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Uncertainty in Finite Element Modeling and Failure Analysis: A Metrology-Based Approach

In this paper, we first review the impact of the powerful finite element method (FEM) in structural engineering, and then address the shortcomings of FEM as a tool for risk-based decision making and incomplete-data-based failure analysis. To illustrate the main shortcoming of FEM, i.e., the computational results are point estimates based on "deterministic" models with equations containing mean values of material properties and prescribed loadings, we present the FEM solutions of two classical problems as reference benchmarks: (RB-101) The bending of a thin elastic cantilever beam due to a point load at its free end and (RB-301) the bending of a uniformly loaded square, thin, and elastic plate resting on a grillage consisting of 44 columns of ultimate strengths estimated from 5 tests. Using known solutions of those two classical problems in the literature, we first estimate the absolute errors of the results of four commercially available FEM codes (ABAQUS, ANSYS, LSDYNA, and MPAVE) by comparing the known with the FEM results of two specific parameters, namely, (a) the maximum displacement and (b) the peak stress in a coarse-meshed geometry. We then vary the mesh size and element type for each code to obtain grid convergence and to answer two questions on FEM and failure analysis in general: (Q-1) Given the results of two or more FEM solutions, how do we express uncertainty for each solution and the combined? (Q-2) Given a complex structure with a small number of tests on material properties, how do we simulate a failure scenario and predict time to collapse with confidence bounds? To answer the first question, we propose an easy-to-implement metrology-based approach, where each FEM simulation in a grid-convergence sequence is considered a "numerical experiment," and a quantitative uncertainty is calculated for each sequence of grid convergence. To answer the second question, we propose a progressively weakening model based on a small number (e.g., 5) of tests on ultimate strength such that the failure of the weakest column of the grillage causes a load redistribution and collapse occurs only when the load redistribution leads to instability. This model satisfies the requirement of a metrology-based approach, where the time to failure is given a quantitative expression of uncertainty. We conclude that in today's computing environment and with a precomputational "design of numerical experiments," it is feasible to "quantify" uncertainty in FEM modeling and progressive failure analysis. [DOI: 10.1115/1.2150843]

1 Introduction

In a survey article presented at the First International Conference on Structural Failure, Product Liability and Technical Insurance (Vienna, 1983), Ross [1] reported a disproportional increase in the annual rate of product liability cases (25%) versus that of the population in the United States (2%). He cited a landmark 1963–1964 California decision [2] in product liability law to alert engineers that it was no longer sufficient to design and manufacture (construct) a product without looking at all conceivable scenarios of failures:

"*A manufacturer is strictly liable in tort when an article he places on the market*, *knowing that it is to be used without inspection for defects*, *proves to have a defect that causes injury to a human being.*"

Rossmanith [3] reported at the same conference, as shown in Fig. 1, that engineers need to be familiar with all possible failure-inducing parameters before committing to a product design and manufacturing (construction) plan. Implicit in his representation is the notion of two types of uncertainty.

Uncertainty-1. Engineers use a deterministic model, a computational tool named a finite element method (FEM), and a number of assumptions based on experience, to configure a product with a series of code-specified safety factors to account for uncertainty due to load, geometry, material property testing, and manufacturing process, in order to estimate an "acceptable" product life. Alternatively, engineers may use a probabilistic model and the associated finite element method (PFEM) to arrive at a distribution of product life that is acceptable to the user. In either case, **uncertainty-1** needs to be expressed and verified, if required, by technical experts. **Uncertainty-1** allows engineers to conduct *risk-based decision making.*

Uncertainty-2. When a failure occurs and a failure analyst is engaged to identify the damage, find the probable causes and assist the proper parties in assessing the damage claim for recovery by insurance, a different type of uncertainty enters the picture.

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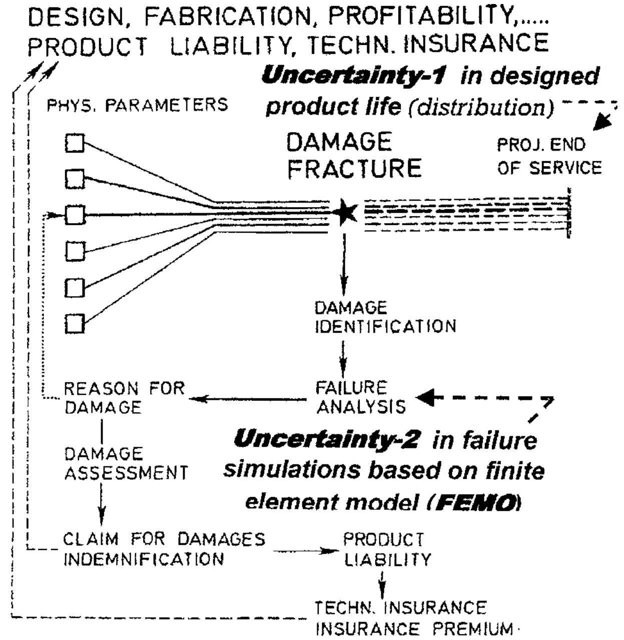


Fig. 1 The addition of two types of uncertainties in a representation of the interrelation of fracture mechanics, failure analysis, product liability, and technical insurance by Rossmanith [3]

Foremost among the uncertainties is the loss of data relevant to the failure. If the analyst uses a deterministic model, the input data is no longer deterministic. Results of failure simulations based on a finite element method with variable input data need to be expressed with **uncertainty-2**, which is required to be not only verified by a panel of experts, but also admissible in a court of law. An analyst manages **uncertainty-2** with *incomplete-data-based failure analysis.*

To update our knowledge of uncertainty in the finite element method (FEM) and failure analysis (FA), we resort to Google Scholar [4] and obtain the following statistics:

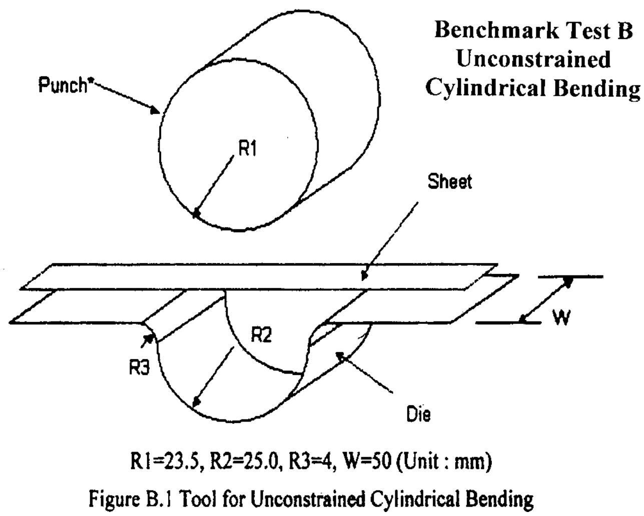
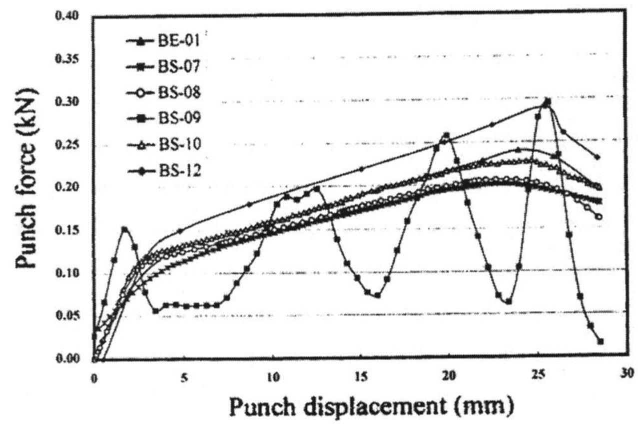


Fig. 2 One of three benchmark test problems of NUMISHEET [71], an international interlaboratory validation exercise for simulating three-dimensional aluminum and steel sheet-forming processes



(a)

SIMULATIONS			
BS-01	Eugenio Oñate Ibañez De Navarra, Alberto Férriz, and Oscar Fruitós CIMNE	CIMNE	Spain
BS-02	Ming Chen	United States Steel Corp.	USA
BS-03	Tony Chang and Wei Wang	Rouge Steel Co.	USA
BS-04	Francis Sabourin	INSA - Lyon	France
BS-05	Satoshi Ikeda	ESTECH Corp.	Japan
BS-06	Margot Klaassen and Eisso Atzema	Corus Research	The Netherlands
BS-07	Siguang Xu, Ramesh Joshi, and Chuan-Tao Wang	General Motors Corp.	USA
BS-08	Jun Park	Hibbitt, Karlsson & Sorensen, Inc.	USA
BS-09	Bin Zhang, Dongsheng Li, and Xianbin Zhou	Beijing Univ. of Aeronautics and Astronautics	P. R. China
BS-10	M. C. Oliveira, J. L. Alves, and L. F. Menezes	Univ. of Coimbra	Portugal
BS-11	Chung-Souk Han and Robert H. Wagoner	The Ohio State Univ.	USA
BS-12	Margot Klaassen and Eisso Atzema	Corus Research	The Netherlands
BS-13	Seung-Geun Lee, Yangwook Choi, and J. K. Lee	The Ohio State Univ.	USA
BS-14	Maki Nagakura, Masato Takamura, and Ohura Kenichi	The Institute of Physical & Chemical Research	Japan
BS-15	Margot Klaassen and Eisso Atzema	Corus Research	The Netherlands
BS-16	Siguang Xu, Ramesh Joshi, and Chuan-Tao Wang	General Motors Corp.	USA
BS-17	Yongming Kong, Wan Cheng, and Decai Jia	Altair Engineering Ltd.	P. R. China

(b)

Fig. 3 (a) Force versus displacement results of five simulations and one experiment (BE-01, Korea) reported in 2002 NUMISHEET [71]. The simulations resulted from five different FEM models, i.e., BS-07 (LS-DYNA3D, 384 elem.), BS-08 (ABAQUS, 4400 elem.), BS-09 (PAM-STAMP, 3600 elem.), BS-10 (DD3 IMP, 7380 elem.), and BS-12 (Indeed 7.3.1, 4000 elem.); (b) details of each team of investigators.

1,240,000(100%) — *Search "failure analysis,"*
81,900(6.6%) — *add "finite element method,"*
14,400(1.2%) — *add "uncertainty,"*
3910(0.3%) — *add "stochastic,"*
1510(0.1%) — *add "structural engineering,"*
31(0.002%) — *add "metrology."*

Four application papers [5–8] are significant in the sense that they represent the current state of the art of how engineers manage uncertainty with the newly found power of laptop, parallel, and network computing.

The purpose of this paper is two-fold. First, we present cases studies to argue that FEM is still inadequate to assist engineers in managing uncertainty. We then propose a metrology-based approach such that both types of uncertainty mentioned above can be addressed. In Sect. 2, we briefly review the impact of FEM in structural engineering practice. In Sect. 3, we trace the steady

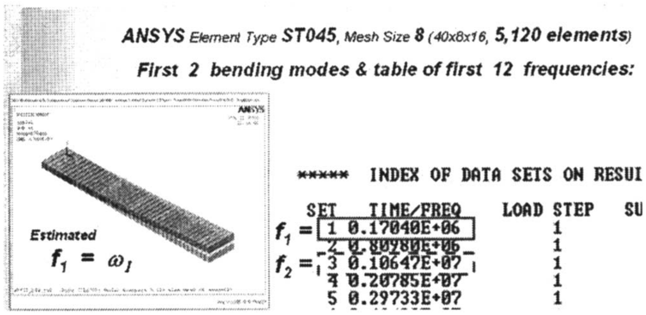


Fig. 4 A typical result of the free vibration solution of the reference benchmark problem, RB-101, using ANSYS [83] with 5120 solid elements (ST45). This run is part of a 96-run design.

progress made in three disciplines, namely, (a) stochastic ordinary and partial differential equations, (b) probabilistic structural mechanics, and (c) a stochastic finite element method, and we conclude that tools to address the uncertainty of FEM and FA are still not yet available.

In Sect. 4, we propose a metrology-based approach to resolve this dilemma. Some typical results from two example problems, RB-101 and RB-301, to illustrate this approach, appear in Sects. 5 and 6. A summary of the main results and some concluding remarks are given in Sect. 7. A list of references is given in Sect. 8.

2 Impact of Finite Element Method (FEM)

The finite element method owes its development to seven sets of "parents:" three from mathematics (M), three from structural engineering (SE), and one from computer technology (CT). Beginning with Gauss in 1795 [9], through Galerkin in 1915 [10], and until Biezeno and Koch in 1923 [11], the method of weighted residuals (Parent-**M1**) was combined with variational methods (Parent-**M2** due to Rayleigh in 1870 [12] and Ritz in 1909 [13]) to give engineers a viable numerical method without computers. Then came Courant in 1943 [14], Prager and Synge in 1947 [15], and Zienkiewicz and Cheung in 1964 [16] to clinch the mathematical side of FEM through the use of piecewise continuous trial functions (Parent-**M3**).

On the structural engineering side, Hrenikoff in 1941 [17], McHenry in 1943 [18], and Newmark in 1949 [19], innovated out of necessity, independently from the work of Courant, Prager, and Synge, to give us the so-called "structural analogue substitution" method (Parent-**SE-1**), which soon led to the development of "direct continuum elements" (Parent-**SE-2**) by Argyris in 1955 [20] and Turner, Clough, and Martin in 1956 [21]. Computer technology was then at its infancy, and code convergence was the major barrier. Beginning in 1966, Bruce Irons [22–24] led the development of the powerful "patch test" (Parent-**SE-3**) that became, under fairly broad restrictions, the necessary and sufficient condition for convergence.

Commercially available FEM began to appear in the 1970s. As computer technology (Parent-CT-1) improved over the last three decades, so did the power and versatility of FEM. Today, its impact in structural engineering is so pervasive that no engineering design or analysis is acceptable without FEM. Its value to the engineering community is comparable to the development of an x ray in the medical and dental professions, where practically no one is willing to consult a dentist if he or she does not offer the patient an x-ray record for diagnosis. There are many good textbooks or references on FEM, and we consider the ones by Zienkiewicz and Taylor [25] and Hughes [26] as among the best. As a tool for engineers, FEM today is unsurpassed in its power and ease of use.

3 Shortcomings of FEM and Its Recent Progress

As a tool for design simulations, FEM is intrinsically used to deliver a so-called "point estimate" of the solution of a "deterministic" model. Three categories of variability in any such problem may be considered: Category V-1 refers to initial and/or boundary data associated with a fixed set of governing equations. Category V-2 refers to material properties that change the governing equa-

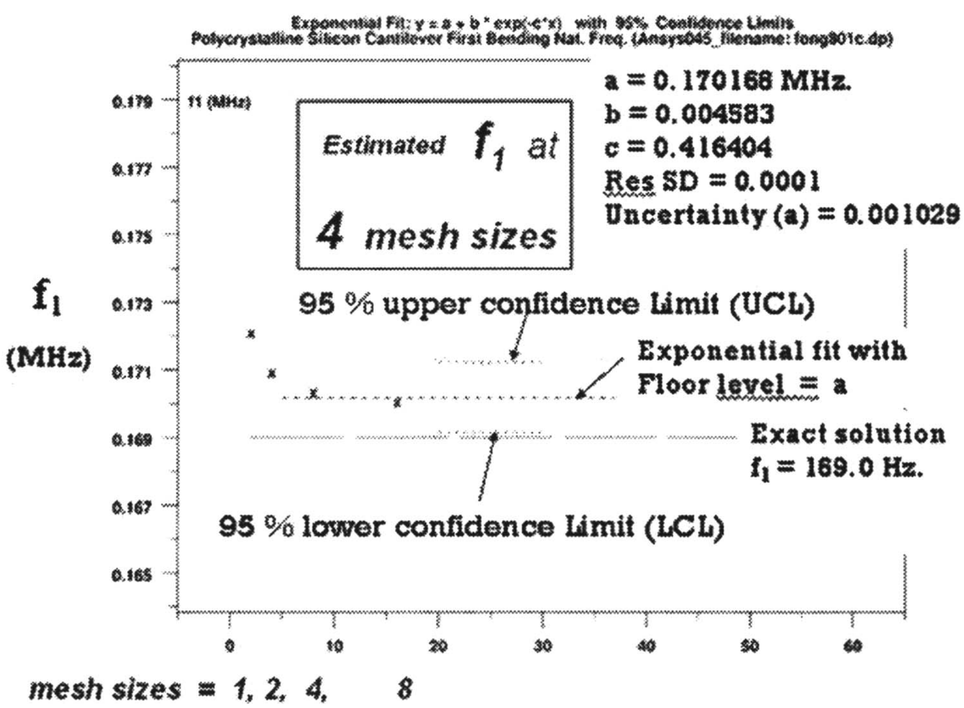


Fig. 5 A typical result of estimating uncertainty of a natural frequency of RB-101 using a three-parameter exponential fit of a grid convergence run. Note the comparison with an exact solution.

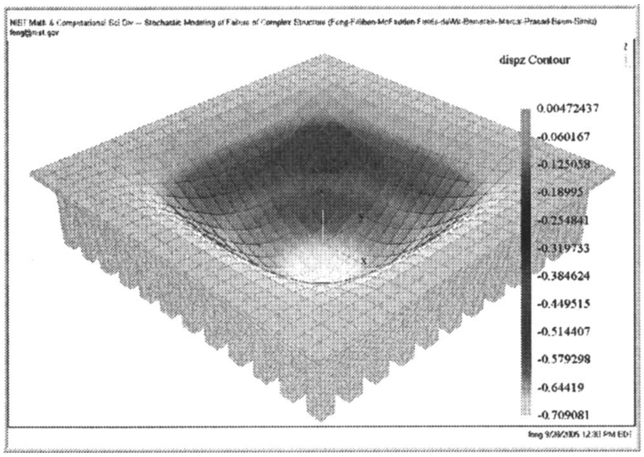


Fig. 6 A typical result of a nonlinear (large deformation) analysis of RB-301 using MPAVE [87]

tions. Category V-3 is common in failure analysis, where both the governing equations and the geometry may change. For example, in a progressive failure scenario, a weak element of a structure has a localized failure and is separated from the main structure. This causes a load redistribution due to a change of geometry, and a modification of the governing equations because the global material property distribution has changed after the removal of a weak element.

To address all three categories of variability in FEM and FA, as mentioned above, let us examine recent progress in three disciplines, namely, (a) stochastic ordinary and partial differential equations (SDE), (b) probabilistic structural mechanics (PSM), and (c) stochastic finite element method (SFEM). First of all, SDE traces its beginning from the 1938 paper of Wiener [27], and the 1976 series of three papers on diffusion problem by Becus and Cozzarelli [28–30]. Two books, one on stochastic wave propagation by Sobczyk in 1985 [31] and the second on stochastic elastic and viscoelastic systems by Potapov in 1999 [32], rounded up the three types of partial differential equations, namely, parabolic, hyperbolic, and elliptic. Using a powerful technique known as the Wiener-Askey polynomial chaos, Xiu and his colleagues [33–36] published a series of papers in 2002–2003 to address stochastic problems in fluids and heat transfer. In 1999, Platen [37] published a monumental paper on numerical methods for SDE that listed 349 references. So far, however, none of this impressive mathematics research has been applied to structures.

Progress in the second discipline, PSM, is more promising. It began in 1939 with Weibull [38] on a statistical theory of breaking strength, and 1946 by Freudenthal [39] on fatigue. The statistical concepts of structural materials advanced by Freudenthal in 1950 [40] led to a 1969 probability-based structure code by Cornell [41] and a structural risk analysis and reliability-based design methodology in the 1970s by Ang [42] and Ang and Tang [43]. Also in the 1970s, a statistical theory of fatigue using Weibull statistics at the microscopic, specimen, and component levels was developed by Fong and his colleagues [44–46]. PSM became a well-established discipline in the 1980s [47] and was the subject of a handbook in 1995 by Sundararajan [48]. However, as succinctly stated by Hess, Bruchman, and Ayyub [49], "*preceding the development of any reliability-based design procedure, relevant variables must be identified and their statistical characteristics need to be defined.*" Without data, Thacker et al. [[50], p. 17] stated the following:

"*Since data were not available to characterize the model inputs sufficiently, it was decided to model all random variables with a coefficient of variation (cov) of ten percent.…default distributions were assigned based on experience, i.e., a lognormal distribution was used to model modulus variables and a normal distribution was used otherwise.*"

Rahman [[51], p. 115] admitted in his Table 1 of statistical properties of random input that he assumed the elastic modulus and the farfield tensile stress of a DENT specimen to be normally distributed with an arbitrary c.o.v. of 5% and 10%, respectively. Without a proper statistical database of material properties, PSM simulations are challengeable in a court of law.

Since SDE and PSM are both ineffective, we naturally turn our attention to the third discipline, namely, the stochastic finite element method (SFEM), and we were not disappointed. Beginning in 1975 with a paper by Cambou [52], we saw steady progress in

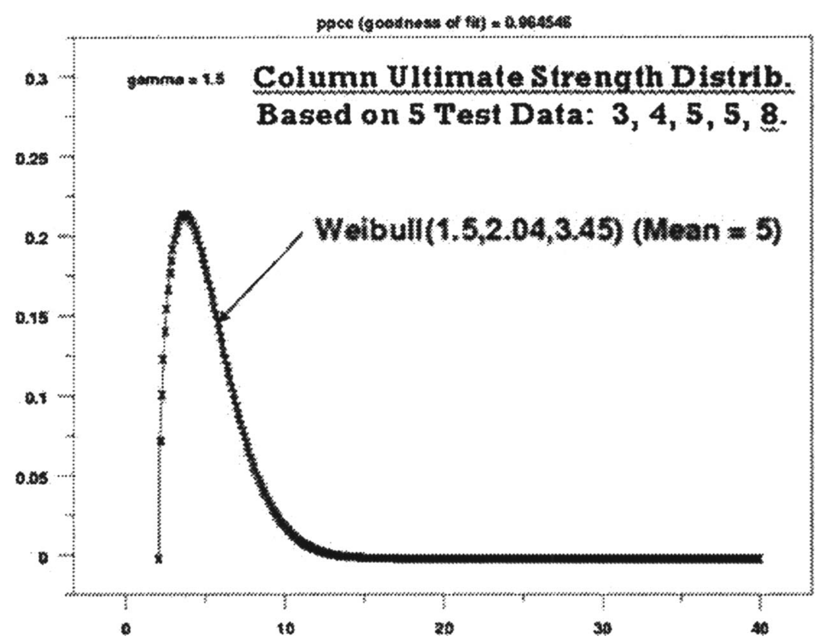


Fig. 7 Plot of a three-parameter Weibull distribution of the column ultimate strength based on 5 test data (3, 4, 5, 5, and 8). Note in small print the goodness of fit equal to 0.964 546 [90].

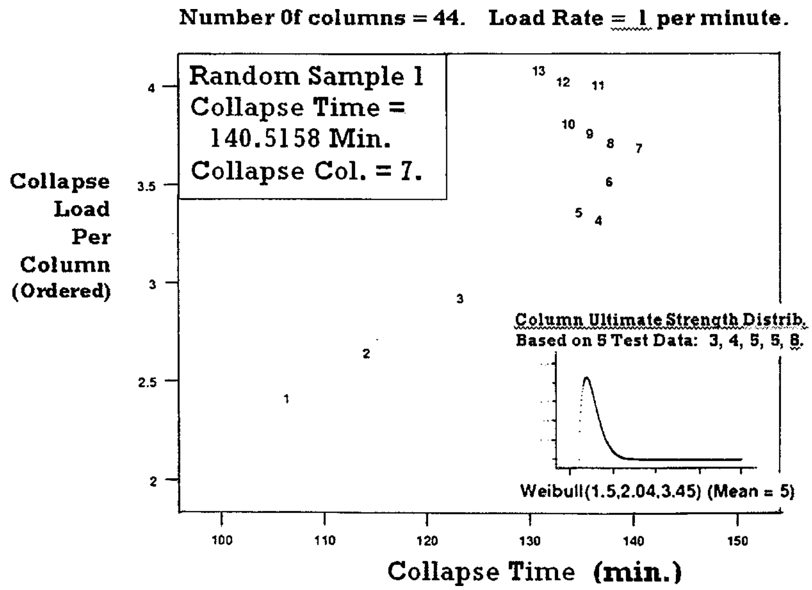


Fig. 8 A typical sample of a progressive failure analysis of the collapse of a 44-column grillage loaded at a constant rate with a 5-test ultimate strength distribution given in Fig. 7 [90]

the 1980s [see, e.g., [53–61]] and the 1990s [see, e.g., [62–67]], ending with a landmark paper entitled "3D SFEM for elasto-plastic bodies" by Anders and Hori in 2001 [68]. Two more contributions in 2004, one by Babuska, Tempone, and Zouraris [69] and the other by Hlavacek, Chleboun, and Babuska [70] supposedly gave us the rigorous foundation to rush into SFEM and implement it in the same way as we tackled FEM in the 1960s. However, this promising tool is still not the answer, as shown in a concluding remark given by Anders and Hori in their remarkable paper [[68], pp. 474–475]:

"*The comprehensive predictions of the proposed SFEM provide design tools in terms of bounding response analysis. However, due to the unpredictability of three-dimensional bifurcation phenomena in random media, to ensure meaningful applicability of the method to a particular problem, the authors advise a preliminary coarse-mesh Monte-Carlo analysis to assess the probabilistic content of potential modes of failure.*"

Thus, we are back to square one, where for most applications involving multiparameter calculations, a Monte Carlo analysis is not a viable option to handle uncertainty within a reasonable time and funding constraint. An example to motivate a new approach to FEM and FA uncertainty is given in the next section.

4 A Metrology-Based Approach to FEM and FA

Since 1991, five international conferences on the numerical simulation of three-dimensional (3-D) aluminum and steel sheet-forming processes (NUMISHEET) have taken place. At the 2002

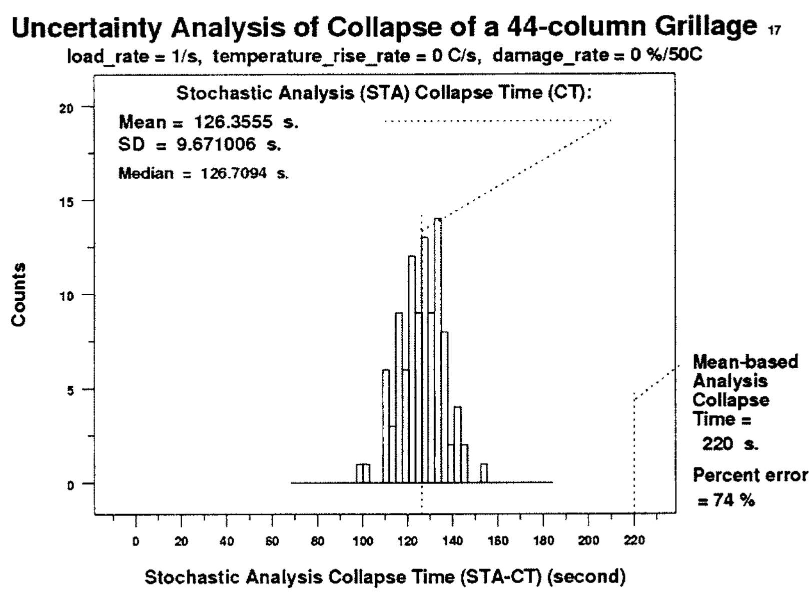


Fig. 9 A typical time-to-collapse distribution based on 100 simulations of progressive failure of a 44-column grillage with one of the sample simulations given in Fig. 8. Note the difference between the simulation mean and the deterministic mean.

conference held in Korea, the results of three benchmark test problems involving 12 experiments and 41 simulations by investigators from 7 countries [71] were reported. Typical results of one of the three test problems are given in Figs. 2 and 3. None of the simulations was reported with an expression of uncertainty, and the goal of validating the numerical simulations of all three test problems was thus unattainable in the sense of Oberkampf, Trucano, and Hirsch [72].

To remedy this situation, we observe that an ISO guide [73] to and a NIST note [74] on the expression of uncertainty in measurement have existed since 1993–1994. We also note that a Bayesian approach to combining results from interlaboratory experiments appeared in 2002 [75]. Instead of resorting to large-scale simulation techniques such as Monte Carlo, we adopted an experimental design technique known as the orthogonal fractional factorial design (see, e.g., [[76], pp. 374–433], [[77], pp. 359–363]), and a basic assumption that any computer-generated simulation is, in fact, a "numerical experiment," such that the ISO guide [73] applies to guide us in estimating the uncertainty of FEM and FA. We named this approach "metrology based" because we implemented it on a suite of benchmarks of known solutions as reported in [78].

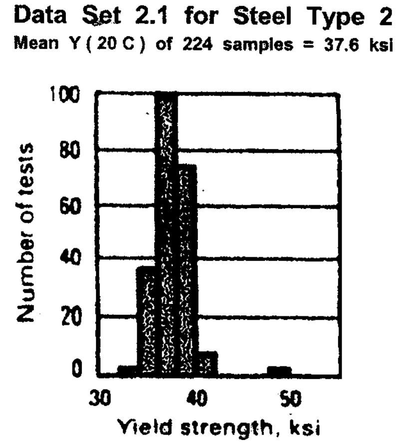


Fig. 10 A histogram of the yield strength of 224 heats of ASTM 285 Grade C steel comparable to ASTM A36) from the purchase record of a single fabricator [88]. This set of data was fitted by Fong et al. [89] with a three-parameter Weibull (γ=1.8).

5 Example RB-101—A Thin Elastic Cantilever Beam

Motivated by a 1998 FEM workshop course notes [79] and the known exact solutions in the literature [80,81], we implemented our metrology-based approach of verification and validation (MV&V) on a very simple problem, namely,

RB-101	The static deformation due to an end load and the free vibration of a thin, isotropic, and linearly elastic cantilever beam.

We planned a numerical experiment of a fractional factorial design [76,77] involving three commercially available FEM codes [82–84], each of which is required to run 4 mesh sizes, 3 element types ($3\times4=12$) with 6 two-level factors ($2^{6-3}=8$) for a total of 96 runs per code. To achieve uniformity in mesh generation and problem specification, we used TrueGrid [85] to generate input files of all three FEM codes. Typical results from one of those runs are given in Figs. 4 and 5.

6 Example RB-301—A Thin Elastic Square Plate

Our next example, RB-301, was motivated by the classical problem of the bending of a thin, isotropic, and linearly elastic plate, of which exact solutions for both linear (small strains) and nonlinear (large deformation) formulations under uniformly distributed loads are known [86]. To accentuate the nonlinear aspect of the problem, we added a fourth FEM code named MPAVE, which was recently developed by Marcal [87] as a follow-up of his earlier version named MARK. A typical result of nonlinear analysis by MPAVE is given in Fig. 6.

Motivated by two studies on the distributional properties of yield and ultimate strengths of selected steels [88,89], we modified the geometry of RB-301 to that of a thin square steel plate resting on a grillage of 44 concrete columns with a distribution of ultimate compressive strengths based on five tests. Using a NIST-developed statistical analysis software [90], we show in Figs. 7–9 some typical results of a time-to-collapse analysis and Figs. 10 and 11 an unsymmetrical plate failure pattern even when the loading and boundary conditions are symmetric.

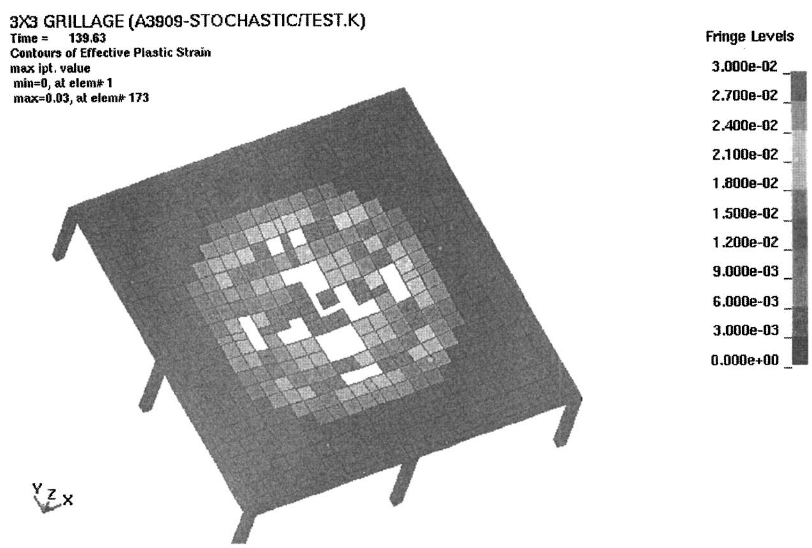


Fig. 11 A typical unsymmetrical pattern of failure of a thin square plate loaded at its center with a point load when the material is given a failure strain distribution of the Weibull type

7 Summary of Results and Concluding Remarks

The metrology-based approach to a quantitative expression of uncertainty in finite element modeling and failure analysis as introduced in this paper is based on five specific ideas, namely, (1) standard reference benchmarks such as RB-101 and RB-301 mentioned in the previous sections, (2) orthogonal fractional factorial design of numerical experiments as discussed in [76,77], (3) the ISO guides [73,74] to the expression of uncertainty in experiments, (4) a three-parameter exponential-fit of at least four grid-convergence run results, and (5) the Bayesian approach to combining results from multiple methods [75].

To illustrate the feasibility of this new approach, we show a few examples in this expository paper as motivations to answer the following two generic questions on engineering uncertainty:

(Q-1) Given the results of two or more FEM solutions, how do we express uncertainty for each solution and the combined?

(Q-2) Given a complex structure with a small number of tests on material properties, how do we simulate a failure scenario and predict time to collapse with confidence bounds?

We show in Sects. 5 and 6 that our approach is easy to implement, and the results are physically meaningful. In particular, the 74% error shown in one example run (Fig. 9) for the difference between the result of a deterministic mean-based model and the mean of the stochastic model serves as a reminder that engineering design with a deterministic model may be challenged in court as being "unsafe," as it overestimates the time to the onset of failure of complex structures. Further implementation of this approach and new research based on incomplete-data analysis [91,92] are being conducted and will be reported as the results become available.

Acknowledgment

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