 1, 2) and Kirillov (1995, § 1.2, pp. 69, 70, 74).

7. Relation to hypergeometric and generalized hypergeometric functions

From (2.1) and the integral representation of the hypergeometric function, it follows that the dilogarithm may be expressed as the derivative of the hypergeometric function with respect to one of the parameters,

$$\mathrm{L}_2(z) = \lim_{b\to 0,\, a\to 0}\left(\frac{1}{b}\frac{\partial_2 F_1(a,b;b+1;z)}{\partial a}\right). \tag{7.1}$$

An alternative expression for the dilogarithm in terms of the hypergeometric function is

$$\mathrm{L}_2(z) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\{{}_2F_1(\epsilon,\epsilon;1+\epsilon;z)-1\}. \tag{7.2}$$

This expression has been used in Andrews *et al.* (1999) to derive the transformation of the dilogarithm (equation (3.3)) using the transformation of the hypergeometric function. From (2.1), it follows that the dilogarithm satisfies a second-order linear inhomogeneous differential equation

$$z(1-z)\,\mathrm{L}_2''(z) + (1-z)\,\mathrm{L}_2'(z) = 1. \tag{7.3}$$

From (3.1), the dilogarithm may be written as a generalized hypergeometric function ${}_3F_2$,

$$\mathrm{L}_2(z) = z\,{}_3F_2(1,1,1;2,2;z). \tag{7.4}$$

For further relations between the dilogarithm and hypergeometric functions, see Andrews *et al.* (1999, § 2.6, pp. 102–107 and ex. 38, 39, p. 131). The relation between the dilogarithm and Appell's function F_3 is examined in Sanchis-Lozano (1997). The main result given in that reference is

$$\tfrac{1}{2}uvF_3(1,1,1,1,3;u,v) = \mathrm{L}_2(u) + \mathrm{L}_2(v) - \mathrm{L}_2(u+v-uv), \tag{7.5}$$

in which $|\arg(1-u)| < \pi$, $|\arg(1-v)| < \pi$ and $|\arg(1-u)(1-v)| < \pi$.

8. Functions closely related to the dilogarithm

There are two functions that are directly related to the dilogarithm: the inverse tangent integral and Clausen's integral. We give here only the relation of these functions to the dilogarithm; for additional details, we give references below.

(*a*) *Inverse tangent integral*

The inverse tangent integral is the imaginary part of the dilogarithm of purely imaginary argument. For $-1 \leqslant y \leqslant 1$, we have, from (3.1),

$$\mathrm{L}_2(\mathrm{i}y) = \left(-\frac{y^2}{2^2}+\frac{y^4}{4^2}-\frac{y^6}{6^2}+\cdots\right) + \mathrm{i}\left(\frac{y}{1^2}-\frac{y^3}{3^2}+\frac{y^5}{5^2}-\cdots\right). \tag{8.1}$$

The real part of $\mathrm{L}_2(\mathrm{i}y)$ is, from (3.11),

$$\mathrm{Re}(\mathrm{L}_2(\mathrm{i}y)) = \tfrac{1}{2}(\mathrm{L}_2(\mathrm{i}y) + \mathrm{L}_2(-\mathrm{i}y)) = \tfrac{1}{4}\,\mathrm{L}_2(-y^2). \tag{8.2}$$

The imaginary part of $L_2(iy)$ is called the inverse tangent integral,

$$\mathrm{Ti}_2(y) = \frac{y}{1^2} - \frac{y^3}{3^2} + \frac{y^5}{5^2} - \cdots . \tag{8.3}$$

Since, for $|y| \leqslant 1$,

$$\arctan(y) = \frac{y}{1} - \frac{y^3}{3} + \frac{y^5}{5} - \cdots , \tag{8.4}$$

we can define the inverse tangent integral by

$$\mathrm{Ti}_2(y) = \int_0^y \frac{\arctan(u)}{u}\,\mathrm{d}u. \tag{8.5}$$

The integral (8.5) then defines $\mathrm{Ti}_2(y)$ for all real y; the arctangent function is taken to lie in the range $-\frac{1}{2}\pi < \arctan(y) < \frac{1}{2}\pi$. For $\mathrm{Ti}_2(y)$, the relation similar to that given for the dilogarithm in (3.2), valid for all real values of y, is

$$\mathrm{Ti}_2\left(\frac{1}{y}\right) = \mathrm{Ti}_2(y) - \mathrm{sgn}(y)\frac{\pi}{2}\ln|y|. \tag{8.6}$$

For more complicated relations and generalizations of the inverse tangent integral, see Lewin (1981, ch. 2, 3).

(*b*) *Clausen's integral*

Clausen's integral, $\mathrm{Cl}_2(\theta)$, is the imaginary part of the dilogarithm with argument on the unit circle,

$$\begin{aligned} L_2(\mathrm{e}^{\mathrm{i}\theta}) &= \sum_1^\infty \frac{\cos n\theta}{n^2} + \mathrm{i}\sum_1^\infty \frac{\sin n\theta}{n^2} \\ &= \tfrac{1}{6}\pi^2 - \tfrac{1}{4}|\theta|(2\pi - |\theta|) + \mathrm{i}\,\mathrm{Cl}_2(\theta), \quad |\theta| \leqslant 2\pi. \end{aligned} \tag{8.7}$$

From (2.1), with the change of integration variable $t = \mathrm{e}^{\mathrm{i}\phi}$, the integral for $\mathrm{Cl}_2(\theta)$ is obtained (for details, see Lewin 1981, ch. 4),

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \ln|2\sin\tfrac{1}{2}\phi|\,\mathrm{d}\phi. \tag{8.8}$$

(i) *Periodic properties*

$$\mathrm{Cl}_2(2n\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\,\mathrm{Cl}_2(\theta), \tag{8.9}$$

$$\mathrm{Cl}_2(\pi + \theta) = -\,\mathrm{Cl}_2(\pi - \theta). \tag{8.10}$$

(ii) *Duplication formula*

$$\mathrm{Cl}_2(\theta) - \mathrm{Cl}_2(\pi - \theta) = \tfrac{1}{2}\,\mathrm{Cl}_2(2\theta). \tag{8.11}$$

9. Generalizations of the dilogarithm function

(a) *Polylogarithm*

The polylogarithm function, $\mathrm{L}_n(z)$, may be defined by

$$\mathrm{L}_n(z) = \int_0^z \frac{\mathrm{L}_{n-1}(t)}{t}\,\mathrm{d}t, \quad \text{with } \mathrm{L}_0(z) = \frac{z}{1-z}, \tag{9.1}$$

where $n = 1, 2, 3, \dots$. In particular, it follows that

$$\mathrm{L}_1(z) = -\ln(1-z). \tag{9.2}$$

Corresponding to (2.1), we have the integral representation for the polylogarithm, valid for all z not on the cut,

$$\mathrm{L}_{n+2}(z) = -\frac{(-1)^n}{n!}\int_0^1 \frac{\ln^n t \ln(1-zt)}{t}\,\mathrm{d}t, \quad n = 0, 1, 2, \dots. \tag{9.3}$$

Corresponding to (3.1), we have, for $|z| \leqslant 1$ and $n = 2, 3, \dots$ ($|z| < 1$ for $n = 0, 1$),

$$\mathrm{L}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{9.4}$$

We note that the function $\mathrm{L}_n(z)$ is denoted by $\mathcal{L}_n(z)$ in Gröbner & Hofreiter (1975), by $S_n(z)$ in Mignaco & Remiddi (1969), Kölbig *et al*. (1970) and Barbieri *et al*. (1971, 1972*a*, *b*) and by $\mathrm{Li}_n(z)$ in Lewin (1981).

(i) *Analytic continuation*

Corresponding to (3.2) for the dilogarithm, we have

$$\mathrm{L}_n(z) + (-1)^n \mathrm{L}_n\left(\frac{1}{z}\right) = -\frac{(2\pi \mathrm{i})^n}{n!} B_n\left(\frac{\ln z}{2\pi \mathrm{i}}\right), \quad n = 0, 1, 2, \dots, \tag{9.5}$$

where $B_n(z)$ is the Bernoulli polynomial of order n (see Erdélyi 1953, § 1.13, pp. 35–39).

(ii) *Quadratic transformations*

Corresponding to (3.11) for the dilogarithm, we have

$$\mathrm{L}_n(z) + \mathrm{L}_n(-z) = 2^{1-n}\,\mathrm{L}_n(z^2), \quad n = 0, 1, 2, \dots \tag{9.6}$$

(see Gröbner & Hofreiter 1975, p. 73 (5a); Mignaco & Remiddi 1969; Kölbig *et al*. 1970, p. 46, eqn (3.15); Barbieri *et al*. 1971, 1972*a*, *b*). More generally, corresponding to (3.12) for the dilogarithm, we have

$$\sum_{k=0}^{m-1} \mathrm{L}_n(\omega^k z) = \frac{1}{m^{n-1}}\,\mathrm{L}_n(z^m), \quad \text{where } \omega = \mathrm{e}^{2\pi \mathrm{i}/m}, \quad m, n = 1, 2, 3, \dots \tag{9.7}$$

(see Lewin 1981, p. 197, eqn (7.41); Gröbner & Hofreiter 1975, p. 73, eqn (5)).

(iii) *Numerical values*

We have

$$\mathrm{L}_{2n}(1) = (-1)^{n-1}\frac{(2\pi)^{2n}B_n}{2(2n)!}, \quad n = 1, 2, \ldots, \tag{9.8}$$

$$\mathrm{L}_{2n}(-1) = -\left(1 - \frac{1}{2^{2n-1}}\right)\mathrm{L}_{2n}(1). \tag{9.9}$$

Here, B_n are the Bernoulli numbers ($B_0 = 1$, $B_1 = -1$, $B_2 = \frac{1}{6}$, etc. (see Erdélyi 1953, § 1.13, pp. 29, 30)).

(*b*) *Nielsen's generalized polylogarithms*

Nielsen's generalized polylogarithms, $S_{n,p}(z)$, are defined by

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{\ln^{n-1} t \ln^p(1-zt)}{t}\,\mathrm{d}t, \quad n, p = 1, 2, 3, \ldots. \tag{9.10}$$

From (9.10), one may obtain, by differentiation and partial integration, the difference-differential equation for $S_{n,p}(z)$,

$$\frac{\mathrm{d}}{\mathrm{d}z}S_{n,p}(z) = \frac{S_{n-1,p}(z)}{z}, \quad n \geqslant 2, \tag{9.11}$$

which may also be written in the form

$$S_{n,p}(z) = \int_0^z \frac{S_{n-1,p}(t)}{t}\,\mathrm{d}t. \tag{9.12}$$

If one further defines

$$S_{0,p}(z) \equiv \frac{(-1)^p}{p!}\ln^p(1-z), \tag{9.13}$$

then (9.11) and (9.12) are valid for $n, p \geqslant 1$. The polylogarithm is a special case of Nielsen's generalized polylogarithm,

$$\mathrm{L}_n(z) = S_{n-1,1}(z), \quad n = 2, 3, \ldots. \tag{9.14}$$

For details on Nielsen's generalized polylogarithms see Nielsen (1909), Mignaco & Remiddi (1969), Kölbig *et al.* (1970), Kölbig (1986) and Barbieri *et al.* (1971, 1972*a*, *b*).

(*c*) *Jonquière's function*

Jonquière's function, also referred to as a polylogarithm of non-integral order, is defined for complex s and z, as in (9.4), by

$$\mathrm{L}_s(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^s}, \quad |z| < 1. \tag{9.15}$$

The function $\mathrm{L}_s(z)$ is denoted by $\zeta(s, z)$ in the original work of Jonquière (1888, 1889*a*–*c*), by $F(z.s)$ in Erdélyi (1953, § 1.11, pp. 30, 31), where many of its properties are given, and by $\mathrm{Li}_s(z)$ in Lewin (1981, § 7.12, pp. 236–238) and in Lee (1997). The function $\mathrm{L}_s(z)$ satisfies, in particular, the relations (9.1) and (9.6) for the polylogarithm on replacing n by s.

(*d*) *Lerch's function*

Lerch's function is defined for complex z, α and s by

$$\Phi(z,s,\alpha)=\sum_{m=0}^{\infty}\frac{z^m}{(m+\alpha)^s},\quad |z|<1,\quad \alpha\neq 0,-1,-2,\ldots. \tag{9.16}$$

Jonquière's function is a particular case of Lerch's function, obtained when $\alpha=1$,

$$\mathrm{L}_s(z)=z\Phi(z,s,1). \tag{9.17}$$

For details on Lerch's function, see Erdélyi (1953, § 1.11, pp. 27–32) and Lerch (1887).

10. Historical notes

The integral

$$-\int_0^z \frac{\ln(1-t)}{t}\,\mathrm{d}t$$

has an extensive history that long predates it being named and being referred to as the dilogarithm. We give here some of the outstanding references to its early consideration. A careful examination of the original publications has enabled a critical appreciation of some of the early work in the literature on this function. The integral first appears in 1696 in the correspondence of Leibniz in a series of letters that are part of an extensive correspondence with Jacob and Johann Bernoulli (Leibniz 1855).† Leibniz expresses the integral in the form of a power series and discusses recursion relations for integrals of the form

$$\int_0^z t^{\mu}\ln^n(1+t)\,\mathrm{d}t,$$

noting that the case $\mu=-1$ must be excluded. The first study of the properties of the integral appears in an article by Landen (1760). He defines a series of functions that are identical to the polylogarithms defined here, giving both the recursion relation (9.1) and the Taylor series expansion (9.4). The transformations (3.2) and (3.3), as well as the transformation (9.5) for arbitrary n, are derived. In a memoir published 20 years later, Landen (1780) derives the values of the dilogarithm given here in (6.5) and (6.6), as well as the transformation that follows from (3.3) and (3.5) on elimination of $\mathrm{L}_2(z)$ between them. All of the published literature credits Euler with being the first to study the integral, referring to his work published in 1768‡, and calling the integral the Euler dilogarithm, a name given much later by Hill (1828). We have found there only the transformation listed here as (3.3), along with the numerical values given in (6.2) and (6.4), all of which were given in the earlier work of Landen (1760).

† Letter IX, pp. 56–62, January 1697; Letter XXXVIII, pp. 334–336, November 1696; Letter XXXIX, pp. 337–338, November 1696; Letter XLI, pp. 347–354, December 1696.

‡ Note that page numbers differ in the various later editions of Euler (1768), though the content is identical. In the 3rd edn (1824), this material is in vol. 1, pp. 110–112. In vol. 11 of *Leonardi Euleri opera omnia*, series I, pub. Lipsiae et Berolini, Typis et in Aedibus B. G. Teuberni (1913), the material is in *Institutiones Calculi Integralis*, vol. 1, pp. 113–114.

However, the first comprehensive detailed study of the function is the essay of Spence (1809), which was generally not referenced on the continent for several decades. (This work was, along with other manuscripts left by Spence, re-edited by John Herschel.) Spence (1820) defines various orders of logarithmic transcendents with the symbol

$$\overset{n}{\mathrm{L}}(1 \pm x),$$

which are essentially the polylogarithms defined here, i.e.

$$\overset{n}{\mathrm{L}}(1 \pm x) = -\,\mathrm{L}_n(\pm x).$$

He derives many of the transformations given here, among them (3.2), (3.3) and (3.11) for the dilogarithm, a number of functional relations involving two variables for both the dilogarithm and trilogarithm ($n = 3$), as well as (9.5), (9.6) and the recursion relation (9.1) for arbitrary n. His work also includes a study of the properties of the inverse tangent integral given here in § 8 *a*.

11. Applications in physics and mathematics

Numerous references to the occurrence of dilogarithms and polylogarithms in physical problems are given in Lewin (1981, § 1.12, pp. 31–35). An extensive list of references in which dilogarthms and polylogarithms appear in several fields of mathematics, among them number theory, geometry, representation theory and algebraic K-theory, are given in Kirillov (1995) and Oesterlé (1993). The paper by Zagier (1989), on the dilogarithm function in geometry and number theory, is worthy of note in being accessible to the non-specialist. In the field of statistical mechanics, the chemical potential of free Fermi and Bose gases is expressed in terms of polylogarithms in Lee (1995). We note in addition that the appearance of the dilogarithm is inherent to all higher-order calculations in quantum electrodynamics. This may be seen in the calculation of electron form factors in Mignaco & Remiddi (1969), Barbieri *et al*. (1971, 1972*a*, *b*) and Roskies *et al*. (1990), and in the calculation of radiative corrections in Mo & Tsai (1969), Maximon & Tjon (2000) and Passarino & Veltman (1979). The integrals essential to all of these calculations are evaluated in terms of dilogarithms in 't Hooft & Veltman (1979) and Passarino & Veltman (1979).

The author thanks Professor Richard Askey for bringing Kirillov (1995) to his attention. It is a particular pleasure to acknowledge the assistance given by the librarians and staff of a number of libraries for making original works of the eighteenth and nineteenth century available. Most especially these include the National Museum of Natural History Branch Library, the Dibner Library of the History of Science and Technology of the Smithsonian Museum of American History, the American University Library Special Collections, the library of the US Naval Observatory and the Lund University Library, Department of Cultural Heritage Collections. Finally, the author is deeply indebted to Professor Frank W. J. Olver for incisive and helpful comments concerning many details of this article, in particular his analysis concerning analytic continuation, which has been integrated into § 3 *b*.

References

Abel, N. H. 1881 Note sur la fonction $\psi x = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots + \frac{x^n}{n^2} + \cdots$. In *Oeuvres complètes de Niels Henrik Abel*, Nouvelle édition publée aux frais de l'État Norvégien, par MM. L. Sylow et S. Lie, vol. 2, pp. 189–193. Christiana: Grondahl & Son. (Reprinted by Johnson Reprint Corp, New York, 1965.)

Abramowitz, M. & Stegun, I. A. (eds) 1972 *Handbook of mathematical functions with formulas, graphs and mathematical tables*, 10th edn. National Bureau of Standards in Applied Mathematics, series 55. Washington, DC: US Government Printing Office.

Andrews, G. E., Askey, R. & Roy, R. 1999 Special functions. In *Encyclopedia of mathematics and its applications*, vol. 71. Cambridge University Press.

Barbieri, R., Mignaco, J. A. & Remiddi, E. 1971 Fourth-order radiative corrections to electron-photon vertex and the Lamb-shift value. *Nuovo Cim.* A **6**, 21–28.

Barbieri, R., Mignaco, J. A. & Remiddi, E. 1972*a* Electron form factors up to fourth order. I. *Nuovo Cim.* A **11**, 824–864.

Barbieri, R., Mignaco, J. A. & Remiddi, E. 1972*b* Electron form factors up to fourth order. II. *Nuovo Cim.* A **11**, 865–916.

Erdélyi, A. (ed.) 1953 *Higher transcendental functions*, vol. 1. McGraw-Hill.

Euler, L. 1768 *Institutionum calculi integralis*, vol. 1, pp. 124–126. Petropoli: Impensis Academiae Imperialis Scientiarum.

Gröbner, W. & Hofreiter, N. 1975 *Integraltafel–Zweiter Teil–Bestimmte Integrale*, 5th edn. Springer.

Hill, C. J. 1828 Ueber die Integration logarithmisch-rationaler Differentiale. *J. Reine Angew. Math.* **3**, 101–159.

Hill, C. J. 1830 *Specimen exercitii analytici, functionum integralum* $\int_0^x \frac{\mathrm{d}x}{x} L(1+2xC\alpha+x^2) = \mathrm{D}^\alpha x$ *tum quoad amplitudinem, tum quoad modulum comparandi modum exhibentis*, p. 9. Lund: Academia Carolina (Lund University).

Hölder, O. 1928 Über eine von Abel untersuchte Transzendente und eine merkwürdige Funktionalbeziehung. *Ber. über Verhandlungen Sächsischen Akad. Wissenschaften Leipzig, Math. Phys. Klasse* **80**, 312–325.

Jonquière, A. 1888 Ueber eine Klasse von Transcendenten, welche durch mehrmahlige Integration rationaler Funktionen entstehen. *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar* **45**, 522–531.

Jonquière, A. 1889*a* Note sur la série $\sum_{n=1}^{n=\infty} \frac{x^n}{n^s}$. *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar* **46**, 257–268.

Jonquière, A. 1889*b* Ueber einige Transcendente, welche bei der wiederholten Integration rationaler Funktionen auftreten. *Bihang till Kongl. Svenska Vetenskaps-Akademiens Handlingar* **15**, 1–50.

Jonquière, A. 1889*c* Note sur la série $\sum_{n=1}^{n=\infty} \frac{x^n}{n^s}$. *Bull. Soc. Math. France* **17**, 142–152.

Kirillov, A. N. 1995 Dilogarithm identities. *Prog. Theor. Phys.* **118**, 61–142.

Kölbig, K. S. 1986 Nielsen's generalized polylogarithms. *SIAM J. Math. Analysis* **17**, 1232–1258.

Kölbig, K. S., Mignaco, J. A. & Remiddi, E. 1970 On Nielsen's generalized polylogarithms and their numerical calculation. *BIT* **10**, 38–74.

Landen, J. 1760 A new method of computing sums of certain series. *Phil. Trans. R. Soc. Lond.* **51**, 553–565.

Landen, J. 1780 *Mathematical memoirs respecting a variety of subjects; with an appendix containing tables of theorems for the calculation of fluents*, vol. 1. London: J. Nourse.

Lee, M. H. 1995 Polylogarithmic analysis of chemical potential and fluctuations in a D-dimensional free Fermi gas at low temperatures. *J. Math. Phys.* **36**, 1217–1231.

Lee, M. H. 1997 Polylogarithms and Riemann's ζ function. *Phys. Rev.* E **56**, 3909–3912.

Leibniz, G. W. 1855 *Leibnizens mathematische Schriften* (ed. C. I. Gerhardt), vol. III. Halle: H. W. Schmidt.

Lerch, M. 1887 Note sur la fonction $\mathfrak{K}(w,x,s) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{2k\pi \mathrm{i} x}}{(w+k)^s}$. *Acta Math.* **11**, 19–24.

Lewin, L. 1981 *Polylogarithms and associated functions.* Elsevier.

Lewin, L. (ed.) 1991 *Structural properties of polylogarithms*, vol. 37. Providence, RI: American Mathematical Society.

Maximon, L. C. & Tjon, J. A. 2000 Radiative corrections to electron-proton scattering. *Phys. Rev.* C **62**, 054320.

Mignaco, J. A. & Remiddi, E. 1969 Fourth-order vacuum polarization contribution to the sixth-order electron magnetic moment. *Nuovo Cim.* A **60**, 519–529.

Mo, L. W. & Tsai, Y. S. 1969 Radiative corrections to elastic and inelastic ep and μp scattering. *Rev. Mod. Phys.* **41**, 205–235.

Nielsen, N. 1909 Der Eulersche Dilogarithmus und seine Verallgemeinerungen. *Nova Acta Abh. d. Kaiserl. Leop.-Carol. Deutschen Akademie d. Naturf.* **90**, 123–211.

Oesterlé, J. 1993 Polylogarithmes. *Séminaire Bourbaki Astérisque* **216**, 49–67.

Passarino, G. & Veltman, M. 1979 One-loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model. *Nucl. Phys.* B **160**, 151–207.

Roskies, R. Z., Remiddi, E. & Levine, M. J. 1990 Quantum electrodynamics. In *Advanced series on directions in high energy physics* (ed. T. Kinoshita), vol. 7, pp. 162–217. World Scientific.

Sanchis-Lozano, M. A. 1997 Simple connections between generalized hypergeometric series and dilogarithms. *J. Comput. Appl. Math.* **85**, 325–331.

Spence, W. 1809 *An essay of the theory of the various orders of logarithmic transcendents; with an inquiry into their applications to the integral calculus and the summation of series.* London: J. Murray.

Spence, W. 1820 *Mathematical essays by the late William Spence, Esq.* (ed. J. F. W. Herschel). Edinburgh: Oliver and Boyd.

't Hooft, G. & Veltman, M. 1979 Scalar one-loop integrals. *Nucl. Phys.* B **153**, 365–401.

Zagier, D. 1989 The dilogarithm function in geometry and number theory. In *Number theory and related topics (papers presented at the Ramanujan Colloquium, Bombay 1988).* Oxford University Press.