THE DLMF PROJECT: A NEW INITIATIVE IN CLASSICAL SPECIAL FUNCTIONS

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NIST (formerly, National Bureau of Standards) has started an ambitious project that aims to produce a successor to Abramowitz and Stegun's *Handbook of Mathematical Functions*, published by the National Bureau of Standards in 1964 and reprinted by Dover in 1965. Both editions continue to sell briskly and are widely cited in the scientific literature. However, with the many advances in the theory, computation and application of special functions that have occurred since 1960, a new standard reference is badly needed. NIST intends to satisfy this need by providing a Digital Library of Mathematical Functions (DLMF) as a free Web site together with an associated book and CD-ROM. The Web site will provide many capabilities that are impossible to provide in print media alone.

1 Introduction

Perhaps the most widely-used catalog of special functions is the NBS *Handbook of Mathematical Functions*,[1] published in 1964 by the National Bureau of Standards. This book, whose sales number in the hundreds of thousands, may well be the most-frequently cited source in all archival scientific literature. It was found from *Science Citation Index*, for example, that the NBS Handbook was cited over 7000 times in the interval 1992-1996, and that *the number of its citations is growing faster than the Index as a whole.* This demonstrates the continuing need, across many branches of science and engineering, for reliable information on special functions—especially since the NBS Handbook is badly out-of-date in both content and structure.

The NBS Handbook has never been revised, although there have been numerous advances in basic theory, computational methods, and domains of application since its publication. Its structure is that of a static reference volume; though still very useful, it does not meet the needs of modern users for information that can be conveniently exploited in a highly computer-oriented technical environment. Such needs have been communicated regularly to NIST (formerly, National Bureau of Standards). Recently the author and his colleagues have developed a structured response to these needs by talking with others, hosting workshops, speaking at meetings of professional societies, producing follow-up reports, and establishing e-mail feedback mechanisms.

The project that this process has identified is the subject of this paper: the development of a knowledge base, the Digital Library of Mathematical Functions (DLMF), that is intended to become the successor to the NBS Handbook. Via free access over the World Wide Web, using standard Web browsers, individuals will be able to obtain validated mathematical information in a semantic-based representation that incorporates metadata, interactive features, and extensive linkages. The Web server will be constructed, maintained and operated by NIST. Background information on the project is provided in the NIST reports.[2,3]

The DLMF will incorporate the simple look-up capability of the current NBS Handbook—and much more. The mathematical data will be more comprehensive and will be delivered directly to the desktop working environment. Users will be offered capabilities that are unavailable anywhere today: construction of technical documents with mathematical content that is linked directly to NIST standards, search and retrieval of complex mathematical information, and interaction with a virtual community of authors and users. Such capabilities are of broad benefit to the scientific community, both by the strengthening of discipline-specific mathematical techniques and for the facilitation that a common standard can provide for the exchange of information and techniques among disciplines.

2 The DLMF Project

2.1 Origins

The DLMF Project can be traced back to October 1996 when a working group of several NIST staff began meeting informally to discuss the opportunities presented by the rapid expansion of Web technology that might be applicable to the presentation of mathematical reference information, particularly with respect to special functions. It was clear from the beginning that a project resulting in a genuine replacement for the NBS Handbook would be a very large undertaking. Nevertheless, a general approach was developed, and it was decided in December 1996 to obtain a preliminary determination of the degree of support among scientific colleagues for a new NIST-led project in special functions.

Of the 367 individuals contacted, 46% replied. All agreed that a project to rewrite and modernize the NBS Handbook would be worthwhile. Very few reservations were expressed. Respondents suggested functions not presently included that should be covered and mentioned topics that need improved coverage. Comments on such subjects as format, the desirability of maintaining a printed version, dissemination on CD-ROM's, the issue of software vs. tables, the need for high numerical precision, computational complexity of algorithms, the new Bateman project, and the role of software packages were received also.

Encouraged by this feedback, the NIST working group held an invitational workshop July 28–31, 1997. A specific project plan was developed by involving the participants in roundtable discussions on six different topics: *approach to the project* (philosophy, Web implementation, tables, graphics, symbolics, software); *organization of the project* (editors, associate editors, authors, validators, Web implementors); *funding* (NIST, National Science Foundation, other agencies, proposal writing, administration of funds); *chapters* (existing, proposed, format, sample, authors, validators); *roles* (physics, statistics, numerical analysis, approximation theory, interval analysis); *future phases* (hard copy, CD-ROM's, updating, benchmark algorithms). The non-NIST participants were: *R. A. Askey*, Department of Mathematics, University of Wisconsin; *A. R. Barnett*, Department of Physics and Astronomy, University of Manchester; *C. Brezinski*, Laboratoire d'Analyse Numerique et d'Optimisation, Université des Sciences et Technologies de Lille; *B. C. Carlson*, Department of Mathematics, Iowa State University; *W. Gautschi*, Computer Science Department, Purdue University; *K. S. Kölbig*, CERN, Geneva; *L. C. Maximon*, Department of Physics, George Washington University; *A. B. Olde Daalhuis*, Department of Mathematics and Statistics, University of Edinburgh, United Kingdom; *I. Olkin*, Department of Statistics, Stanford University; *R. B. Paris,* Mathematical Sciences Division, University of Abertay Dundee; and *N. M. Temme*, Centrum voor Wiskunde en Informatica, Amsterdam. Their contributions were invaluable in determining the management structure and work plan for the project, the proposed chapter layout, and the naming of the Board of Associate Editors.

2.2 Content

A significant mathematical research activity is required to collect mathematical results from the literature, select the relevant ones, verify them, and arrange them into a coherent structure. Among the areas to be investigated are: *new fields of application*, such as soliton theory and nonlinear dynamics, in which the classical special functions appear [4,5] or are redefined [6] in a different guise; *additional functions of practical importance*, such as Carlson's elliptic integrals,[7] discrete orthogonal polynomials,[8] new statistical distribution functions,[9,10,11] Painlevé transcendents,[12] and basic hypergeometric functions;[13] *analytical developments* such as uniform asymptotics,[14,15,16] asymptotics via distribution theory,[16] and greatly increased attainable accuracy from asymptotic expansions by re-expansion of remainder terms [17,18,19,20] or by application of nonlinear sequence transformations;[21,22,23] *discovery of new properties*, including integral representations, integrals, addition formulas and generating functions;[24,25,26,27,28,29,30] *current computational methodology*, such as interval analysis,[31,32,33] Padé approximations,[34] boundary-value methods,[35,36,37,38] and computer algebra;[39] and *identification of software*. The list of references in this paragraph is not exhaustive but only indicative of the kinds of references that will be examined in the preparation of the DLMF.

An important feature of the content development work is to highlight important unsolved mathematical and computational problems in special functions, thus encouraging future research into their solution. For example, asymptotic representations for functions with several parameters are very difficult to obtain and many that would be of great utility are still unknown. The whole task is difficult and needs to be done with great insight and experience.

Tables of numerical function values remain important for the construction and testing of mathematical software and for the generation of computer graphics. The voluminous tables in the NBS Handbook were constructed primarily as aids to computation by interpolation, and they are almost totally inadequate for current needs. A future phase of the DLMF Project will provide the capability for users to generate tables of certified function values over arbitrary ranges of the independent variables and to arbitrary precision. The initial phase will provide a small number of newly constructed tables together with references to publications that contain tables.

Graphs, such as the ones in the NBS Handbook, are of great utility in helping users understand the qualitative properties of special functions, and have even stimulated original discoveries of exact relationships. The power of the typical modern desktop computer makes possible a much richer interaction with graphical representations. For example, 3-dimensional surfaces of mathematical functions can be explored interactively through the use of VRML (Virtual Reality Modeling Language). For rendering of functions, substantial difficulties arise with representation of extreme data, falsification of smooth boundaries, and proper inclusion of important features such as zeros. These problems have already been attacked, with some success, by NIST staff.[40]

2.3 Editorial Supervision, Authorship and Validation

The Executive Committee for the whole project consists of the four Principal Editors: *Daniel W. Lozier*, General Editor (NIST); *Frank W. J. Olver*, Mathematics Editor (NIST and University of Maryland); *Charles W. Clark*, Scientific Applications Editor (NIST); and *Ronald F. Boisvert*, Information Technology Editor (NIST). They are being supported by a highly qualified staff of NIST mathematicians, physicists and computer scientists.

The Executive Committee is also being actively assisted by the Board of Associate Editors: *R. A. Askey*, University of Wisconsin (special functions); *M. V. Berry*, University of Bristol, United Kingdom (physics); *W. Gautschi*, Purdue University (numerical analysis); *L. C. Maximon*, George Washington University (physics); *M. Newman*, University of California, Santa Barbara (combinatorics and number theory); *I. Olkin*, Stanford University (statistics); *P. Paule*, Johannes Kepler University, Linz (symbolic computing); *W. P. Reinhardt*, University of Washington (chemistry); *N. M. Temme*, Centrum voor Wiskunde en Informatica, Amsterdam (special functions); and *J. Wimp*, Drexel University (special functions). This Board will confer regularly and provide critical advice on all technical aspects of the project, such as reviewing subject coverage, identifying prospective authors and validators, providing general oversight and guidance on matters of format and presentation, and assuring that all project goals are met.

As was the case with the NBS Handbook, the chapters will be written by individual authors, or in some cases several authors, drawn from the USA and abroad. They are being selected by the Executive Committee. Members of the Executive Committee and the Board of Associate Editors are eligible to serve as authors. The responsibilities of the authors includes research of the literature of the past 40 years and of recent software packages. Provisional authors have already been selected for some chapters; see Section 3 below.

Each chapter will also be assigned an independent validator to check the accuracy of its content. This was not done for the NBS Handbook. If it had it been, it is certain that some errors now known, and possibly others still unknown, would have been avoided.

The authors and validators are being engaged and remunerated under contract to NIST. These contracts are being negotiated and administered by the Executive Committee with assistance from NIST administrative and legal personnel. Final payment is contingent upon satisfactory completion of all contractual obligations.

2.4 Web Site

The DLMF Project aims to provide *universal access* to validated mathematical information in a Web framework with rich structure and many interactive features. The DLMF is not intended to serve as a general-purpose mathematical software system, such as Mathematica, Maple or Matlab, but its contents should be structured so as to be accessible to any such program. Therefore, to the greatest possible extent, it will utilize Web facilities that have been established as open standards. The following capabilities have been identified as attainable project objectives:

- Transference of formulas, graphs, tables and text into word processors, thus avoiding the tedious and error-prone process of manual copying.
- Translation of formulas into code that is usable by mathematical software systems.
- On-demand generation of tables of numerical values, intended primarily for use in constructing and testing mathematical software.
- On-demand generation of graphs, including still images, animations and interactive graphics, intended for scientific visualization and pedagogical applications.
- Advanced search capabilities, with the ability to locate mathematical formulas quickly and easily.
- Links to software and references to algorithms for computing special functions.
- A convenient on-line facility for users to provide commentary on each formula, graph, table or numbered subsection in a permanent, universally accessible record.
- Continuous updating to incorporate corrections, additions and extensions.

Existing Web facilities are sufficiently well-established to provide some of this functionality. However, difficulties exist that are being considered by various groups such as the World Wide Web Consortium. A particularly critical issue is the representation of mathematical formulas within a computer database. The representation must include semantic information that is sufficient for unambiguous translation from one format to another; the LaTeX encoding, for example, was not designed for this and is not sufficient. Another issue is the current unsatisfactory Web practice of displaying formulas as unscalable images.

The Web site is being constructed at NIST and will be maintained indefinitely as a free public resource. NIST is tracking and contributing to efforts to solve difficulties such as the ones identified above, and the DLMF Project will adopt consensus standards as they emerge.

2.5 Application Modules

Application modules are auxiliary Web sites constructed as models for the practical utilization of special functions in science and engineering. Such modules, ranging in scope from training manuals to review monographs, express mathematics in the form that is preferred by the target discipline. Discipline-specific definitions and re-normalizations are traced back, via active links, to a primary definition in the DLMF. For example, Hermite polynomials are encountered in applications such as signal processing, statistical data analysis, electromagnetic wave propagation, quantum mechanics, approximation of functions, and numerical quadrature. These separate applications of Hermite polynomials utilize different normalizations, and the conversions between separate definitions, though straightforward in principle, can be tedious and prone to error.

The DLMF will be universally accessible and will contain primary definitions of virtually any special function likely to be of interest in applications. Eventually, engines might be built that could perform automatic formula verification, translation, or searching for identical or similar formulas across disciplines. NIST physicists have begun work on prototype application modules as a research effort, with the goals of exploring alternative forms of integration of physical and mathematical data, and of addressing issues of ease of development, ease of use, accessibility, and communication across disciplines.

2.6 Sample Chapter

To provide a model for authors, and to help gauge budgetary needs, a sample chapter on Airy functions [41] has been drafted. The chapter is not quite complete. It incorporates 7 pages of Chapter 10 of the NBS Handbook and also contains a more than equal amount of new material. Some further updating may be needed, especially for developments of the past four years, and final validation is required.

The mock-up Web implementation of the sample chapter is accessible at http://math.nist.gov/DigitalMathLib. Among its features are:

- A comprehensive 'formularium' for Airy functions and generalizations.
- 'About' links that give metadata such as references, notes, attributes, and LaTeX encodings.
- Color graphics, including curves, surfaces and user-controllable visualizations (in VRML) of Airy functions and their derivatives.
- Links to Web sites where software can be obtained over the Internet.

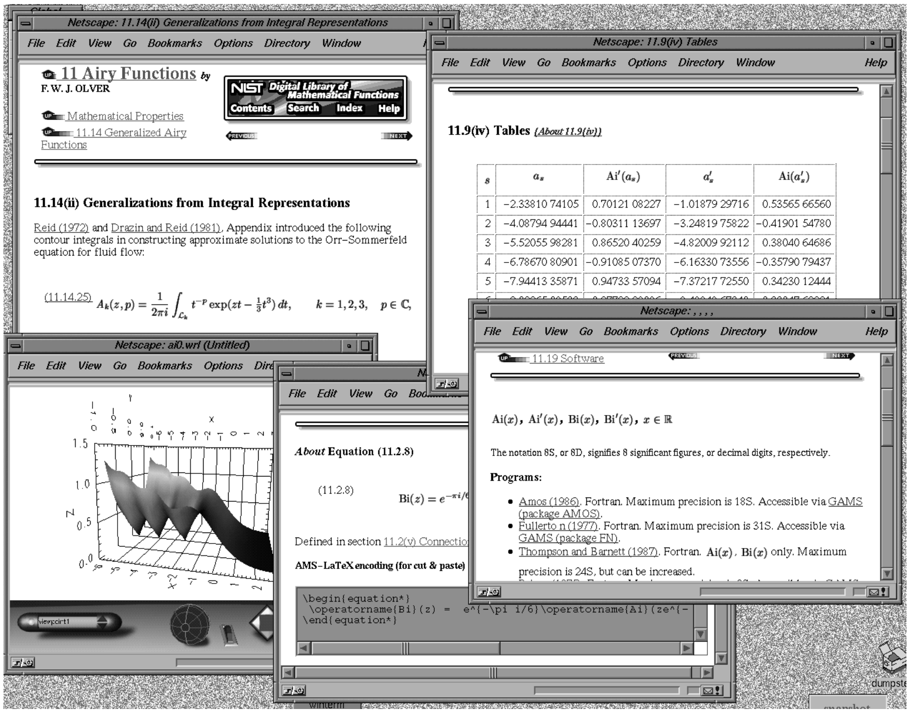

Figure 1. Representative screens from the DLMF sample chapter on Airy functions at http://math.nist.gov/DigitalMathLib. Copyright ©1998 by Frank William John Olver. Used by permission.

- Short tables of zeros and other special values.
- Images of physical phenomena with captions that describe their connection with Airy functions.
- A link to a prototype application module for quantum mechanics.

The first five of these features are illustrated in Figure 1.

2.7 Book Publication

Many persons have expressed a strong preference for a book as well as a Web site. The Executive Committee is looking into possiblities for book publication, with the principal goal being extremely low cost so as to make the book nearly as widely accessible as the free Web site. The possibility of including a CD-ROM with the book to provide some of the additional capabilities of the Web site is being considered also.

2.8 Time Frame

Authors will be required to complete their writing by the end of the second summer after receiving their contracts. Validators are to complete their work no later than the following summer. Web implementation and overall editing will be taking place at all stages, and the whole project should be in final form on the Web by September 2003. It will be partially available before this. Publication in CD-ROM and book form will also occur at approximately the same time.

3 Chapters and Authors

A tentative list of 40 chapter titles, together with prospective authors and validators for some of them, was drawn up after the first meeting of the Executive Committee and the Board of Associate Editors at NIST on March 3–4, 1999.

Subsequently the Executive Committee obtained informal commitments for the tentative authorship of 24 chapters; see Tables 1 and 2. The tentative authors who are not members of the Executive Committee or Board of Associate Editors are: *G. E. Andrews*, Pennsylvania State University; *T. M. Apostol*, California Institute of Technology; *C. Brezinski*, Université des Sciences et Technologies de Lille; *B. C. Carlson*, Iowa State University; *F. Chyzak*, INRIA Rocquencourt; *K. Dilcher*, Dalhousie University; *T. M. Dunster*, San Diego State University; *R. Koekoek*, Delft University of Technology; *A. Olde Daalhuis*, University of Edinburgh; *R. B. Paris*, University of Abertay Dundee; *M. J. Seaton*, University College, London; and *R. F. Swarttouw*, Free University Amsterdam. It is necessary to emphasize that the chapters, chapter titles and authors are subject to change.

The chapters will be enriched with copious meta-information, active links within the DLMF, additional links to pertinent external sites, indexes, a powerful search mechanism, advanced user-controllable graphical renderings, active references to software, and eventually active references to built-in algorithms that can produce numerical tables and symbolic transformations on demand.

Table 1. Tentative list of chapters and authors. See also Table 2.

Ch.	Tentative Title	Tentative Author(s)
1	Mathematical and Physical Constants	
2	Algebraic and Analytical Methods	
3	Asymptotic Approximations	*Olver*
3A	Integrals with Coalescing Saddles	*Berry*
4	Numerical Methods	*Gautschi & Brezinski*
5	Computer Algebra	*Paule & Chyzak*
6	Elementary Functions	
7	Exponential Integral, Logarithmic Integral, Sine and Cosine Integrals	*Temme*
8	Gamma Function	*Askey*
9	Error Functions, Dawson's Integral, Fresnel Integrals	*Temme*
10	Incomplete Gamma Functions and Generalized Exponential Integral	*Paris*
11	Airy Functions	*Olver*
12	Bessel Functions	*Olver & Maximon*
13	Struve Functions and Anger-Weber Functions	*Paris*
14	Legendre Functions and Spherical Harmonics	*Dunster*
15	Confluent Hypergeometric Functions	*Wimp*
16	Coulomb Wave Functions	*Seaton*
17	Parabolic Cylinder Functions	*Temme*
18	Hypergeometric Functions	*Olde Daalhuis*
19	Generalized Hypergeometric Functions and Meijer G-Function	*Askey*

4 Conclusions

The DLMF Project has two general goals:

- To develop an up-to-date account of the kind of information that is found in the NBS Handbook of Mathematical Functions by engaging leading researchers and users of special functions in a complete assessment of the field as it pertains to current and future needs, with emphasis on relevant mathematical properties, computational issues, and typical applications.

Table 2. Tentative list of chapters and authors. See also Table 1.

Ch.	Tentative Title	Tentative Author(s)
20	Classical Orthogonal Polynomials	*Koekoek & Swarttouw*
21	Other Orthogonal Polynomials	*Koekoek & Swarttouw*
22	Elliptic Integrals	*Carlson*
23	Jacobian Elliptic Functions and Theta Functions	
24	Weierstrass Elliptic Functions	
25	Bernoulli and Euler Polynomials.	*Dilcher*
26	Zeta Function	*Apostol*
27	Combinatorial Analysis	
28	Number and Group-Theoretic Functions	
29	Statistical Methods and Distributions	*Olkin*
30	Mathieu Functions	
31	Lamé Functions. Spheroidal Wave Functions	
32	Heun Functions and Hill's Equation	
33	Painlevé Transcendents	
34	Nonlinear Integrable Evolution Equations	
35	Basic Hypergeometric Functions and q-Series	*Andrews*
36	Dilogarithms and Polylogarithms	
37	Wavelets	
39	Scientific Applications	

- To present the newly developed body of information as a validated, interactive, and richly linked Digital Library of Mathematical Functions, which will deliver mathematical reference information in a highly usable format directly to the scientist's desktop computer from a Web site to be operated indefinitely as a free public service.

Meeting these goals will require an intensive period of development followed by a long-term commitment to the operation and upkeep of the Web site. NIST is uniquely qualified to undertake this important project because, unlike industry or academia, it is a U. S. Government laboratory with the mission to develop and provide critical parts of the nation's scientific and technological infrastructure.

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References

1. M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U. S. Government Printing Office, Washington, D. C., 1964.
2. D. W. Lozier. Toward a Revised NBS Handbook of Mathematical Functions. NISTIR 6072, National Institute of Standards and Technology, Gaithersburg, MD 20899 USA, September 1997.
3. D. W. Lozier, B. R. Miller, and B. V. Saunders. Design of a digital mathematical library for science, technology and education. To appear in Proc. IEEE Advances in Digital Libraries Conference, Baltimore, Maryland, May 19–21, 1999. Preprint: Technical Report NISTIR 6297, February 1999.
4. F. K. Abdullaev. *Solitons in Inhomogeneous Media*. John Wiley & Sons, New York, 1994.
5. A. Hasegawa. *Solitons in Optical Communications*. Oxford University Press, 1995.
6. M. J. Seaton. Quantum defect theory. *Rep. Progr. Phys.*, 46:167–, 1983.
7. B. C. Carlson. Elliptic integrals of the first kind. *SIAM J. Math. Anal.*, 8:231–242, 1977.
8. A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. *Classical Orthogonal Polynomials of a Discrete Variable*. Springer-Verlag, Berlin, 1991.
9. T. P. Hutchinson and C. D. Lai. *Continuous Bivariate Distributions*. Rumsby Scientific Publishing, Adelaide, South Australia, 1990.
10. N. L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous Univariate*

Distributions, volume 2. John Wiley & Sons, New York, second edition, 1995.
11. N. L. Johnson, S. Kotz, and N. Balakrishnan. *Discrete Multivariate Distributions*. John Wiley & Sons, New York, 1997.
12. K. Iwasaki, H. Kimura, S. Shimomura, and T. Kimura. *From Gauss to Painlevé: A Modern Theory of Special Functions*. Vieweg, Braunschweig, 1991.
13. G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Cambridge University Press, 1990.
14. N. Bleistein and R. A. Handelsman. *Asymptotic Expansions of Integrals*. Holt, Rinehart and Winston, New York, 1975.
15. F. W. J. Olver. *Asymptotics and Special Functions*. Academic Press, New York, 1974. Reprinted by AK Peters, Wellesley, Massachusetts, 1997.
16. R. Wong. *Asymptotic Approximations of Integrals*. Academic Press, New York, 1989.
17. M. V. Berry. Uniform asymptotic smoothing of Stokes's discontinuities. *Proc. Roy. Soc. London Ser. A*, 422:7–21, 1989.
18. M. V. Berry and C. Howls. Hyperasymptotics for integrals with saddles. *Proc. Roy. Soc. London Ser. A*, 434:657–675, 1991.
19. A. B. Olde Daalhuis and F. W. J. Olver. Hyperasymptotic solutions of second-order linear differential equations I. *Methods Appl. Anal.*, 2:173–197, 1995.
20. F. W. J. Olver. Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms. *SIAM J. Math. Anal.*, 22:1475–1489, 1991.
21. C. Brezinski and M. Redivo Zaglia. A new presentation of orthogonal polynomials with applications to their computation. *Numer. Algorithms*, 1:207–221, 1991.
22. E. J. Weniger. Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series. *Comput. Phys. Rep.*, 10:189–371, 1989.
23. J. Wimp. *Sequence Transformations and Their Applications*. Academic Press, New York, 1981.
24. R. A. Askey. *Orthogonal Polynomials and Special Functions*. SIAM, Philadelphia, 1975.
25. L. Debnath. *Integral Transforms and Their Applications*. CRC Press, Boca Raton, Florida, 1995.
26. I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products*. Academic Press, 5th edition, 1994.

27. T. H. Koornwinder. The addition formula for Jacobi polynomials I. Summary of results. *Proc. Konink. Nederl. Akad. Wetensch.*, 34:188–191, 1972.
28. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series*. Gordon and Breach, second printing with corrections, 1986.
29. H. M. Srivastava and H. L. Manocha. *A Treatise on Generating Functions*. Horwood, Chichester, 1984.
30. N. M. Temme. *Special Functions. An Introduction to the Classical Functions of Mathematical Physics*. John Wiley & Sons, New York, 1996.
31. G. Alefeld and J. Herzberger. *Introduction to Interval Computations*. Academic Press, New York, 1983.
32. R. Hammer, M. Hocks, U. Kulisch, and D. Ratz. *Numerical Toolbox for Verified Computing. I: Basic Numerical Problems, Theory, Algorithms, and Pascal-XSC Programs*. Springer-Verlag, Berlin, 1993.
33. R. E. Moore. *Methods and Applications of Interval Analysis*. SIAM, Philadelphia, 1979.
34. G. A. Baker, Jr. and P. Graves-Morris. *Padé Approximants*. Cambridge University Press, second edition, 1996.
35. F. W. J. Olver. Numerical solution of second-order linear difference equations. *J. Res. Nat. Bur. Standards Sect. B*, 71:111–129, 1967.
36. D. W. Lozier. Numerical Solution of Linear Difference Equations. NBSIR 80-1976, National Bureau of Standards, Gaithersburg, MD 20899 USA, March 1980.
37. D. W. Lozier and F. W. J. Olver. Airy and Bessel functions by parallel integration of ODEs. In R. F. Sincovec, D. E. Keyes, M. R. Leuze, L. R. Petzold, and D. A. Reed, editors, *Proceedings of the Sixth SIAM Conference on Parallel Processing for Scientific Computing, vol. 2*, pages 531–538. SIAM, Philadelphia, 1993.
38. J. Wimp. *Computation with Recurrence Relations*. Pitman, London, 1984.
39. M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. AK Peters, Wellesley, Massachusetts, 1996.
40. Q. Wang and B. Saunders. Interactive 3D Visualization of Mathematical Functions Using VRML. Technical Report NISTIR 6289, National Institute of Standards and Technology, Gaithersburg, MD 20899 USA, February 1999.
41. F. W. J. Olver. Airy Functions, Chapter 11, Digital Library of Mathematical Functions Project. National Institute of Standards and Technology, Gaithersburg, MD 20899-8910 USA, 1998. http://math.nist.gov/DigitalMathLib/.