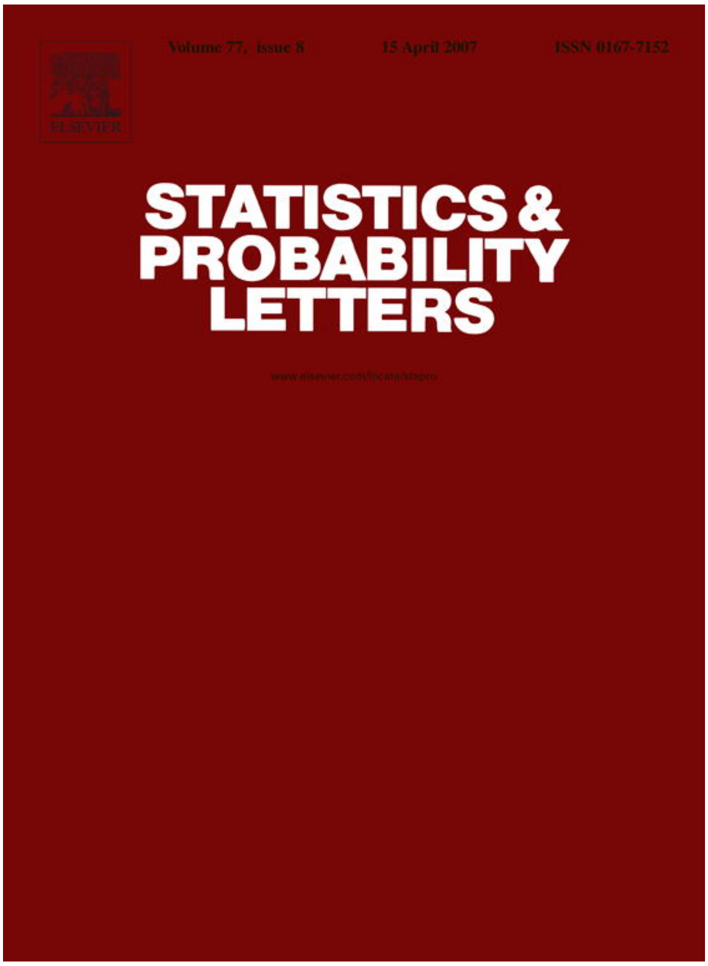
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Conservative confidence intervals based on weighted means statistics

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Abstract

For weighted means estimators of the common mean of several normal populations associated (conservative) confidence intervals are constructed. These intervals are compared to several traditional confidence bounds. Monte Carlo simulation results of these comparisons are reported.

Keywords: Common mean; Degree of equivalence; Fairweather procedure; Graybill–Deal estimator; Interlaboratory study; Unbiased estimators

1. Introduction

The common mean estimation problem is one of the traditionally difficult and challenging problems of mathematical statistics. It appears, for example, when examining data from interlaboratory studies or when analyzing balanced incomplete block designs with uncorrelated random block effects and fixed treatment effects.

Let there be p laboratories, each of them measuring the unknown underlying reference value μ common to all laboratories. In the simplest model the measurements y_{ij}, $i = 1, \ldots, p$; $j = 1, \ldots, n_i$, are of the form

$$y_{ij} = \mu + e_{ij} \tag{1}$$

with independent $e_{ij} \sim \mathrm{N}(0, \kappa_i^2)$. All parameters μ, κ_i^2, $i = 1, \ldots, p$ are unknown, and the goal is to estimate μ or, more importantly, to provide a confidence interval for μ.

Put $Y_i = \sum_j y_{ij}/n_i$, and denote by $u_i^2 = \sum_j (y_{ij} - Y_i)^2/[(n_i - 1)n_i]$, the best unbiased estimate of the variance, $\sigma_i^2 = \kappa_i^2/n_i$, of Y_i. The vector $(Y_1, \ldots, Y_p, u_1^2, \ldots, u_p^2)$ forms an (incomplete) sufficient statistic with a well-known distribution.

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2. Weighted means statistics and confidence intervals

The goal here is to explore confidence intervals based on weighted means statistics useful for the common mean estimation. When all variances σ_i^2 are known, the best unbiased estimator of μ is a weighted means statistic

$$\tilde{Y} = \sum_1^p \omega_i Y_i, \tag{2}$$

where $\omega_i = \omega_i^{\text{opt}} = \sigma_i^{-2}/(\sum_1^p \sigma_k^{-2})$ are normalized weights. In this situation it is also the maximum likelihood estimator, and without normality assumption (but when all variances are known) this is the best linear (in Y_i) unbiased estimator of μ. We investigate the possibility of using estimators (2) to obtain confidence intervals for μ. The suggestion is to employ a quadratic form $\sum_1^p q_i(Y_i - \tilde{Y})^2$ with positive coefficients q_i to estimate $var(\tilde{Y})$ for fixed weights ω_i.

We need the following formula for the distribution function of the squared standard normal variable Z,

$$P(Z^2 \leqslant z) = \frac{1}{\pi}\int_0^1 \frac{[1 - \mathrm{e}^{-z/(2u)}]\,\mathrm{d}u}{\sqrt{u(1-u)}}, \quad z > 0$$

(see Abramovitz and Stegun, 1972, Chapters 7, 7.4.9) which implies that with independent standard normal variables $Z_1, \ldots, Z_p$ and positive coefficients $a_2, \ldots, a_p$,

$$\begin{aligned} P\left(Z_1^2 > \sum_2^p a_k Z_k^2\right) &= \frac{1}{\pi}\int_0^1 \frac{E\mathrm{e}^{-(\sum a_k Z_k^2)/(2u)}\,\mathrm{d}u}{\sqrt{u(1-u)}} \\ &= \frac{1}{\pi}\int_0^1 \frac{\mathrm{d}u}{\sqrt{u(1-u)\prod_k (1 + a_k/u)}}. \end{aligned} \tag{3}$$

In particular with T_{p-1} denoting a t-distributed random variable with $p-1$ degrees of freedom

$$P(|T_{p-1}| > t\sqrt{p-1}) = \frac{1}{\pi}\int_0^1 \frac{u^{(p-2)/2}\,\mathrm{d}u}{\sqrt{(1-u)(u+t^2)^{p-1}}},$$

so that as $t \to \infty$,

$$P(|T_{p-1}| > t\sqrt{p-1}) \sim \frac{1}{\pi t^{p-1}}\int_0^1 \frac{u^{(p-2)/2}\,\mathrm{d}u}{\sqrt{1-u}} = \frac{\Gamma(p/2)}{\sqrt{\pi}\Gamma((p+1)/2)t^{p-1}}.$$

Let $\sum_1^p q_i(Y_i - \tilde{Y})^2$ be a quadratic estimator of $var(\tilde{Y})$. Put $\sum q_i = q, \gamma = \sum_i \omega_i^2/q_i \geqslant 1/q$.

Theorem 2.1. *For $t \to \infty$,*

$$\lim t^{p-1} \sup_{\sigma_1^2,\ldots,\sigma_p^2} P\left(\left|\frac{\tilde{Y} - \mu}{\sqrt{\sum_1^p q_i(Y_i - \tilde{Y})^2}}\right| > t\right) = \lim t^{p-1} P\left(|T_{p-1}| > t\sqrt{(p-1)\left(\gamma p^p \prod_1^p q_i\right)^{1/(p-1)}}\right). \tag{4}$$

In other words, the smallest coverage probability of the confidence interval for μ, $\tilde{Y} \pm t\sqrt{\sum_1^p q_i(Y_i - \tilde{Y})^2}$, when t is large, is attained at the t-distribution with $p-1$ degrees of freedom. We will show that this happens when the variances σ_i^2 are inversely proportional to the coefficients q_i, $\sigma_i^2 \sim t^2/q_i$. Clearly the coverage probability above is invariant under simultaneous scale transformations, so that its infimum is attained asymptotically at any positive multiple of $q_1^{-1}, \ldots, q_p^{-1}$.

Proof. One has

$$P\left(|\tilde{Y}-\mu|>t\sqrt{\sum_1^p q_i(Y_i-\tilde{Y})^2}\right) = P\left(\sum_1^p q_i\left(\sigma_i Z_i - \sum_k \omega_k\sigma_k Z_k\right)^2 - t^{-2}\left(\sum_1^p \omega_i\sigma_i Z_i\right)^2 < 0\right) = P(Z^{\mathrm{T}}DZ<0)$$

with the vector Z formed by independent standard normal $Z_1,\ldots,Z_p$, and the matrix D of the form

$$D=\Sigma^{1/2}[I+aa^{\mathrm{T}}-bb^{\mathrm{T}}]\Sigma^{1/2}=\Sigma+\Sigma^{1/2}aa^{\mathrm{T}}\Sigma^{1/2}-\Sigma^{1/2}bb^{\mathrm{T}}\Sigma^{1/2}.$$

Here $\Sigma^{1/2}$ is the diagonal matrix given by the elements $\tau_i=\sigma_i\sqrt{q_i}$, $i=1,\ldots,p$;

$$a=\frac{1}{\sqrt{q-t^{-2}}}\left(\frac{\omega_1}{\sqrt{q_1}}(q-t^{-2})-\sqrt{q_1},\ldots,\frac{\omega_p}{\sqrt{q_p}}(q-t^{-2})-\sqrt{q_p}\right)^{\mathrm{T}},$$

$$b=\frac{1}{\sqrt{q-1/t^2}}(\sqrt{q_1},\ldots,\sqrt{q_p})^{\mathrm{T}}.$$

Thus, D is congruent via the diagonal matrix $\Sigma^{1/2}$ to the matrix

$$F=I+aa^{\mathrm{T}}-bb^{\mathrm{T}},$$

which does not depend on $\sigma_1,\ldots,\sigma_p$.

We look first at the case when a and b are linearly independent, i.e. ω's are not proportional to q's, so that $\gamma>1/q$. Clearly the linear operator corresponding to F leaves the subspace L, spanned by the vectors a and b, and its orthogonal complement invariant. Its restriction to this complement is the identity transformation, and the restriction on L has the determinant, $d=-\gamma/t^2$, and the trace, $\mathrm{tr}=(q-1/t^2)\gamma$. Thus, F has eigenvalues 1 with multiplicity $p-2$, $\mu_1=\mathrm{tr}/2-\sqrt{\mathrm{tr}^2/4-d}<0$, $\mu_2=\mathrm{tr}/2+\sqrt{\mathrm{tr}^2/4-d}>1$, and D has exactly one negative eigenvalue λ_1 and $p-1$ positive eigenvalues λ_k, $k=2,\ldots,p$.

The characteristic polynomial of the matrix D has the form

$$\phi_D(\lambda) = -\det(\lambda I-\Sigma)\det\begin{pmatrix}1+a^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}a & a^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}b\\ b^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}a & -1+b^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}b\end{pmatrix}$$
$$= \det(\lambda I-\Sigma)[(1+a^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}a)(1-b^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}b)+(a^{\mathrm{T}}(\lambda\Sigma^{-1}-I)^{-1}b)^2]$$

(see Harville, 1997, Theorem 18.1.1). By Sylvester's law of inertia (Horn and Johnson, 1985, Theorem 4.5.8), $\phi_D(\lambda)$ must have exactly one simple negative root, say, λ_1. It is convenient to normalize matrix D so that $\lambda_1=-1$, which means that $\tau_1,\ldots,\tau_p$ are subject to the condition

$$\left(\sum\frac{a_i^2}{\tau_i^{-2}+1}+1\right)\left(\sum\frac{b_i^2}{\tau_i^{-2}+1}-1\right)=\left(\sum\frac{a_ib_i}{\tau_i^{-2}+1}\right)^2. \tag{5}$$

Then according to (3),

$$\begin{aligned}
&P(Z^{\mathrm{T}}DZ<0)\\
&=P\left(Z_1^2>\sum_{k=2}^p\lambda_k Z_k^2\right)\\
&=\frac{1}{\pi}\int_0^1\frac{u^{p/2-1}\,\mathrm{d}u}{\sqrt{(1-u)\prod_2^p(u+\lambda_k)}}=\frac{1}{\pi}\int_0^1\frac{u^{p/2-1}\,\mathrm{d}u}{|\phi_D(-u)|^{1/2}}\\
&=\frac{1}{\pi}\int_0^1\frac{u^{p/2-1}\,\mathrm{d}u}{[\prod(u+\tau_i^2)[(\sum(a_i^2/(u\tau_i^{-2}+1))+1)t(\sum(b_i^2/(u\tau_i^{-2}+1))-1)-(\sum(a_ib_i/(u\tau_i^{-2}+1)))^2]]^{1/2}}.
\end{aligned}$$

When $t\to\infty, \sum b_i^2\to 1, \sum a_ib_i=\sum b_i^2-1\to 0$, but $\sum a_i^2\to q\gamma-1>0$. One gets from (5), $\tau_i^2\to\infty$, so that $\sum b_i^2\tau_i^{-2}/(\sum b_i^2-1)\to 1$. It follows that for $0<u<1$,

$$\left(\sum\frac{a_i^2}{u\tau_i^{-2}+1}+1\right)\left(\sum\frac{b_i^2}{u\tau_i^{-2}+1}-1\right)-\left(\sum\frac{a_ib_i}{u\tau_i^{-2}+1}\right)^2 = (1-u)q\gamma\left(\sum b_i^2-1\right)[1+\mathrm{o}(1)],$$

with the o(1) term which is uniform in u. As before,

$$\lim t^{p-1}\sup_{\sigma_1^2,\dots,\sigma_p^2} P(Z^{\mathrm{T}}DZ<0)=\frac{1}{\pi\sqrt{q\gamma}\prod\hat\tau_i(\sum b_i^2-1)^{1/2}}\int_0^1\frac{u^{p/2-1}\,\mathrm{d}u}{\sqrt{1-u}}$$

and the values $\hat\tau_i$ minimizing $\prod\tau_i$ under condition (5) are $\hat\tau_i^2\approx pb_i^2/(\sum b_i^2-1)$. We conclude that

$$\begin{aligned}\lim t^{p-1}\sup_{\sigma_1^2,\dots,\sigma_p^2} P\left(|\tilde Y-\mu|>t\sqrt{\sum_1^p q_i(Y_i-\tilde Y)^2}\right) &= \frac{(\sum b_i^2-1)^{(p-1)/2}}{\pi\prod b_i\sqrt{q\gamma}}\int_0^1\frac{u^{p/2-1}\,\mathrm{d}u}{\sqrt{1-u}}\\ &=\frac{\Gamma(p/2)}{\sqrt{\pi}\Gamma((p+1)/2)t^{p-1}q^{p/2}p^{p/2}\gamma^{1/2}\prod\sqrt{q_i/q}}.\end{aligned}$$

If vectors a and b are linearly dependent, i.e. $\omega_i=q_i/q$, the matrix

$$F=I-(1+1/(qt^2))bb^{\mathrm{T}}/(b^{\mathrm{T}}b)=I-\left(1-\frac{1}{q^2t^4}\right)bb^{\mathrm{T}}$$

has eigenvalues: 1 with multiplicity $p-1$, and $1-(1+1/(qt^2))=-1/(qt^2)<0$. A similar argument shows that (4) holds. □

Theorem 2.1 shows that the smallest coverage probability of the approximate $(1-\alpha)$-confidence interval

$$\tilde Y\pm\frac{t_{\alpha/2}(p-1)\sqrt{\sum_1^p q_i(Y_i-\tilde Y)^2}}{\sqrt{(p-1)(\gamma p^p\prod_1^p q_i)^{1/(p-1)}}}, \tag{6}$$

is attained at the t-distribution with $p-1$ degrees of freedom. Notice that the width of this interval is invariant under multiplication of the q's by a common positive factor.

Theorem 2.1 holds for a more general quadratic form $\sum_{i,k}q_{ik}(Y_i-\tilde Y)(Y_k-\tilde Y)$, with a symmetric matrix $Q=\{q_{ik}\}$. Indeed with d denoting the vector of ones, and ω the vector of weights, put in the proof $q=d^{\mathrm{T}}Qd$ and $b=(q-1/t^2)^{-1/2}Q^{1/2}d$, so that $a=\sqrt{q-1/t^2}Q^{-1/2}\omega-b$, and $\gamma=\omega^{\mathrm{T}}Q^{-1}\omega$.

When $p=2$, (4) is an equality which holds for all t such that

$$t^{-2}\leqslant q-\max_i q_i(2\omega_i-1)\omega_i^{-2}.$$

This condition means positivity of the diagonal elements of F; without it the supremum in (4) is 1 for any p. Bakirov (1987) has shown that (4) is an equality for all p when $t^2>p-1, \omega_i\equiv p^{-1}$ (i.e. $\tilde Y$ is the sample mean $\bar Y$) and $q_i\equiv 1/[p(p-1)]$ (i.e. an unbiased multiple of the sample variance estimates the variance of $\tilde Y$). In fact, in this situation the minimal value of (4) under condition (5) occurs when for some $m=0,1,\dots,p-1$, $\tau_{i_1}^2=\dots=\tau_{i_m}^2=0$ and all remaining $\tau_{i_k}^2\equiv pt^2/((p-1)(p-m)-mt^2)$. It follows that

$$\begin{aligned}f(t)=f_p(t)&=\sup_{\sigma_1^2,\dots,\sigma_p^2}P\left(|\bar Y-\mu|>t\sqrt{\frac{\sum_1^p(Y_i-\bar Y)^2}{p(p-1)}}\right)\\ &=\max_{k:k(p-1)>t^2(p-k)}P\left(|T_{k-1}|>\frac{t\sqrt{p(k-1)}}{\sqrt{k(p-1)-t^2(p-k)}}\right)\end{aligned}$$

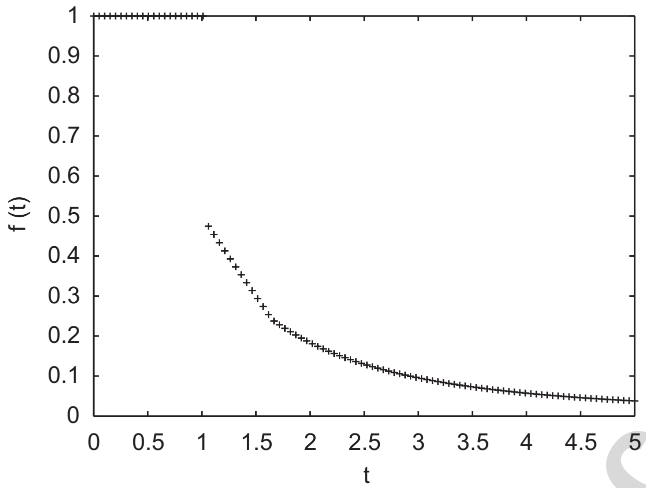

Fig. 1. Plot of $f(t)$ for $p = 3$.

is attained at the t-distribution with $p-1$ degrees of freedom. In particular, $f(t) = 1$ for $t < 1$, and $f(t) = P(|T_1| > \sqrt{pt^2/(2(p-1)-(p-2)t^2)}) = 1 - 2\pi^{-1} \arctan(\sqrt{pt^2/(2(p-1)-(p-2)t^2)}), 1 \leqslant t^2 < 2(p-1)/(p-2)$, with $f(1) = 0.5$, and $f(t) = P(|T_{p-1}| > t)$ for $t^2 > p-1$. Plot of this function when $p = 3$ is shown in Fig. 1.

3. Graybill–Deal estimator: unbiased estimation of the variance and of degree of equivalence

In practice the weights ω_i are estimated by the available u_i^2, but the conservativeness of the interval (6) holds in this case as well. One of the traditional (weighted means) estimators of μ suggested by Graybill and Deal (1959) has the form

$$\tilde{Y}^0 = \frac{\sum_i (Y_i/u_i^2)}{\sum_i (1/u_i^2)} = \sum \omega_i^0 Y_i \tag{7}$$

with $\omega_i^0 = u_i^{-2}/\sum_{k=1}^p u_k^{-2}$. Although the variance of this statistic does not have a very simple form, an unbiased estimate of this variance is known. See Sinha (1985) and Voinov and Nikulin (1993, pp. 194–196).

The latter authors express the unbiased estimate $\widehat{var}$ of the variance of $\tilde{Y}^0$ in terms of the hypergeometric function

$$F(1,2;c;z) = \sum_{n=0}^{\infty} \frac{(n+1)!\Gamma(c)}{\Gamma(n+c)} z^n.$$

More precisely,

$$\widehat{var}(\tilde{Y}^0) = \frac{\sum_i \omega_i^0 F(1,2;(n_i+1)/2, 1-\omega_i^0)}{\sum_i 1/u_i^2}.$$

Note that for $n_i = 3$, $F(1,2;2;1-z) = 1/z$. Then

$$\widehat{var}(\tilde{Y}^0) = \frac{p}{\sum_i 1/u_i^2}.$$

This formula should be compared with the variance estimate $[\sum_i 1/u_i^2]^{-1}$, which is often suggested, but which systematically underestimates $var(\tilde{Y}^0)$. Cochran (1954, p. 126) gives a table of multiples of $[\sum_i 1/u_i^2]^{-1}$, which provide an estimate of $var(\tilde{Y}^0)$. Although the value $\bar{n} = 3$ is not given in this table, this factor seems to be fairly close to p.

In addition to the common mean estimator and its uncertainty in the international key comparisons studies one has to give for each laboratory a characteristic of consistency with the common mean (the so-called degree of equivalence). To estimate this characteristic we use the statistic $Y_i - \tilde{Y}^0$.

Here we give the unbiased estimate of the variance of this statistic.

Proposition 3.1. *The statistic*

$$\widehat{var}(Y_i - \tilde{Y}^0) = u_i^2 - 2\frac{F(1,1;(n_i+1)/2, 1-\omega_i^0)}{\sum_k 1/u_k^2} + \frac{\sum_k \omega_k^0 F(1,2;(n_k+1)/2, 1-\omega_k^0)}{\sum_k 1/u_k^2} \tag{8}$$

is an unbiased estimator of the variance of $Y_i - \tilde{Y}^0$.

Proof. By independence of u_k^2 and Y_k,

$$var(Y_i - \tilde{Y}^0) = \sigma_i^2 E(1-\omega_i^0)^2 + \sum_{k \neq i} \sigma_k^2 E(\omega_k^0)^2 = \sigma_i^2 - 2\sigma_i^2 E\omega_i^0 + \sum_k \sigma_k^2 E(\omega_k^0)^2.$$

The unbiased estimate of the last term in the right-hand side is $\widehat{var}(\tilde{Y}^0)$, the first term is estimated unbiasedly by u_i^2. To estimate the second term, $\sigma_i^2 E\omega_i$, one can use the same technique of integration by parts as given in Theorem 1 in Sinha (1985) to obtain

$$\sigma_i^2 E\omega_i^0 = E\delta_i(u_1^2, \ldots, u_p^2)$$

with

$$\delta_i(u_1^2, \ldots, u_p^2) = \frac{n_i - 1}{2u_i^{n_i-3}} \int_{u_i^{-2}}^{\infty} \frac{\mathrm{d}v}{(v + \sum_{k\neq i} u_k^{-2}) v^{(n_i-1)/2}}.$$

The known facts about the hypergeometric function show that

$$\delta_i(u_1^2, \ldots, u_p^2) = \frac{F(1,1;(n_i+1)/2, 1-\omega_i^0)}{\sum_k 1/u_k^2}$$

so that the unbiased estimate $\widehat{var}(Y_i - \tilde{Y}^0)$ indeed has the form (8). □

4. Simulation results

When $p = 2$, there is a simple formula for the width

$$\Delta = \frac{2t_{\alpha/2}(p-1)\sqrt{\sum_1^p q_i(Y_i - \tilde{Y})^2}}{\sqrt{(p-1)(\gamma p^p \prod_1^p q_i)^{1/(p-1)}}},$$

of the confidence interval (6) which depends neither on the weights ω_i nor on coefficients q_i. Indeed, $\sum_1^2 q_i(Y_i - \tilde{Y})^2 = (\omega_2^2 q_1 + \omega_1^2 q_2)(Y_1 - Y_2)^2 = \gamma \prod_1^2 q_i (Y_1 - Y_2)^2$, so that $\Delta = t_{\alpha/2}(1)|Y_1 - Y_2|$ and $E\Delta = t_{\alpha/2}(1)\sqrt{2(\sigma_1^2 + \sigma_2^2)/\pi}$.

Several confidence procedures for the common mean μ, when $p = 2$, are reviewed by Yu et al. (1999). As most of these procedures may not give an interval, we compared (6) only with the confidence interval based on the Fairweather (1972) procedure. The estimator of μ is based on the weights proportional to $(n_i - 3)/[u_i(n_i - 1)]$, the width of the interval centered at this estimate is determined from a t-approximation with estimated degrees of freedom. This approximation supposes the condition, $\min_i n_i \geqslant 5$ (which is commonly violated in interlaboratory studies), and the estimated degrees of freedom is always larger than 4. It turned out that in the model (1) the average width of the corresponding interval was always smaller than Δ. However, for substantially different σ_i^2 the coverage probability of the Fairweather interval falls below the nominal value, and this fact also holds for larger p. Figs. 1 and 2 portray these characteristics for both intervals as the function of $\lambda = \sigma_2^2/\sigma_1^2$ when $n_1 = n_2 = 5$. Notice that (6) could be used for any n_1, n_2, and in the extreme case $n_1 = n_2 = 1$, (6) outperforms the analogue of the Fairweather procedure based on the Cauchy distribution for the pivot $\sum c_i(Y_i - \mu)u_i^{-1}/\sum c_i u_i^{-1}$. The behavior of the Fairweather interval dramatically deteriorates in a random effects model which is discussed later (Fig. 3).

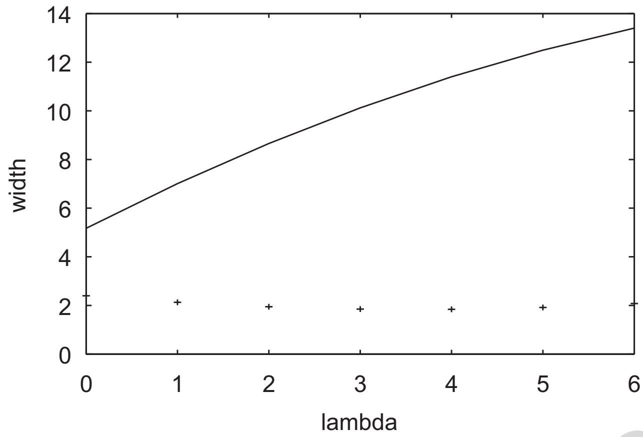

Fig. 2. Plot of widths of Fairweather confidence interval (line marked by +) and (6) (continuous line).

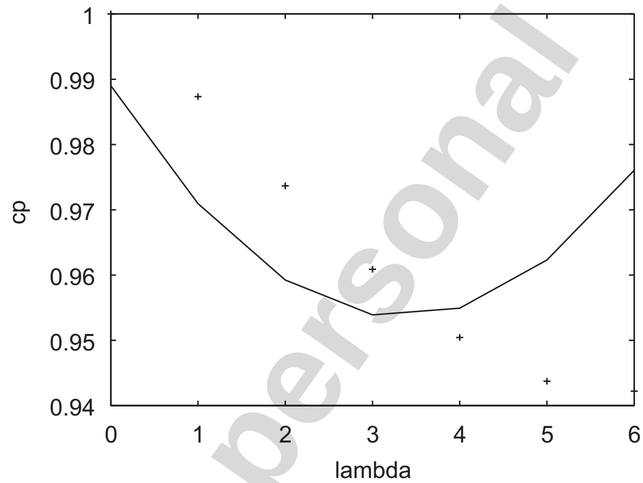

Fig. 3. Plot of coverage probabilities of Fairweather confidence interval and (6) (designations of lines are the same as in Fig. 3).

To compare (6) with other procedures based on estimating the variance of a weighted means statistic $\tilde{Y}$ by $\sum_1^p q_i(Y_i - \tilde{Y})^2$, we investigated an estimator of the variance of $\tilde{Y}$ determined by coefficients

$$q_i^{(0)} = \frac{p}{p-1}\omega_i^2 \tag{9}$$

suggested by Rukhin (2003) or by $q_i^{(0a)} = \omega_i^2$ introduced in Rukhin and Vangel (1998).

Another procedure to estimate $var(\tilde{Y})$ applicable in a more general setting of linear models was put forward by Horn et al. (1975). To estimate this variance, $var(\tilde{Y}) = \sum_1^p \omega_i^2 \, var(Y_i)$, the statistic, $\widehat{var}(\tilde{Y}) = \sum_i \omega_i^2 (Y_i - \tilde{Y})^2/(1-\omega_i)$, has been suggested. In other terms

$$q_i^{(1)} = \frac{\omega_i^2}{1-\omega_i}. \tag{10}$$

One may be also interested in an unbiased estimator of $var(\tilde{Y})$ for fixed ω's, i.e. in coefficients q's such that $E\sum_1^p q_i(Y_i - \tilde{Y})^2 = \sum_1^p \omega_i^2 \sigma_i^2$. The form of these coefficients follows from the formula (2.8) in Cressie (1982),

$$q_i = \frac{\omega_i^2}{1-2\omega_i}\left[1 + \sum_k \frac{\omega_k^2}{1-2\omega_k}\right]^{-1},$$

which gives a non-satisfactory answer when $\max_k \omega_k \geqslant \frac{1}{2}$.

Table 1
The confidence coefficients (cc) of the confidence intervals and their average half-widths (hw)

	$\bar{Y}$	$GD(11)$	$GD(10)$	$GD(9)$	$DL(11)$	$DL(10)$	$DL(9)$	F
hw	2.37	2.42	3.19	2.96	2.32	2.68	2.59	1.20
cc	0.95	0.97	0.98	0.98	0.97	0.98	0.98	0.78

The proof of Theorem 2.1 suggests to look at the special case when $\omega_i = q_i/q$, with the interval,

$$\tilde{Y} \pm \frac{t_{\alpha/2}(p-1)\sqrt{\sum_1^p \omega_i (Y_i - \tilde{Y})^2}}{\sqrt{(p-1)[p^p \prod_1^p \omega_i]^{1/(p-1)}}}. \tag{11}$$

In meta-analysis applications the weights are commonly estimated via $\hat{\omega}_i = (y+u_i)^{-2}/\sum_k (y+u_k)^{-2}$ with a positive y determined by the method of moments in the random effects model

$$y_{ij} = \mu + \ell_i + e_{ij} \tag{12}$$

with independent $\ell_i \sim \mathrm{N}(0, \tau^2)$ and $e_{ij} \sim \mathrm{N}(0, \kappa_i^2)$. The most popular method is due to DerSimonian and Laird (1986). It uses a non-negative $y = y_{\mathrm{DL}}$ from the formula

$$y_{\mathrm{DL}} = \max\left[0, \frac{\sum_i u_i^{-2}(Y_i - \tilde{Y}^0)^2 - p + 1}{\sum_{i=1}^p u_i^{-2} - \sum_{i=1}^p u_i^{-4}[\sum_{i=1}^p u_i^{-2}]^{-1}}\right].$$

Here $\tilde{Y}^0$ is the Graybill–Deal estimator.

In Table 1 we report the results of a Monte Carlo simulation study when the sample sizes of $p = 10$ laboratories are chosen to be 5 to 14. The distribution of the variances, σ_i^2, is taken to be the inverted gamma-distribution with parameters $\alpha = 2$, $\beta = 1/p$, so that $E\sigma_i^2 = p$. The sample means Y_i were simulated as $Y_i = \sigma_i Z_i$ from a standard normal sample $Z_1, \ldots, Z_p$; the sample variances u_i^2 were taken to be realizations of multiples of χ^2-random variables, $u_i^2 \sim \sigma_i^2 \chi^2(n_i - 1)/(n_i - 1)$. We studied the confidence intervals based on the following estimators: the overall sample mean, $\bar{Y}$; the Graybill–Deal (GD) estimator (7) with three intervals based on (9)–(11); the DerSimonian–Laird (DL) estimator with three similar intervals and the Fairweather procedure (F) based on t-approximation with estimated degrees of freedom.

Table 1 gives the simulated values of the widths of the intervals based on these procedures and the corresponding confidence coefficients. Clearly (11) gives a much shorter interval than (10) or (9) although in terms of the width the Fairweather interval is the best overall. However, its confidence coefficient (0.78) is well below the nominal 95% value, and this advantage disappears if the model (1) is replaced by (12). These findings are confirmed by other simulations. Therefore the widely applicable interval (11) can be recommended in situations where a high coverage probability is desired, when the sample variances u_i^2 underestimate the variances of the sample means Y_i, and/or when the sample sizes n_i are small.

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