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LOGARITHMIC CONVEXITY AND THE "SLOW EVOLUTION" CONSTRAINT IN ILL-POSED INITIAL VALUE PROBLEMS*

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Abstract. This paper examines a wide class of ill-posed initial value problems for partial differential equations, and surveys logarithmic convexity results leading to Hölder-continuous dependence on data for solutions satisfying prescribed bounds. The discussion includes analytic continuation in the unit disc, time-reversed parabolic equations in L^p spaces, the time-reversed Navier–Stokes equations, as well as a large class of nonlocal evolution equations that can be obtained by randomizing the time variable in abstract Cauchy problems. It is shown that in many cases, the resulting Hölder-continuity is too weak to permit useful continuation from imperfect data. However, considerable reduction in the growth of errors occurs, and continuation becomes feasible, for solutions satisfying the *slow evolution from the continuation boundary* constraint, previously introduced by the author.

Key words. ill-posed problems, analytic continuation, backwards in time continuation, logarithmic convexity, Hölder-continuity, parabolic equations, growing diffusion coefficients, non self-adjoint problems, Navier–Stokes equations, holomorphic semigroups, subordinated processes, slow evolution from continuation boundary, SECB constraint

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1. Introduction.

The problem of reconstructing the past behavior of a system, given knowledge of its current state, is of interest to many branches of science. For evolution equations, such backwards in time continuation is typically ill-posed in the presence of dissipative terms. Other spatial continuation problems in elliptic or parabolic equations exhibit similar characteristics. This paper is concerned with the Hölder-continuous dependence on data that results when certain ill-posed continuation problems in partial differential equations are stabilized by prescribed bounds [14], [21].

Because the Hölder exponent must decay to zero as the continuation boundary is approached, there is an unavoidable growth of errors originating from imperfect data. In some cases, such errors may preclude continuation into a region of particular interest. The *slow evolution from the continuation boundary* (SECB) constraint introduced in [6], [7] is an a priori statement about the rate of change of the desired solution near the continuation boundary. This information supplements information provided by prescribed bounds on the solution. As a consequence, stronger stability estimates can be obtained for solutions satisfying the SECB constraint than is otherwise possible. This constraint was shown to be effective in controlling the growth of noise in certain *image deblurring* problems, in which backwards in time continuation in diffusion equations involving fractional Laplacians plays a key role. In these problems, the Hölder exponent decays *linearly* to zero.

The present self-contained paper deals with a much wider class of problems. We survey important classes of equations, including the Navier–Stokes equations, where logarithmic convexity inequalities can be shown to hold. Using the theory of holomorphic semigroups, we consider parabolic equations in L^p spaces, as well as a large class of nonparabolic problems, typically involving nonlocal differential operators, that can

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be constructed by *subordination* [8]. The resulting Hölder exponents are particularly instructive in their dependence on the continuation variable. Linear decay to zero is the exception, the behavior being generally sublinear, and rapid exponential decay is possible in some cases. In time-dependent problems, such Hölder exponents are indicative of the rate at which the evolution equation has forgotten the past, and hence, of the subsequent difficulty of reconstructing the past from imperfect knowledge of the present.

It is shown that the SECB constraint, when applicable, becomes progressively more significant the faster the Hölder exponent decays to zero. Considerable error reduction is possible in many cases. Indeed, continuation problems that were heretofore intractable may become amenable to numerical computation, provided their solutions satisfy an SECB constraint. The paper concludes with a simple, explicit example of backwards in time continuation in an evolution equation with exponentially decaying Hölder exponent.

The following problem is important in its own right and serves to motivate the subsequent discussion.

1.1. Analytic continuation in the unit disc. Let $\mathcal{A}$ be the class of complex-valued functions $u(z)$ that are continuous in the closed unit disc and holomorphic in its interior, and let

$$\| u(r) \|_\infty = \max_{0\le\theta\le 2\pi} |u(re^{i\theta})|. \tag{1}$$

Fix a with $0 < a < 1$, and consider the problem of determining $u(re^{i\theta})$ for $a < r < 1$ from approximate knowledge of $u(z)$ on the circle $r = a$. Hadamard's three-circle theorem asserts that $\log \| u(r) \|_\infty$ is a convex function of $\log r$ for $a \le r \le 1$. If

$$\omega(r) = \log r / \log a, \qquad 0 < a \le r \le 1, \tag{2}$$

then

$$\| u(r) \|_\infty \le \| u(1) \|_\infty^{1-\omega(r)} \| u(a) \|_\infty^{\omega(r)}, \qquad 0 < a \le r \le 1. \tag{3}$$

We have equality in (3) for $u(z) = z^n$. This convexity inequality is the basis for stabilizing the ill-posed continuation problem when noisy data are given on $r = a$. Restrict the class of admissible continuations to functions $u(z) \in \mathcal{A}$ satisfying a *prescribed* bound,

$$\| u(1) \|_\infty \le M. \tag{4}$$

Fix $\epsilon > 0$, $\epsilon \ll M$, and let data $f(\theta)$ be given on $r = a$ such that for some $u(z) \in \mathcal{A}$ satisfying (4), we have

$$\| u(a) - f \|_\infty \le \epsilon. \tag{5}$$

If now $u_1(z), u_2(z) \in \mathcal{A}$ are any two objects satisfying (4) and (5), we get from (3)

$$\| u_1(r) - u_2(r) \|_\infty \le 2M^{1-\omega(r)}\epsilon^{\omega(r)}, \qquad a \le r \le 1. \tag{6}$$

For fixed $r_0 < 1$, the difference between any two possible continuations at $r = r_0$ can be made arbitrarily small in the L^∞ norm by giving sufficiently accurate data at $r = a$, i.e., by making $\epsilon > 0$ *sufficiently small* in (5). On the other hand, no matter how small one chooses ϵ in (5), the inequality (6) cannot ensure accurate results at the continuation boundary $r = 1$, since $\omega(1) = 0$. Indeed, with M given in (4), and $\epsilon > 0$ given in (5), consider continuing the function $u(z) \equiv M/2$ from data $f(\theta) = M(1 + a^n e^{in\theta})/2$ at $r = a$, where n is such that $a^n < 2\epsilon/M$. At $r = 1$, the continuation $v(z) = M(1+z^n)/2$ satisfies the prescribed bound, but approximates the desired continuation $u(z) \equiv M/2$ with a relative error of 100% in the L^∞ norm. The inequality (6) establishes Hölder-continuous dependence on the data only on compact subsets of the region where bounds are prescribed. This situation prevails in diverse classes of improperly posed problems in partial differential equations stabilized by a priori bounds. Use of such bounds, together with the analysis of the resulting continuity with respect to the data, was pioneered by Fritz John in a landmark paper [14].

A basic difficulty with the above Hölder-continuity is the following. In most applications, $\epsilon > 0$ is determined by the accuracy of the instrumentation used to acquire the data. While ϵ is usually small, it is *fixed* and *cannot* be made arbitrarily small. In such applications, the dependence of the Hölder exponent $\mu(t)$ on the continuation variable t plays a crucial role. In some cases, such as backwards in time continuation in the heat equation, we have $\mu(t) = t/T$, so that $\mu(t)$ decays linearly to zero as continuation progresses from $t = T > 0$ to the continuation boundary $t = 0$. More typically, μ is *sublinear* in the continuation variable. If μ decays too rapidly to zero, useful continuation becomes impossible, even in regions well away from the continuation boundary. This is the case in (2), for example, when $a > 0$ is small. In the case of evolution equations, as will be seen below, rapid decay of μ to zero can be brought about by various factors, including nonlinearity, non-self-adjointness, diffusion coefficients that grow with time, or adverse spectral properties in the spatial differential operator.

It develops that while prescribed bounds are necessary to stabilize ill-posed initial value problems, they are frequently insufficient to allow continuation far enough into the region of interest. Further a priori information must be provided for this purpose. In this paper we show that knowing the rate of change of the desired solution near the continuation boundary is *slow* can be very helpful.

2. Slow evolution from the continuation boundary (SECB). We consider linear or nonlinear continuation problems in a single variable t, $0 \le t \le T$, with continuation boundary at $t = 0$. In spatial continuation problems with radial symmetry, t is a radial coordinate, e.g., $t = 1 - r$ in (1). In applications involving continuation in the time variable, t is related to time. In an appropriate Banach space X with norm $\| \ \|$, the continuation $u(t)$ is an X-valued function with norm $\| u(t) \|$ for fixed t. Let $u_1(t)$, $u_2(t)$, be any two continuations from the given data $f(x)$ at $t = T$, with $\| u_i(T) - f \| \le \epsilon$, and satisfying a prescribed bound, $\| u_i(0) \| \le M$ at $t = 0$. Here, ϵ, $M > 0$ are both known, and $\epsilon \ll M$. Let $w(t) = u_1(t) - u_2(t)$. We assume $w(t)$ satisfies a convexity inequality

$$\| w(t) \| \le \| w(0) \|^{1-\mu(t)} \| w(T) \|^{\mu(t)}, \qquad 0 \le t \le T \tag{7}$$

with known exponent $\mu(t)$, $0 \le \mu(t) \le 1$. For given K with $0 < K \ll M/\epsilon$, define μ^* by

$$\mu^* = \log\{M/(M - K\epsilon)\}/\log(M/\epsilon). \tag{8}$$

The SECB constraint is expressed as follows: *There exists a known small constant $K > 0$, and a known fixed $s > 0$, with $\mu(s) > \mu^*$, such that $\| u(s) - u(0) \| \le K\epsilon$.*

By continuity as $t \downarrow 0$, given any $\epsilon > 0$, there always exists a sufficiently small $s > 0$ such that the last inequality holds with a small K. However, the requirement that s be known and be such that $\mu(s) > \mu^*$, constitutes further a priori information about the continuation problem. It will turn out to be desirable that $\mu(s) \gg \mu^*$.

There are several sets of circumstances that can result in solutions satisfying SECB. As an example, consider linear parabolic initial value problems with time-independent coefficients, homogeneous boundary conditions, and no forcing term. If the coefficients are small and the initial values are not dominated by very high frequency components, the solution will evolve slowly from these data at the continuation boundary $t = 0$. This situation prevails in some important biomedical image deblurring problems, where the blurring kernel is a Gaussian distribution with small variance. In that case, the blurred image can be viewed as the solution at time $t = T > 0$, of an initial value problem for a diffusion equation with a small diffusivity, the data at $t = 0$ being the desired unblurred image. See [6], [7, Fig. 2]. Despite the small diffusivity, fine scale information that may be of vital significance typically cannot be discerned in the blurred image. Hence the need for deblurring. More generally, in parabolic problems with time-dependent coefficients, consider the case where the coefficients are initially small but grow with time. Again, the solution will evolve slowly from the initial values, while it may change rapidly at later times. See the example in section 8 below. Inhomogeneous boundary conditions provide another mechanism that can produce solutions satisfying SECB, even when the coefficients are not small. Thus, if a body in thermal equilibrium at $t = 0$ is subjected to a boundary heat flux $b(t)$, where, with $b(0) = 0$, $b(t)$ increases slowly in the interval $0 \le t \le T/4$; increases rapidly between $T/4$ and $T/2$; and decreases rapidly to zero between $T/2$ and $3T/4$, the solution at time T will differ considerably from its initial values, while evolving slowly near $t = 0$. Similar behavior can occur in the Navier–Stokes initial value problem. Consider flows in lid-driven cavities as in [20] and the references therein. If the velocity of the driving lid has a time dependence similar to that in the heat flux $b(t)$ above, the solution of the Navier–Stokes system at time $T > 0$ will differ substantially from its initial state, while having evolved slowly near $t = 0$. Examples of SECB may likewise be found in spatial continuation problems.[1]

LEMMA 1. *For $1 > \mu(s) > \mu^*$, let $\Gamma(K, s)$ be the unique root of the transcendental equation*

$$x = K + x^{1-\mu(s)}. \tag{9}$$

Then

$$\begin{aligned} &K + 1 < \Gamma < M/\epsilon, \\ &\{K/\mu(s)\} \le \Gamma \log \Gamma \le \{K/\mu(s)\}\{\Gamma/(\Gamma - K)\}, \\ &\Gamma \log \Gamma \approx K/\mu(s) \le \{\mu^*/\mu(s)\}(M/\epsilon) \log M/\epsilon, \qquad K \ll \Gamma. \end{aligned} \tag{10}$$

Moreover, if $K + 1 \le x_0 \le M/\epsilon$, the iteration $x_{n+1} = K + x_n^{1-\mu(s)}$ converges to Γ.

Proof. The curve $y = x$ intersects the curve $y = K + x^{1-\mu(s)}$ at a single point, Γ. From (8), we have $M/\epsilon = K + (M/\epsilon)^{1-\mu^*}$, so that M/ϵ is the root of (9) when

[1] In spatial continuation for the heat equation in the quarter plane, or *sideways heat equation* problem [10], large, rather than small, diffusivities near $x = 0$ are conducive to slow evolution from that boundary.

$\mu(s) = \mu^*$. The roots of (9) decrease monotonically as $\mu(s)$ increases. Therefore, $\Gamma < M/\epsilon$. Evidently, $\Gamma > 1$, which implies $\Gamma > K + 1$. Using the inequality $w \leq \log\{1/(1-w)\} \leq w/(1-w)$, $0 \leq w < 1$, we get $K/\Gamma \leq \mu(s)\log\Gamma \leq K/(\Gamma - K)$. Thus, $\Gamma\log\Gamma \approx K/\mu(s)$ if $K \ll \Gamma$. Next, $K\epsilon/M \leq \mu^*\log(M/\epsilon)$, which leads to the last inequality in (10). The last statement in Lemma 1 is a standard result called "fixed point iteration." ∎

THEOREM 1. *Let ϵ, M, K be given positive constants with $\epsilon < M$ and $K\epsilon < M$. Let X be a Banach space with norm $\|\ \|$, and let $f \in X$. Let $\mathcal{C}$ be a linear or nonlinear continuation problem from the data f for the X-valued function $u(t)$, $0 \leq t \leq T$, where $\| u(0) \| \leq M$ and $\| u(T) - f \| \leq \epsilon$. Let $\mathcal{C}$ be such that the difference $w(t)$ of any two possible continuations satisfies*

$$\| w(t) \| \leq \| w(0) \|^{1-\mu(t)} \| w(T) \|^{\mu(t)}, \qquad 0 \leq t \leq T, \tag{11}$$

with known $\mu(t)$, $0 \leq \mu(t) \leq 1$. If the solutions of $\mathcal{C}$ also satisfy $\| u(s) - u(0) \| \leq K\epsilon$ for some known $s > 0$ with $\mu(s) > \mu^*$, where μ^* is defined in (8), then

$$\| w(t) \| \leq 2\Gamma^{1-\mu(t)}\epsilon, \qquad 0 \leq t \leq T, \tag{12}$$

where Γ is the constant defined in Lemma 1. Moreover, $\Gamma \ll M/\epsilon$ if $\mu^* \ll \mu(s)$.

Proof. From (11), the difference of any two continuations satisfies

$$\begin{aligned} &\| w(t) \| \leq \Lambda^{1-\mu(t)}\delta^{\mu(t)}, \qquad 0 \leq t \leq T, \\ &\| w(s) - w(0) \| \leq K\delta, \qquad s > 0, \quad \mu(s) > \mu^*, \end{aligned} \tag{13}$$

where $\Lambda = 2M$, $\delta = 2\epsilon$. From

$$\begin{aligned} \| w(t) \| &\leq \| w(0) \|^{1-\mu(t)} \| w(T) \|^{\mu(t)} \\ &\leq \{\| w(s) - w(0) \| + \| w(s) \|\}^{1-\mu(t)} \| w(T) \|^{\mu(t)}, \end{aligned} \tag{14}$$

together with (13), we get

$$\| w(s) \| \leq \left\{K\delta + \Lambda^{1-\mu(s)}\delta^{\mu(s)}\right\}^{1-\mu(s)}\delta^{\mu(s)}. \tag{15}$$

The initial estimate for $w(s)$, $\| w^1(s) \| \leq \Lambda^{1-\mu(s)}\delta^{\mu(s)}$, has been used in (14) to produce a new estimate, $\| w^2(s) \|$, given by (15). We may insert (15) back into (14) to produce a third estimate for $w(s)$, and so on. At the nth step of that iteration, we get $\| w^n(s) \| \leq Z_n^{1-\mu(s)}\delta^{\mu(s)}$, where

$$Z_1 = \Lambda, \qquad Z_k/\delta = K + (Z_{k-1}/\delta)^{1-\mu(s)}, \qquad k > 1. \tag{16}$$

We have $Z_n \to \Gamma\delta$ as $n \uparrow \infty$, where Γ is defined in Lemma 1. Thus, $\| w(s) \| \leq \Gamma^{1-\mu(s)}\delta$. Inserting this back into (14), and using (13), we get

$$\| w(t) \| \leq 2\Gamma^{1-\mu(t)}\epsilon, \qquad 0 \leq t \leq T. \tag{17}$$

Finally, the last inequality in (10) shows that $\Gamma\log\Gamma \ll (M/\epsilon)\log(M/\epsilon)$ provided $\mu(s) \gg \mu^*$. ∎

To illustrate the SECB constraint, we return to analytic continuation in the unit disc, as discussed in the Introduction. Let $M = 4$, $\epsilon = 10^{-5}$, $a = 0.1$, and consider continuing the function $u(z) = 1 + 0.1z$ from data $f(\theta) = 1 + 0.1ae^{i\theta} + 2a^6e^{6i\theta}$ at $r = a$. Let $v(z) = 1 + 0.1z + 2z^6$. Then,

$$\| u(1) \|_\infty = 1.1 < M, \qquad \| u(a) - f \|_\infty = 2 \times 10^{-6} < \epsilon,$$
$$\| v(1) \|_\infty = 3.1 < M, \qquad \| v(a) - f \|_\infty = 0 < \epsilon. \tag{18}$$

Thus, both $u(z)$ and $v(z)$ satisfy the a priori constraints (4), (5). However, at $r = 3/4$, we find $\| u(3/4) - v(3/4) \|_\infty \,/\, \| u(3/4) \|_\infty = 33\%$, and $\| u(1) - v(1) \|_\infty \,/\, \| u(1) \|_\infty = 182\%$ at $r = 1$. These are unacceptable relative errors. Additional a priori information about $u(z)$ near the continuation boundary $r = 1$ can reduce this uncertainty. With $K = 12$ and $s = 0.001$, we have

$$\| u(1) - u(1-s) \|_\infty = 0.1s \le K\epsilon, \tag{19}$$

while

$$\begin{aligned} \| v(1) - v(1-s) \|_\infty &= \max_\theta |0.1se^{i\theta} + 2\{1 - (1-s)^6\}e^{6i\theta}| \\ &= \max_\theta |10^{-4}e^{i\theta} + 1.197 \times 10^{-2}e^{6i\theta}| \\ &> 10^{-2} > K\epsilon. \end{aligned} \tag{20}$$

Therefore, the SECB constraint (19), with $s = 0.001$ and $K = 12$, eliminates $v(z)$ as a possible continuation. With $\omega(r)$ as in (2), let $\mu(t) = \omega(1-t)$, $0 \le t \le 1-a$. Since $\mu(s) = 4.345 \times 10^{-4}$, while $\mu^* = 2.326 \times 10^{-6}$, we have $\mu(s)/\mu^* = 187$, and $\Gamma \log \Gamma \approx K/\mu(s) = 27618$. This gives $\Gamma = 3397$ while $M/\epsilon = 400{,}000 = 118\Gamma$. Let $w(r, \theta)$ be the difference between any two possible continuations from data at $r = a = 0.1$ satisfying (4), (5), with $M = 4$ and $\epsilon = 10^{-5}$. Without the SECB constraint (19), we have $\| w(1) \|_\infty \le 2M = 8$. With the SECB constraint, we have $\| w(1) \|_\infty \le 2\ \Gamma\ \epsilon = 0.0679$.

Theorem 1 leads to the following corollary to the Hadamard three-circle theorem.

THEOREM 2 (corollary). *In the analytic continuation problem in the unit disc, let* $u_1(z)$, $u_2(z)$ *be as in* (6), *let* $0 < s < 1-a$, *let* $\omega(r)$ *be as in* (2), *and let* $\mu(s) = \omega(1-s)$. *If*

$$\| u_i(1) - u_i(1-s) \|_\infty \le K\epsilon, \qquad i = 1, 2, \tag{21}$$

with known K, $0 < K < M/\epsilon$, *and known* s *such that* $\mu(s) > \mu^*$, *where* μ^* *is defined in* (8), *then*

$$\| u_1(r) - u_2(r) \|_\infty \le 2\Gamma^{1-\omega(r)}\epsilon, \qquad a \le r \le 1, \tag{22}$$

where $\Gamma < M/\epsilon$ *is the constant in Lemma* 1. *Moreover,* $\Gamma \ll M/\epsilon$ *if* $\omega(1-s) \gg \mu^*$.

Remark 1. The SECB constraint does not imply differentiability of $u(1, \theta)$, as a function of θ, on the circle $r = 1$. More generally, at the continuation boundary $t = 0$, $u(t)$ need not be differentiable in its remaining variables in order to satisfy an SECB constraint. This point is emphasized in [6], [7, Fig. 2], and again in the example in section 8 below.

3. An approach to logarithmic convexity. The following method has been widely used to obtain continuous dependence inequalities in linear and nonlinear initial value problems, typically in a Hilbert space setting [2], [16], [17]. Let H be a

Hilbert space, and let $\mathcal{S}$ be an initial value problem for a system of partial differential equations, with solutions $u(t) \in H$ for each $t \in (0,T]$. Let $F(t)$ be a real-valued twice continuously differentiable function of t, defined on the set of solutions $u(t)$ of $\mathcal{S}$ and satisfying

$$\begin{aligned} &F(t) \geq 0, \qquad F(t) = 0 \Longleftrightarrow u(t) = 0, \qquad 0 \leq t \leq T, \\ &F(t)F''(t) - \{F'(t)\}^2 \geq -a_1\, F(t)F'(t) - a_2\, F^2(t), \qquad 0 < t < T, \end{aligned} \tag{23}$$

where a_1 and a_2 are constants. If $a_1 = a_2 = 0$ in (23), then

$$F(t) \leq \{F(0)\}^{(T-t)/T}\{F(T)\}^{t/T}, \qquad 0 \leq t \leq T. \tag{24}$$

More typically, $a_1 \neq 0$ in (23). In that case, let

$$m = -a_2/a_1, \qquad \mu(t) = \{e^{-a_1 t} - 1\}\{e^{-a_1 T} - 1\}^{-1}, \qquad 0 \leq t \leq T. \tag{25}$$

Then (see [2], [17])

$$e^{-mt}F(t) \leq \{F(0)\}^{1-\mu(t)}\{e^{-mT}F(T)\}^{\mu(t)}, \qquad 0 \leq t \leq T. \tag{26}$$

We now give two examples of the use of this technique in L^2, with $F(t) = \| u(t) \|^2$. Many other examples, and choices for $F(t)$, may be found in [2], [16], [24], and the references therein.

4. Self-adjoint parabolic problems with time-dependent coefficients. Let Ω be a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$. For $x \in R^n$ and $t \geq 0$, let $a(t;u,v)$ and $\dot{a}(t;u,v)$ be *symmetric* bilinear forms on $H_0^m(\Omega)$ given by

$$\begin{aligned} a(t;u,v) &= \sum_{|p|,|q| \leq m} \int_\Omega a_{pq}(x,t) D^q u \overline{D^p v}\, dx, \\ \dot{a}(t;u,v) &= \sum_{|p|,|q| \leq m} \int_\Omega \dot{a}_{pq}(x,t) D^q u \overline{D^p v}\, dx, \end{aligned} \tag{27}$$

where the coefficients a_{pq} depend smoothly on x and t, $\overline{a}_{pq} = a_{qp}$, and $\dot{a}_{pq}$ denotes $\partial a_{pq}/\partial t$. We assume $a(t;u,v)$ to be uniformly strongly coercive on $H_0^m(\Omega)$, i.e., there exists $\alpha > 0$, independent of t, such that

$$a(t;v,v) \geq \alpha \| v \|_m^2, \qquad v \in H_0^m(\Omega). \tag{28}$$

Both $a(t;u,v)$ and $\dot{a}(t,u,v)$ are continuous on $H_0^m(\Omega) \times H_0^m(\Omega)$, uniformly in t; i.e., there exist $\beta,\ \gamma > 0$, independent of t, such that

$$|a(t;u,v)| \leq \beta \| u \|_m \| v \|_m, \quad |\dot{a}(t;u,v)| \leq \gamma \| u \|_m \| v \|_m, \quad u,v \in H_0^m(\Omega). \tag{29}$$

The bilinear form $a(t,u,v)$ defines a positive self-adjoint operator $A(t)$ in $L^2(\Omega)$, [28], with domain $D_A = H^{2m}(\Omega) \cap H_0^m(\Omega)$ such that

$$\begin{aligned} &(A(t)v,v) = a(t:v,v), \qquad (\dot{A}(t)v,v) = \dot{a}(t;v,v), \qquad v \in D_A, \\ &|(\dot{A}(t)v,v)| \leq (\gamma/\alpha)(A(t)v,v), \qquad v \in D_A, \end{aligned} \tag{30}$$

where (,) denotes the scalar product in $L^2(\Omega)$. The operator $A(t)$ is the closed extension of the strongly elliptic symmetric differential operator

$$A(x,t,D)u = \sum_{|p|,|q|\le m} (-1)^{|p|} D^p(a_{pq}(x,t)D^q u), \qquad x\in\Omega, \quad t>0, \tag{31}$$

with zero Dirichlet data on $\partial\Omega$. We distinguish two cases: a) the case where diffusion is constant or *decreases* with time and $(\dot{A}(t)v,v)\le 0$, and b) the case where diffusion *increases* at least some of the time and $(\dot{A}(t)v,v)\le\gamma \parallel v \parallel_m^2$. The theorem below is due to Agmon and Nirenberg [1].

THEOREM 3. *Let α and γ be the positive constants in* (28) *and* (29). *Let $u(t)\in L^2(\Omega)$ be a solution of $u_t = -A(t)u,\ t>0$. If $(\dot{A}(t)v,v)\le 0,\ 0<t\le T$, let $\mu(t)=t/T$. If $(\dot{A}(t)v,v)\le\gamma\parallel v\parallel_m^2,\quad 0<t\le T$, let $c=\gamma/\alpha$ and let $\mu(t)=\{e^{ct}-1\}\{e^{cT}-1\}^{-1}$. Then,*

$$\parallel u(t)\parallel\le\parallel u(0)\parallel^{1-\mu(t)}\parallel u(T)\parallel^{\mu(t)}, \qquad 0\le t\le T. \tag{32}$$

Proof. With $F(t)=\parallel u(t)\parallel^2$, we have $F'(t)=-2(A(t)u,u)$, and

$$\begin{aligned} FF''-\{F'\}^2 &= -2(\dot{A}(t)u,u)F+4\parallel A(t)u\parallel^2\parallel u\parallel^2 -4|(A(t)u,u)|^2 \\ &\ge -2(\dot{A}(t)u,u)F \end{aligned} \tag{33}$$

on using Schwarz's inequality. If $(\dot{A}(t)u,u)\le 0$, we have the case $a_1=a_2=0$ in (23) and the result follows from (24). If $(\dot{A}(t)u,u)\le\gamma\parallel u\parallel_m^2$, we use (30) to obtain $-2(\dot{A}(t)u,u)F\ge(\gamma/\alpha)FF'$ in (33). This is the case $a_2=0,\ a_1=-\gamma/\alpha$ in (23), and the result follows from (26) with $m=0$. ∎

Evidently, growing diffusion coefficients can result in exponential decay in $\mu(t)$. In section 8 below, we study a simple explicit example where this is indeed the case. Applying Theorem 1 to the Agmon–Nirenberg result, we have the following corollary.

THEOREM 4 (corollary). *In Theorem* 3, *let positive constants ϵ, M, be given, with $\epsilon<M$. Let $f\in L^2(\Omega)$ be given data at time $T>0$, and let $u_1(t)$, $u_2(t)$ be two solutions of $u_t=-A(t)u+g(t),\ 0<t\le T$, such that $\parallel u_i(T)-f\parallel\le\epsilon$, and $\parallel u_i(0)\parallel\le M,\ i=1,2$. Let $w(t)=u_1(t)-u_2(t)$. Then, with $\mu(t)$ as in Theorem* 3,

$$\parallel w(t)\parallel\le 2M^{1-\mu(t)}\epsilon^{\mu(t)}, \qquad 0\le t\le T. \tag{34}$$

If, in addition, $\parallel u_i(s)-u_i(0)\parallel\le K\epsilon,\ i=1,2$, with known K, $0<K<M/\epsilon$, and known $s>0$ such that $\mu(s)>\mu^*$, where μ^* is defined in (8), *then*

$$\parallel w(t)\parallel\le 2\Gamma^{1-\mu(t)}\epsilon, \qquad 0\le t\le T, \tag{35}$$

where $\Gamma<M/\epsilon$ is the constant in Lemma 1. *Moreover, $\Gamma\ll M/\epsilon$ if $\mu(s)\gg\mu^*$.*

5. Navier–Stokes equations backwards in time. With $i=1,3$, and summation convention understood, consider the Navier–Stokes system in a bounded domain $\Omega\subset R^3$, with smooth boundary $\partial\Omega$,

$$\begin{aligned} &\left.\begin{aligned} u_{i,t} &= \nu\Delta u_i - u_j u_{i,j} - \rho^{-1}p_{,i} + G_i(x,t) \\ u_{j,j} &= 0 \end{aligned}\right\} \qquad (x,t)\in\Omega\times(0,T], \\ &u_i(x,T)=f_i(x), \quad x\in\Omega, \qquad u_i=g_i(x,t), \quad (x,t)\in\partial\Omega\times[0,T]. \end{aligned} \tag{36}$$

Here, differentiation is denoted by a comma, ν is the kinematic viscosity, ρ is the constant density, p the unknown pressure, $u_i(x,t)$ is the ith component of fluid velocity,

$G_i(x,t)$ is a prescribed body force per unit mass, and $g_i(x,t)$ are prescribed boundary values. In [18], Knops and Payne study the stability of reconstructing the solution of (36) on $[0,T)$, under small perturbations of the solution values $f_i(x)$ at some positive time T. Let P and Q be prescribed positive constants. A function $u_i(x,t)$ is said to belong to the set $\mathcal{P}$ provided

$$\sup_{\Omega\times[0,T]} u_i u_i \leq P^2, \tag{37}$$

while it belongs to the set $\mathcal{Q}$ whenever

$$\sup_{\Omega\times[0,T]} \{u_i u_i + (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) + u_{i,t} u_{i,t}\} \leq Q^2. \tag{38}$$

In [25], the same stability problem is studied under weaker constraints. Let $u_i^1(x,t)$ and $u_i^2(x,t)$ denote classical solutions of (36) corresponding to terminal data $f_i^1(x)$ and $f_i^2(x)$ at time $T > 0$. Let $v_i(x,t) = (u_i^1 - u_i^2)(x,t)$. Define the spatial L^2 norm of $v_i(x,t)$ at time t by

$$\| v(t) \| = \left\{ \int_\Omega v_i(x,t) v_i(x,t) dx \right\}^{1/2}. \tag{39}$$

Knops and Payne [18] show that if $u_i^1(x,t) \in \mathcal{P}$ and $u_i^2(x,t) \in \mathcal{Q}$, and if $F(t) = \| v(t) \|^2$,

$$\begin{aligned} F(t)F''(t) - \{F'(t)\}^2 &\geq 2\nu^{-1}(P^2+1)F(t)F'(t) - Q^2\{2\nu^{-2}(P^2+1)+1\}F^2(t) \\ &= -a_1 F(t)F'(t) - a_2 F^2(t). \end{aligned} \tag{40}$$

Hence, with

$$\begin{aligned} &c = -a_1 = 2\nu^{-1}(P^2+1), \\ &\mu(t) = (e^{ct}-1)(e^{cT}-1)^{-1}, \\ &w_i(x,t) = e^{-mt} v_i(x,t), \qquad m = -a_2/2a_1, \qquad 0 \leq t \leq T, \end{aligned} \tag{41}$$

it follows from (40) and (26) that

$$\| w(t) \| \leq \| w(0) \|^{1-\mu(t)} \| w(T) \|^{\mu(t)}, \qquad 0 \leq t \leq T. \tag{42}$$

Applying Theorem 1 to the Knops–Payne result (42), we have the following corollary.

THEOREM 5 (corollary). *For the given positive ϵ, M, with $\epsilon < M$, let $w_i(x,t)$ in* (41) *satisfy $\| w(0) \| \leq M$, $\| w(T) \| \leq \epsilon$, and let $\mu(t)$ be as in* (41). *Then*

$$\| w(t) \| \leq M^{1-\mu(t)} \epsilon^{\mu(t)}, \qquad 0 \leq t \leq T. \tag{43}$$

If, in addition, $\| w(s) - w(0) \| \leq K\epsilon$, with known K, $0 < K < M/\epsilon$, and known $s > 0$ such that $\mu(s) > \mu^* \equiv \log\{M/(M-K\epsilon)\}/\log(M/\epsilon)$, then

$$\| w(t) \| \leq \Gamma^{1-\mu(t)} \epsilon, \qquad 0 \leq t \leq T, \tag{44}$$

where $\Gamma < M/\epsilon$ is the unique root of $x - x^{1-\mu(s)} - K = 0$. Moreover, $\Gamma \ll M/\epsilon$ if $\mu(s) \gg \mu^*$.

For large c, the rapid exponential decay of $\mu(t)$ as t decreases from $t = T$ makes it unlikely that the most general solutions satisfying the constraints (37) or (38) can be continued very far into the past. However, it may be possible to continue solutions that have evolved slowly near $t = 0$. Consider the following example. Let $P = 1$, $\nu = 10^{-1}$, $T = 0.25$, $M = 20$, and $\epsilon = 10^{-6}$. Then, $c = 40$ and $\mu(t) = \{e^{40t} - 1\}\{e^{10} - 1\}^{-1}$, $0 \leq t \leq 0.25$. In particular, $\mu(T/2) = 6.693 \times 10^{-3}$. Consequently, (43) gives

$$\| w(T/2) \| \leq 19.603 \times 0.911681 = 17.872 \tag{45}$$

On the other hand, suppose that the solutions to be reconstructed are known to have evolved slowly enough near $t = 0$ that with $s = 0.01T$ and $K = 10$, we have $\| w(s) - w(0) \| \leq K\epsilon$. Then $\mu(s) = 4.775 \times 10^{-6}$, while $\mu^* = 2.974 \times 10^{-8}$. Thus, $\{\mu(s)/\mu^*\} = 160.55$. From $\Gamma \log \Gamma \approx K/\mu(s)$, we find $\Gamma \approx 173,600$ and $M/\epsilon = 115\Gamma$. From (44), we get

$$\| w(T/2) \| \leq 160,134 \times 10^{-6} = 0.16. \tag{46}$$

Thus, the difference between any two solutions satisfying the SECB constraint is over one hundred times smaller at $t = T/2$ than it is in the more general case of (45).

6. Holomorphic semigroups and evolution equations. Let X be a complex Banach space, let A be a closed linear operator with domain D_A dense in X, and consider the evolution equation $u_t = -Au$, $t > 0$, for the X-valued function $u(t)$. We assume that $-A$ generates a holomorphic semigroup e^{-tA} in an open sector of the complex t-plane, $\Sigma_\phi = \{\text{Re } t > 0, \ |\text{Arg } t| < \phi\}$, for some fixed ϕ, $0 < \phi \leq \pi/2$. Moreover, for any $0 < \sigma < \phi$, e^{-tA} is strongly continuous at $t = 0$ within $\Sigma_{\phi-\sigma}$, reduces to the identity operator at $t = 0$, and satisfies $\| e^{-tA} \| \leq B_\sigma < \infty$ for $t \in \overline{\Sigma}_{\phi-\sigma}$. Thus, e^{-tA} is a bounded holomorphic semigroup as defined in [15].

Parabolic initial boundary value problems constitute the best-known area of application of holomorphic semigroups. We briefly sketch this connection below, and refer the reader to [12] and [28] for a complete treatment. Less well known are applications to a wide class of nonparabolic equations, typically involving nonlocal partial differential operators, that are obtained by "subordination" in well-posed Cauchy problems [4], [11], [8]. This class of problems, mentioned in section 6.1, is drawing increasing interest from physical scientists working in certain areas of fractal analysis.

Let Ω be a bounded domain in R^n with a sufficiently smooth boundary $\partial\Omega$. For $x \in R^n$, let $A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$ be a linear partial differential operator with coefficients $a_\alpha(x)$ continuous in the closure of Ω. If $A(x, D)$ is strongly elliptic, and zero Dirichlet data are given on $\partial\Omega$, a closed linear operator A in $L^2(\Omega)$, with dense domain $D_A = H^{2m}(\Omega) \cap H_0^m(\Omega)$, can be defined by

$$(Au)(x) = A(x, D)u(x), \qquad u \in D_A. \tag{47}$$

Moreover, as shown in [12], [28], for some $k \geq 0$ the linear operator $-(A+kI)$ generates a bounded holomorphic semigroup in $L^2(\Omega)$. If $A(x, D)$ is a symmetric differential operator, then $A + kI$ is self-adjoint, and we may choose $\phi = \pi/2$ in Σ_ϕ.

More general boundary conditions can be handled and parabolic equations of order $2m$ can be considered in $L^p(\Omega)$, $1 \leq p < \infty$. Let $H^{j,p}(\Omega)$ denote the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to j exist and belong to $L^p(\Omega)$. Let $\{B_j\}_{j=1}^m$ be m boundary operators of respective orders $m_j < 2m$, given by

$$B_j(x,D) = \sum_{|\alpha|\le m_j} b^j_\alpha(x) D^\alpha, \tag{48}$$

and consider the boundary value problem

$$\begin{aligned} A(x,D)u &= g, \qquad x \in \Omega, \\ B_j(x,D)u &= 0, \qquad x \in \partial\Omega, \quad 1 \le j \le m. \end{aligned} \tag{49}$$

A closed linear operator A with dense domain $D_A = H^{2m,p}(\Omega; \{B_j\})$, consisting of the closure in $H^{2m,p}(\Omega)$ of the set of functions $u \in C^{2m}(\overline{\Omega})$ that satisfy the boundary conditions in (49), can be defined via

$$(Au)(x) = A(x,D)u, \qquad u \in D_A. \tag{50}$$

If the system B_j is *normal*, and satisfies further *complementary* conditions, and if $A(x,D)$ is strongly elliptic, one obtains a *regular* elliptic boundary value problem, $(A, \{B_j\}, \Omega)$, such that for some $k \ge 0$, the linear operator $-(A+kI)$ generates a bounded holomorphic semigroup in $L^p(\Omega)$. See [12], [28].

6.1. Subordinated semigroups. Let $H(y)$ denote the Heaviside unit step function, and consider the family $p_y(t)$ given by

$$p_y(t) = \frac{tH(y)e^{-t^2/4y}}{\sqrt{4\pi y^3}}, \qquad t > 0. \tag{51}$$

For each fixed $t > 0$, $p_y(t)$ is a probability density function on $y \ge 0$, and $p_y(t)$ tends to the Dirac δ-function $\delta(y)$ as $t \downarrow 0$. Moreover, if $*$ denotes convolution with respect to y, then $p_y(t) * p_y(s) = p_y(t+s)$, for $s,\ t \ge 0$. The Laplace transform with respect to y of $p_y(t)$ is given by

$$\mathcal{L}\{p_y(t)\} \equiv \int_0^\infty e^{-yz} p_y(t)dy = e^{-t\sqrt{z}}, \qquad \text{Re } z > 0. \tag{52}$$

The "inverse Gaussian" family in (51) is just one example of an *infinitely divisible* family of probability density functions on the half-line $y \ge 0$, [11].

Let $T(t) = e^{-tA},\ t \ge 0$, be a uniformly bounded, not necessarily holomorphic, C_0 semigroup on a complex Banach space X. Using (51), one may construct a new C_0 semigroup $U(t)$ on X, with $\| U(t) \| \le \| T(t) \| \le B < \infty, \quad t \ge 0$, by means of

$$U(0) = I, \quad U(t)g = \int_0^\infty p_y(t) T(y) g \; dy, \qquad t > 0, \quad g \in X. \tag{53}$$

Indeed, it turns out that $U(t) = e^{-tA^{1/2}}$ and that $U(t)$ can be extended to a *bounded holomorphic semigroup* in some sector Σ_ω.

The construction in (53) amounts to randomization of the time variable t in the original semigroup $T(t)$. A wide variety of infinitely divisible families $q_y(t)$ may be used in (53). The new semigroup $U(t)$ is said to be "subordinated" to $T(t)$ through the "directing process" $q_y(t)$ [11]. This concept originated in [4] and was subsequently refined into a functional calculus in [26], [23], and [3]. The observation that $U(t)$ is holomorphic whenever the directing process $q_y(t) = \mathcal{L}^{-1}\{e^{-tz^\alpha}\}, \quad 0 < \alpha < 1$, was made in [29]. In that case, $U(t) = e^{-tA^\alpha}$. Subordinated processes and fractional differential operators are of interest in polymer science [9], while diffusion equations with fractional Laplacians play a role in image deblurring [6]. Further applications are discussed in [5, pp. 140–156] and [11].

An arbitrary infinitely divisible family $q_y(t)$ on $y \geq 0$ can be characterized in terms of its Laplace transform [11]. We have $\mathcal{L}\{q_y(t)\} = e^{-t\psi(z)},\ t \geq 0$, where the *exponent* $\psi(z)$ is holomorphic for Re $z > 0$ and continuous for Re $z \geq 0$, with Re $\psi(z) \geq 0$. Moreover, $\psi(0) = 0$, and $\psi'(x)$ is completely monotone for $x > 0$. In [8], the results of [29] are extended. A necessary and sufficient condition on $q_y(t)$ is given, in order that the subordinated semigroup $U(t) = e^{-t\psi(A)}$ be holomorphic on X, whenever $T(t)$ is C_0 and uniformly bounded on X. In addition, a *necessary* condition on the exponent $\psi(z)$ is obtained for that to be the case. In [13], a *sufficient* condition on $\psi(z)$ is given that ensures analyticity of $U(t)$. As a consequence of [29], [8], and [13], a rich class of exponents $\psi(z)$ is known, with the property that $-\psi(A)$ generates a bounded holomorphic semigroup on X whenever $-A$ generates a uniformly bounded C_0 semigroup on X. As one example, consider the symmetric hyperbolic system,

$$u_t = \sum_{i=1}^{n} a_i(x) u_{x_i} + b(x)u, \qquad x \in R^n, t > 0, \tag{54}$$
$$u(x,0) = f(x),$$

where $u(x)$ is an N-component vector, $a_i(x),\ b(x)$ are $N \times N$ matrices with boundedly differentiable entries on R^n, and $a_i(x)$ is Hermitian. The differential operator on the right-hand side of (54) can be extended into a closed densely defined linear operator $-A$ in $L^2(R^n)^N$. As shown in [28], for some $k \geq 0,\ -(A+kI)$ generates a contraction semigroup on $L^2(R^n)^N$. It follows from [29], [8] that if

$$\psi_1(A) = (A+kI)^\alpha,\ 0 < \alpha < 1, \qquad \psi_2(A) = \mathrm{Log}\{A + (k+1)I\}, \tag{55}$$

then each of $-\psi_1(A), -\psi_2(A)$, generates a holomorphic semigroup on $L^2(R^n)^N$. If $\{\alpha_n\}_{n=1}^\infty$ and $\{a_n\}_{n=1}^\infty$ are any two sequences satisfying $a_n \geq 0,\quad a_1 > 0,\quad 1 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots > 0,\quad \sum_{n=1}^\infty a_n/\alpha_n < \infty$, and if

$$\psi_3(A) = \sum_{n=1}^{\infty} a_n (A+kI)^{\alpha_n}, \tag{56}$$

it follows from [13] that $-\psi_3(A)$ generates a holomorphic semigroup on $L^2(R^n)^N$. None of the $\psi_i(A),\ i = 1, 3$, are elliptic operators when $-A$ is the differential operator on the right-hand side of (54). This shows that holomorphic semigroup theory encompasses a class of initial value problems in partial differential equations that is considerably wider than the class of parabolic problems.

7. Logarithmic convexity and holomorphic semigroups. In Banach space, approaches different from those used in sections 3–5 appear necessary to obtain logarithmic convexity inequalities. Following the basic work in [19], further convexity results were obtained in [1], [12], and [22]. Theorems 6 and 7 below are a reformulation of results originating with these authors.

For any $a \geq 0$, and $0 < \xi \leq 1$, let $S(a,\xi)$ be the set in the complex τ-plane given by

$$S(a,\xi) = \{\tau = t + is;\quad t \geq a;\quad |s| \leq (t-a)\tan(\pi\xi/2)\}. \tag{57}$$

Let $T > 0$. Then, $S(T,\xi) \subset S(0,\xi)$. Let $G(T,\xi) = S(0,\xi) \setminus S(T,\xi)$, and let $\Lambda_L,\ \Lambda_R$ be, respectively, the left and right boundary arcs of $G(T,\xi)$. Let $\omega_\xi(t,s)$ be the unique bounded continuous function on $\overline{G}(T,\xi)$ which is harmonic in the interior of $G(T,\xi)$, equals zero on Λ_L, and equals one on Λ_R. Let $\mu_\xi(t) = \omega_\xi(t,0),\quad 0 \le t \le T$.

LEMMA 2. $\mu_1(t) = t/T$, *and, if* $0 < \xi < \eta \le 1,\quad \mu_\xi(t) < \mu_\eta(t),\quad 0 < t < T$.

Proof. Let $H(\xi,\eta) = S(0,\eta) \setminus S(T,\xi)$. Then $G(T,\xi) \subset H(\xi,\eta)$, and $G(T,\eta) \subset H(\xi,\eta)$. Let Λ'_L be the left boundary arc of $H(\xi,\eta)$, and let Λ'_R be the right boundary arc of $G(T,\eta)$. Let $\tilde{\omega}(t,s)$ be the unique bounded continuous function on $\overline{H}(\xi,\eta)$ which is harmonic in the interior of $H(\xi,\eta)$, equals zero on Λ'_L, and equals one on Λ_R. The harmonic function $\tilde{\omega} - \omega_\xi$ in $G(T,\xi)$ has value zero on Λ_R, is nonnegative on Λ_L and hence must be strictly positive in the interior of $G(T,\xi)$. Therefore $\mu_\xi(t) < \tilde{\omega}(t,0),\quad 0 < t < T$. A similar argument, applied to the harmonic function $\omega_\eta - \tilde{\omega}$ in $G(T,\eta)$, shows that $\mu_\eta(t) > \tilde{\omega}(t,0),\quad 0 < t < T$. Finally, if $\xi = 1$, then $\overline{G}(T,1)$ is the vertical strip $0 \le \mathrm{Re}\ \tau \le T$, and $\omega_1(t,s) = t/T$. $\square$

We now consider the evolution equation $u_t = -Au,\ t > 0$, in a complex Banach space X with norm $\|\quad\|$, under the assumption that $-A$ generates a bounded holomorphic semigroup in an open sector Σ_ϕ in the complex $\tau = t + is$ plane. With $0 < \alpha\pi/2 < \phi \le \pi/2$, let $S(0,\alpha)$, defined in (57), be a closed subsector of Σ_ϕ, and let $\| e^{-\tau A} \| \le B_\alpha < \infty,\quad \tau \in S(0,\alpha)$. Introduce the equivalent norm $\|\quad\|_\alpha$ on X defined by

$$\| x \|_\alpha \equiv \sup_{\tau \in S(0,\alpha)} \| e^{-\tau A} x \|, \qquad x \in X. \tag{58}$$

Then, as is easily verified,

$$\| x \| \le \| x \|_\alpha \le B_\alpha \| x \|, \quad x \in X, \qquad \| e^{-\tau A} \|_\alpha \le 1, \quad \tau \in S(0,\alpha). \tag{59}$$

THEOREM 6. *Let X be a complex Banach space with norm $\|\quad\|$, let $u(t)$ be a solution of $u_t = -Au,\quad 0 < t \le T$, where $-A$ generates a bounded holomorphic semigroup on X. Then, with $\|\quad\|_\alpha$ as in* (58) *and $\mu_\alpha(t)$ as in Lemma* 2*,*

$$\| u(t) \|_\alpha \le \| u(0) \|_\alpha^{1-\mu_\alpha(t)} \| u(T) \|_\alpha^{\mu_\alpha(t)}, \qquad 0 \le t \le T, \tag{60}$$

and

$$\| u(t) \| \le B_\alpha \| u(0) \|^{1-\mu_\alpha(t)} \| u(T) \|^{\mu_\alpha(t)}, \qquad 0 \le t \le T. \tag{61}$$

Proof. Let l be a linear functional on X with $|l|_\alpha = 1$, where $|\ |_\alpha$ denotes the norm on X^* corresponding to the norm $\|\quad\|_\alpha$ on X. Let $h(\tau) = l(e^{-\tau A}u(0))$ for $\tau \in S(0,\alpha)$. We have that $h(\tau)$ is continuous and bounded on $S(0,\alpha)$, with $|h(\tau)| \le \| u(0) \|_\alpha$, and $h(\tau)$ is holomorphic in the interior of $S(0,\alpha)$. The same is true for $h(\tau)$ in $S(T,\alpha)$, with $|h(\tau)| \le \| u(T) \|_\alpha$. This follows from $e^{-\tau A} = e^{-(\tau-T)A}e^{-TA}$ for $\tau \in S(T,\alpha)$. Let $G(T,\alpha)$ and $\omega_\alpha(t,s)$ be as defined above, and consider the function $v(t,s)$ in $G(T,\alpha)$ where

$$v(t,s) = \log|h(\tau)| - \omega_\alpha(t,s) \log \| u(T) \|_\alpha + (\omega_\alpha(t,s) - 1) \log \| u(0) \|_\alpha\ . \tag{62}$$

The function $v(t,s)$ is upper semicontinuous and bounded above on $G(T,\alpha)$, subharmonic in the interior of $G(T,\alpha)$, and nonpositive on the left and right boundary arcs of $G(T,\alpha)$. Therefore $v(t,s) \le 0$ on $\overline{G}(T,\alpha)$. Using

$$\| u(\tau) \|_\alpha = \sup_{l \in X^*,\ |l|_\alpha = 1} |h(\tau)|, \tag{63}$$

we obtain

$$\| u(\tau) \|_\alpha \leq \| u(0) \|_\alpha^{1-\omega_\alpha(\tau)} \| u(T) \|_\alpha^{\omega_\alpha(\tau)}, \qquad \tau \in \overline{G}(T,\alpha), \tag{64}$$

which implies, on using (59),

$$\| u(\tau) \| \leq B_\alpha \| u(0) \|^{1-\omega_\alpha(\tau)} \| u(T) \|^{\omega_\alpha(\tau)}, \qquad \tau \in \overline{G}(T,\alpha). \tag{65}$$

Finally, (60), (61), follow from the above on putting $\tau = t$. ∎

Remark 2. The inequality (61) follows from (60) but not vice versa. From Lemma 2, we see that $\mu_\alpha(t)$ is sublinear in t, and this sublinearity becomes more severe as α becomes smaller. The choice of α depends on the spectrum of the spatial operator A. Since $-A$ generates a holomorphic semigroup in the open sector Σ_ϕ, the spectrum of A must be contained in the closed sector $Arg\,|z| \leq \beta = \pi/2 - \phi$ in the right half-plane. As β increases, ϕ, and hence α, must decrease. Theorem 6 does not yield the explicit dependence of $\mu_\alpha(t)$ on t, which is necessary for applying the SECB constraint. The next result is more useful in that regard.

THEOREM 7. *With $u(t)$ and α as in Theorem 6, let $0 < \sigma < \alpha < 1$, and let*

$$\begin{aligned} &\lambda = \inf_{0 \leq \theta \leq \pi/2} \{\cos\sigma\theta\, [1 - \tan\sigma\theta / \tan(\alpha\pi/2)] / (\cos\theta)^\sigma\}, \\ &\rho_\sigma(t) = (\lambda t/T)^{1/\sigma}, \qquad 0 \leq t \leq T. \end{aligned} \tag{66}$$

Then,

$$\| u(t) \|_\alpha \leq \| u(0) \|_\alpha^{1-\rho_\sigma(t)} \| u(T) \|_\alpha^{\rho_\sigma(t)}, \qquad 0 \leq t \leq T, \tag{67}$$

and

$$\| u(t) \| \leq B_\alpha \| u(0) \|^{1-\rho_\sigma(t)} \| u(T) \|^{\rho_\sigma(t)}, \qquad 0 \leq t \leq T. \tag{68}$$

Proof. Note that λ in (66) satisfies $0 < \lambda < 1$ and may be found graphically given α and σ. Let $Y > 0$, let l be a linear functional on X with $|l|_\alpha = 1$, and let $h(\tau) = l(e^{-\tau A} u(0))$ for $\tau \in S(0,\alpha)$. As in Theorem 6, $h(\tau)$ is continuous and bounded on $S(0,\alpha)$ (resp., $S(Y,\alpha)$) and holomorphic in its interior, with $|h(\tau)| \leq \| u(0) \|_\alpha$, (resp., $|h(\tau)| \leq \| u(Y) \|_\alpha$). Let $0 < \sigma < \alpha$, let V be the vertical strip $0 \leq \mathrm{Re}\,\tau \leq Y$, and consider the function $\psi(\tau) = h(\tau^\sigma)$ for $\tau \in V$. We have that $\psi(\tau)$ is continuous and bounded on V, holomorphic in its interior, with $|\psi(\tau)| \leq \| u(0) \|_\alpha$. A more precise estimate for $|\psi(\tau)|$ on the line $\mathrm{Re}\,\tau = Y$ will now be obtained. We first show that with λ as in (66),

$$\mathrm{Re}\,\tau = Y \Longrightarrow \tau^\sigma \in S(\lambda Y^\sigma, \alpha). \tag{69}$$

Indeed, with $\tau = Y + is = re^{i\theta}$, $0 \leq |\theta| < \pi/2$, we have $\tau^\sigma = r^\sigma(\cos\sigma\theta + i\,\sin\sigma\theta)$, and $Y = r\cos\theta$. Therefore, $\tau^\sigma \in S(\lambda Y^\sigma, \alpha)$ if and only if

$$r^\sigma |\sin\sigma\theta| \leq \{r^\sigma \cos\sigma\theta - \lambda(r\cos\theta)^\sigma\} \tan(\alpha\pi/2), \qquad 0 \leq |\theta| < \pi/2, \tag{70}$$

i.e., if and only if $\forall\, 0 \leq \theta < \pi/2$, we have

$$\lambda \leq \cos\sigma\theta\, \{1 - \tan\sigma\theta / \tan(\alpha\pi/2)\} / (\cos\theta)^\sigma. \tag{71}$$

But this is guaranteed from the definition of λ. It follows that

$$|\psi(\tau)| \leq \| u(0) \|_\alpha, \quad \text{Re } \tau = 0, \qquad |\psi(\tau)| \leq \| u(\lambda Y^\sigma) \|_\alpha, \quad \text{Re } \tau = Y. \tag{72}$$

We may now apply the "three lines theorem," [27, p. 244], to $\psi(\tau)$ in the strip V and conclude that

$$|\psi(y)| \leq \| u(0) \|_\alpha^{1-y/Y} \| u(\lambda Y^\sigma) \|_\alpha^{y/Y}, \qquad 0 \leq y \leq Y. \tag{73}$$

Using (63), we obtain

$$\| u(y^\sigma) \|_\alpha \leq \| u(0) \|_\alpha^{1-y/Y} \| u(\lambda Y^\sigma) \|_\alpha^{y/Y}, \qquad 0 \leq y \leq Y. \tag{74}$$

Putting $t = y^\sigma,\ T = \lambda Y^\sigma,\ \rho_\sigma(t) = (\lambda t/T)^{1/\sigma}$ in (74) gives

$$\| u(t) \|_\alpha \leq \| u(0) \|_\alpha^{1-\rho_\sigma(t)} \| u(T) \|_\alpha^{\rho_\sigma(t)}, \qquad 0 \leq t \leq T/\lambda. \tag{75}$$

Since $T/\lambda > T$, (75) implies (67) which implies (68). ∎

Remark 3. When X is a Hilbert space, $\rho_\sigma(t)$ in (66) may be viewed as expressing the penalty for non-self-adjointness in the spatial operator A. When A is self-adjoint, we have $\rho(t) = t/T$. If the spectrum of A leaves the nonnegative real axis and expands into the sector $Arg\ |z| \leq \pi/2 - \phi$, $\rho_\sigma(t)$ decays to zero faster than t/T, through the exponent $1/\sigma$. It is remarkable that (66) actually holds in any complex Banach space X. The next theorem summarizes the main results of this section.

THEOREM 8 (corollary). *Let X be a complex Banach space with norm $\| \ \|$. Let $-A$ generate a holomorphic semigroup $e^{-\tau A}$ on X, satisfying $\| e^{-\tau A} \| \leq B_\alpha < \infty$, in a closed sector $|Arg\ \tau| \leq \alpha\pi/2$ of the complex $\tau = t + is$ plane, for suitable α with $0 < \alpha < 1$. Let $\| \ \|_\alpha$ be the equivalent norm on X defined in* (58), (59). *Let $0 < \sigma < \alpha$, and let λ and $\rho_\sigma(t)$ be as in* (66). *For given ϵ, M, with $\epsilon < M$, let $f \in X$ be given data at time $T > 0$, and let $u_i(t),\ i = 1, 2$, be two solutions of $u_t = -Au + g(t),\ 0 < t \leq T$, with $\| u_i(T) - f \| \leq \epsilon/B_\alpha$, and $\| u_i(0) \| \leq M/B_\alpha$. Finally, let $w(t) = u_1(t) - u_2(t)$. Then*

$$\| w(t) \| \leq \| w(t) \|_\alpha \leq 2M^{1-\rho_\sigma(t)} \epsilon^{\rho_\sigma(t)}, \qquad 0 \leq t \leq T. \tag{76}$$

If, in addition, $\| u_i(s) - u_i(0) \| \leq K\epsilon/B_\alpha,\ i = 1, 2$, with known $K,\ 0 < K < M/\epsilon$, and known $s > 0$ such that $\rho_\sigma(s) > \mu^*$, where μ^* is defined in (8), *then*

$$\| w(t) \| \leq \| w(t) \|_\alpha \leq 2\Gamma^{1-\rho_\sigma(t)} \epsilon, \qquad 0 \leq t \leq T, \tag{77}$$

where $\Gamma < M/\epsilon$ is the unique root of $x - K - x^{1-\rho_\sigma(s)} = 0$. Moreover, $\Gamma \ll M/\epsilon$ if $\rho_\sigma(s) \gg \mu^*$.

Proof. From (59), we have $\| w(0) \|_\alpha \leq 2M$, $\| w(T) \|_\alpha \leq 2\epsilon$. Hence, (76) follows from (67). Likewise, $\| w(s) - w(0) \|_\alpha \leq 2K\epsilon$. Applying Theorem 1 with the $\| \ \|_\alpha$ norm on X, we obtain (77) from (67). ∎

8. An example. In the Navier–Stokes equations, where the Hölder exponent $\mu(t)$ in (41) depends on $1/\nu$, it is not known whether or not there can be equality in the Knops–Payne inequality (42). However, the following example demonstrates that rapid exponential decay in $\mu(t)$ can be realized in quite simple problems. With positive constants $a,\ c,\ Q$, consider the 1-D parabolic initial value problem in $L^2(0, \pi)$,

$$\begin{aligned} &u_t = ae^{ct} u_{xx}, \qquad 0 < x < \pi, \quad t > 0, \\ &u(0, t) = u(\pi, t) = 0, \qquad t \geq 0, \\ &u(x, 0) = Q\ \sin mx, \qquad 0 \leq x \leq \pi. \end{aligned} \tag{78}$$

The unique solution of (78) is

$$u(x,t) = Q\ e^{-am^2(e^{ct}-1)/c}\ \sin mx, \qquad t \geq 0. \tag{79}$$

Moreover, $u(x,t)$ satisfies

$$\| u(t) \| = \| u(0) \|^{1-\mu(t)} \| u(T) \|^{\mu(t)}, \qquad 0 \leq t \leq T, \tag{80}$$

where $\mu(t) = \{e^{ct} - 1\}\{e^{cT} - 1\}^{-1}$ and $\| \quad \|$ is the norm on $L^2(0,\pi)$. This shows that Theorem 3 is sharp. By choosing $c > 0$ sufficiently large in (78), we can expect to simulate some of the difficulties that would attend backwards in time continuation in the Navier–Stokes equations.

Let $a = 2 \times 10^{-5}$, let $c = 10$, and, for any positive integer m, let

$$g_m(t) = e^{-am^2(e^{ct}-1)/c}, \qquad t \geq 0. \tag{81}$$

Let $p = \sqrt{2/\pi}$. With $M = 10$, and $\epsilon = 2 \times 10^{-7}$, consider the initial data

$$u(x,0) = p\sqrt{(1-\epsilon^2)/2}\ M \sin 2x + p \sum_{n=1}^{\infty} b_{2n+1} \sin(2n+1)x, \tag{82}$$

where

$$\sum_{n=1}^{\infty} b_{2n+1}^2 = \epsilon^2 M^2/2, \qquad \sum_{n=1}^{\infty} n^q b_{2n+1}^2 = \infty \qquad \forall\ q > 0. \tag{83}$$

Thus, $u(x,0)$ is an L^2 function on $(0,\pi)$ which is not in $H^q(0,\pi)$ for any $q > 0$, and $\| u(0) \| = M/\sqrt{2}$. We may think of the second term in (82) as representing highly localized, nondifferentiable singularities that are superimposed onto the first term. With these initial data in (78), the unique solution is

$$u(x,t) = p\sqrt{(1-\epsilon^2)/2}\ M g_2(t)\ \sin 2x + p \sum_{n=1}^{\infty} b_{2n+1}\ g_{2n+1}(t)\ \sin(2n+1)x. \tag{84}$$

Given an a priori L^2 bound for $u(x,t)$ at $t = 0$, consider recovering the solution (84) on $0 \leq t < 1$, from approximate data $f(x)$ at $t = 1$, with $\| u(1) - f \| \leq \epsilon$. Let

$$\| u(0) \| \leq M = 10 \tag{85}$$

be this prescribed bound, and let the data $f(x)$ at $t = 1$ be given by

$$f(x) = u(x,1) + p(M/\sqrt{2})\ g_{20}(1)\ \sin 20x. \tag{86}$$

Then

$$\begin{aligned} &\| u(1) - f \| = (M/\sqrt{2}) g_{20}(1) = 1.574 \times 10^{-7} < \epsilon, \\ &\| u(1) - f \| \ / \ \| u(1) \| < g_{20}(1)\{\sqrt{1-\epsilon^2}\ g_2(1)\}^{-1} = 2.66 \times 10^{-8}. \end{aligned} \tag{87}$$

Evidently, the given data $f(x)$ approximates $u(x,1)$ extremely closely in both absolute and relative terms. However, if $v(x,t)$ is the function

$$v(x,t) = u(x,t) + p(M/\sqrt{2})\ g_{20}(t)\ \sin 20x, \qquad 0 \leq t \leq 1, \tag{88}$$

then $v(x, 1) = f(x)$, $v(x, t)$ is a solution, and, since $\| v(0) \| = M$, $v(x, t)$ is an equally valid continuation. Noteworthy is the substantial qualitative difference between $u(x, t)$ and $v(x, t)$, which emerges as early as $t = 1/2$, as continuation unfolds backwards from $t = 1$. While $g_2(1) = 0.8384$ and $g_{20}(1) = 2.226 \times 10^{-8}$, we find $g_2(1/2) = 0.9988$ and $g_{20}(1/2) = 0.8888$. Consequently, while the primary component in $u(x, t)$ is the large amplitude $\sin 2x$ oscillation for $0 \le t \le 1$, the $\sin 2x$ and $\sin 20x$ terms have approximately *equal* amplitudes in $v(x, t)$, for $0 \le t \le 1/2$. Clearly, Hölder-continuous data dependence is simply too weak to distinguish $u(x, t)$ from $v(x, t)$ in this example, even though $\| u(1/2) - v(1/2) \| = 6.285$ is roughly the same size as $\| u(0 \|$.

We shall show that an SECB constraint can easily distinguish between $u(x, t)$ and $v(x, t)$, although neither function is differentiable in x at $t = 0$. Indeed, with $K = 35$ and $s = 0.01$, we find

$$\begin{aligned} \| u(s) - u(0) \|^2 &= (1 - \epsilon^2)(M^2/2)(1 - g_2(s))^2 + \sum_{n=1}^{\infty} b_{2n+1}^2 \, (1 - g_{2n+1}(s))^2, \\ &\le (1 - \epsilon^2)(M^2/2)(1 - g_2(s))^2 + \epsilon^2 M^2/2, \\ &= (6.115 \times 10^{-6})^2 < K^2 \, \epsilon^2. \end{aligned} \tag{89}$$

On the other hand, with $s = 0.01$,

$$\begin{aligned} \| v(s) - v(0) \|^2 &= \| u(s) - u(0) \|^2 + (M^2/2)(1 - g_{20}(s))^2, \\ &> (M^2/2)(1 - g_{20}(s))^2 = (5.949 \times 10^{-4})^2, \\ &> (2974)^2 \, \epsilon^2. \end{aligned} \tag{90}$$

Therefore, the SECB constraint

$$\| u(0.01) - u(0) \| \le 35 \, \epsilon \tag{91}$$

eliminates $v(x, t)$ in (88) as a possible continuation, while allowing $u(x, t)$ in (84). Here, $\mu(s)/\mu^* = 121$, $\Gamma = 554,235$, and $M/\epsilon = 90\Gamma$. It follows from Theorem 1 and (80) that if $u_1(x, t)$ is any other continuation satisfying (91), then $\| u(t) - u_1(t) \| \le 2\Gamma^{1-\mu(t)}\epsilon$, $0 \le t \le 1$. Hence, $\| u(1/2) - u_1(1/2) \| \le 0.203$, and $\| u(0) - u_1(0) \| \le 0.222$. Since $\| u(1/2) \| > \{(1 - \epsilon^2)/2\}^{1/2} \, M g_2(1/2) = 7.063$, and $\| u(0) \| = 7.071$, the maximum L^2 relative errors in approximating $u(x, t)$ at $t = 1/2$ and at $t = 0$, are, respectively, 2.87% and 3.14%. Without the SECB constraint (91), these relative errors are, respectively, 251% and 283%.

REFERENCES

[1] S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math., 16 (1963), pp. 121–239.

[2] K. A. Ames and B. Straughan, *Non-standard and Improperly Posed Problems*, Academic Press, New York, 1997.

[3] A. V. Balakrishnan, *An operational calculus for infinitesimal generators of semigroups*, Trans. Amer. Math. Soc., 91 (1959), pp. 330–353.

[4] S. Bochner, *Diffusion equation and stochastic processes*, Proc. Nat. Acad. Sci. USA, 35 (1949), pp. 368–370.

[5] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Springer-Verlag, New York, 1967.

[6] A. S. Carasso, *Overcoming Hölder continuity in ill-posed continuation problems*, SIAM J. Numer. Anal., 31 (1994), pp. 1535–1557.

[7] A. S. Carasso, *Error bounds in nonsmooth image deblurring*, SIAM J. Math. Anal., 28 (1997), pp. 656–668.

[8] A. S. Carasso and T. Kato, *On subordinated holomorphic semigroups*, Trans. Amer. Math. Soc., 327 (1991), pp. 867–877.
[9] J. F. Douglas, *Some applications of fractional calculus to polymer science*, in Advances in Chemical Physics, 102, I. Prigogine and S. Rice, eds., Wiley, New York, 1997.
[10] L. Eldén, *Numerical solution of the sideways heat equation by difference approximation in time*, Inverse Problems, 11 (1995), pp. 913–923.
[11] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd ed., Wiley, New York, 1971.
[12] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.
[13] Y. Fujita, *A sufficient condition for the Carasso-Kato theorem*, Math. Ann., 297 (1993), pp. 335–341.
[14] F. John, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math., 13 (1960), pp. 551–585.
[15] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer-Verlag, New York, 1980.
[16] R. J. Knops, ed., *Symposium on Non-Well-Posed Problems and Logarithmic Convexity*, Lecture Notes in Math. 316, Springer-Verlag, New York, 1973.
[17] R. J. Knops, *Logarithmic convexity and other techniques applied to problems in continuum mechanics*, in Symposium on Non-Well-Posed Problems and Logarithmic Convexity, R. J. Knops, ed., Lecture Notes in Math. 316, Springer-Verlag, New York, 1973, pp. 41–54.
[18] R. J. Knops and L. E. Payne, *On the stability of solutions of the Navier–Stokes equations backward in time*, Arch. Rational Mech. Anal., 29 (1968), pp. 331–335.
[19] S. G. Kreĭn and O. I. Prozorovskaya, *Analytic semigroups and incorrect problems for evolutionary equations*, Dokl. Akad. Nauk. SSSR, 13 (1960), pp. 277–280.
[20] H. C. Kuhlmann, M. Wanschura and H. J. Rath, *Flow in two-sided lid-driven cavities: non-uniqueness, instabilities, and cellular structures*, J. Fluid Mech., 336 (1997), pp. 267–299.
[21] K. Miller, *Least squares methods for ill-posed problems with a prescribed bound*, SIAM J. Math. Anal., 1 (1970), pp. 52–74.
[22] K. Miller, *Logarithmic convexity results for holomorphic semigroups*, Pacific J. Math., 58 (1975), pp. 549–551.
[23] E. Nelson, *A functional calculus using singular Laplace integrals*, Trans. Amer. Math. Soc., 88 (1958), pp. 400–413.
[24] L. E. Payne, *Improperly Posed Problems in Partial Differential Equations*, CBMS-NSF Reg. Conf. Series in Appl. Math., 22, SIAM, Philadelphia, PA, 1975.
[25] L. E. Payne, *Some remarks on ill-posed problems for viscous fluids*, Internat. J. Engrg. Sci., 30 (1992), pp. 1341–1347.
[26] R. S. Phillips, *On the generation of semigroups of linear operators*, Pacific J. Math., 2 (1952), pp. 343–369.
[27] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
[28] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.
[29] K. Yosida, *Fractional powers of infinitesimal generators and the analyticity of the semigroups generated by them*, Proc. Japan Acad., 36 (1960), pp. 86–89.