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Integral representations for elliptic functions

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Abstract

We derive new integral representations for constituents of the classical theory of elliptic functions: the Eisenstein series, and Weierstrass' $\wp$ and ζ functions. The derivations proceed from the Laplace–Mellin representation of multipoles, and an elementary lemma on the summation of 2D geometric series. In addition, we present results concerning the analytic continuation of the Eisenstein series to an entire function in the complex plane, and the value of the conditionally convergent series, denoted by $\widetilde{E}_2$ below, as a function of summation over increasingly large rectangles with arbitrary fixed aspect ratio.[1]
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1. Introduction

In this paper we revisit the classical theory of elliptic functions as developed by Eisenstein and Weierstrass. Both of these researchers represented the meromorphic functions appearing in their theories as summations over a given lattice of elementary pole functions of a prescribed order. Our fundamental observation is that pole functions may be repre-

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sented by exponentially-damped, oscillatory integrals. These representations depend on the complex half-planes in which the singularities lie, and are natural variants of the classical Mellin, or Laplace–Mellin, formulas which are valid for isolated poles lying in the right half plane. More recently, such integral representations have resurfaced in the development of fast multipole methods where they are referred to as "plane-wave" representations [4,7]. A key feature of these representations is that the pole centers appear in the exponents of the integrands. As a consequence the lattice summations are transformed into geometric series which may be summed explicitly underneath the integral. The result is a new class of integral representations for the Eisenstein series and other meromorphic functions of Weierstrass' theory.

A brief summary of the paper follows. In the first section we review the definitions of the Eisenstein series E_n and the Weierstrass functions $\wp$ and ζ. We will analyze a generalization of Eisenstein's series which we denote by $\widetilde{E}_s$, the differences being: first, we consider $s = \sigma + \mathrm{i}t \in \mathbb{C}$, and second, we define $\widetilde{E}_s$ as a limit over lattice squares of increasing size, a significant point when $\Re(s) \leqslant 2$ and the sums are not absolutely convergent. In addition, in this preliminary section we provide elementary derivations of the requisite plane-wave formulas for general pole functions of the form $f(\omega) = \omega^{-s}$, and a summation identity for a two-dimensional geometric series.

In the next section we derive an integral representation for $\widetilde{E}_s$ for the case $\Re(s) > 2$. Integral representations for Eisenstein's E_n naturally follow for $s = n \geqslant 3$. We interpret these formulas as the natural lattice analogues to the well-known representation for Riemann's zeta function (denoted with the subscript ζ_R so as to distinguish it from Weierstrass' function of the same name)

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \lambda^{s-1} \frac{1}{1 - \mathrm{e}^{-\lambda}} \mathrm{e}^{-\lambda}\, \mathrm{d}\lambda, \quad \Re(s) > 1. \tag{1}$$

For example, in the case of a square lattice we derive the following integral expression

$$\begin{aligned} E_k(\mathrm{i}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m + n\mathrm{i})^k} \\ &= \frac{8}{(k-1)!} \int_0^{\infty} \lambda^{k-1} \frac{\cos^2(\lambda/2)}{1 - 2\mathrm{e}^{-\lambda} \cos(\lambda) + \mathrm{e}^{-2\lambda}} \mathrm{e}^{-\lambda}\, \mathrm{d}\lambda, \end{aligned} \tag{2}$$

for k divisible by 4, $E_k(\mathrm{i}) = 0$ otherwise. The similarity between (1) and (2) is clear. For more general lattices, we replace i by τ, $k \in \mathbb{N}$ by $s \in \mathbb{C}$, and the single trigonometric ratio in (2) by a sum of analogous ratios denoted by $f_1(\tau, \lambda)$ and $f_2(\tau, \lambda)$ defined in (15) and (17). The general expression is given in Theorem 5.

Subsequently, we derive an alternative representation for $\widetilde{E}_s$ as a contour integral from which we deduce that the sums $\widetilde{E}_s$ admit an analytic continuation as an *entire* function in the complex plane. As a corollary, we prove the existence of a finite limit for $\widetilde{E}_2$. We discuss $\widetilde{E}_2$ and its relation to Eisenstein's, E_2. As the limiting processes defining these two conditionally convergent series are distinct, so too are the limiting values. More generally, we derive a closed form expression relating $\widetilde{E}_2$ to a sum over a rectangular box of fixed aspect ratio. In addition to providing the connection to E_2, this result is a generalization of similar formulas appearing, for example, in [6,13].

In the following section we derive analogous integral formulas for Weierstrass' $\wp$ and ζ functions. We conclude the paper with a brief discussion of these integral representations in relation to previous research in the theories of lattice sums, and elliptic functions.

We note that a subset of the results presented below appeared previously in a slightly different form [8].

2. Preliminaries

We review the definitions of the Eisenstein series and the Weierstrass $\wp$ and ζ functions. Furthermore, we derive elementary lemmas concerning plane-wave representations and a geometric series identity, both of which we will use repeatedly in the subsequent sections.

2.1. The Eisenstein series and elliptic functions

We are given a general lattice $\Lambda \subset \mathbb{C}$ defined by $\Lambda = \{m \cdot \mu + n \cdot \nu \mid m, n \in \mathbb{Z}\}$ where the generators μ, ν are complex numbers such that the lattice ratio, $\tau = \nu/\mu$, is not real. We define the classical Eisenstein series (see, for example, [14] and [11])

$$E_n = \lim_{N\to\infty} \sum_{n=-N}^{N} \lim_{M\to\infty} \sum_{m=-M}^{M} \frac{1}{(m \cdot \mu + n \cdot \nu)^n}, \quad n \geqslant 1; \tag{3}$$

the elimination of the term $m = n = 0$ is implicit here and below. From elementary estimates one finds that the series (3) are absolutely convergent for $n \geqslant 3$, and are absolutely divergent for $n = 1$ or 2. The later implies that the limiting operation specified in (3) plays a non-trivial role in the definition of these two sums. Eisenstein proved that the procedure (3) yields finite values of E_n for these cases. As $E_1 = 0$ trivially, from the point of view of convergence, the only interesting sum is E_2.

Eisenstein was cognizant of this and he derived many identities which connect his summation process for E_2 to others [14]. We choose yet a different summation convention and define $\widetilde{E}_s$ as the limit of partial sums over "lattice-squares" of increasing size. We generalize further in considering complex exponents. Specifically, we define $\widetilde{E}_s$ by

$$\widetilde{E}_s = \lim_{K\to\infty} \sum_{|m|,|n|\leqslant K} \frac{1}{(m \cdot \mu + n \cdot \nu)^s}, \tag{4}$$

which we consider, initially, for $\Re(s) > 2$. For non-integer s we situate the branch of the function ζ^s along the "negative diagonal" of the lattice, $\{z = -t(\mu + \nu) \mid t > 0\}$. For all z in the closure of this cut plane we have

$$\theta \leqslant \arg(z) \leqslant \theta + 2\pi, \quad \text{where } \theta = \operatorname{Arg}(-\mu - \nu). \tag{5}$$

(For the principal branch we fix $|\operatorname{Arg}(z)| < \pi$.) We further enforce the convention that points of the lattice lying along the diagonal are considered symmetrically,

$$\frac{1}{(-m\mu - m\nu)^s} = \frac{1}{2(m|\mu+\nu|)^s}\left(\frac{1}{\mathrm{e}^{\mathrm{i}\theta s}} + \frac{1}{\mathrm{e}^{\mathrm{i}(\theta+2\pi)s}}\right).$$

We return to this point later.

We will derive integral representations for $\widetilde{E}_s$. Restricting s to the positive integers, our formula yields an integral representation for the classical Eisenstein series $\widetilde{E}_n = E_n$, for $n \geqslant 3$. As for the conditionally convergent series, it is straightforward to verify that $\widetilde{E}_1 = 0$ directly from (4). For $s = 2$ that the limit (4) exists follows as a consequence of the integral representations for $\widetilde{E}_s$, $\Re(s) > 2$. In addition, we derive a formula which connects our limiting value to Eisenstein's. Even more, we prove that $\widetilde{E}_s$ admits an analytic continuation to $s \in \mathbb{C}$ as an entire function. For an alternative treatment of extending the sense of the sums (3) see, for example, [10].

Some fifteen years after Eisenstein, in 1862 Weierstrass commenced his study of doubly-periodic functions. Following Weierstrass, we define the usual $\wp$ function

$$\wp(x, \Lambda) = \frac{1}{x^2} + \sum_{\omega \in \Lambda\setminus\{0\}} \left(\frac{1}{(x-\omega)^2} - \frac{1}{\omega^2}\right). \tag{6}$$

In addition, Weierstrass defined his ζ function as an indefinite integral of $\wp$ and developed the following summation representation:

$$\zeta(x, \Lambda) = -\int^x \wp(s, \Lambda)\,\mathrm{d}s = \frac{1}{x} + \sum_{\omega \in \Lambda\setminus\{0\}} \left(\frac{1}{(x-\omega)} + \frac{1}{\omega} + \frac{x}{\omega^2}\right). \tag{7}$$

The absolutely convergent sums (6) and (7) will serve as the starting points for the derivation of the integral representations for $\wp$ and ζ below.

We conclude with a comment on the choice of generators for Λ. Note that $\widetilde{E}_s$, $\wp$, ζ satisfy simple rescalings with respect to μ,

$$\begin{cases} \widetilde{E}_s(\mu, \nu) = \frac{1}{\mu^s}\widetilde{E}_s(1, \tau), \\ \wp(x \mid \mu, \nu) = \frac{1}{\mu^2}\wp\left(\frac{x}{\mu} \mid 1, \tau\right), \\ \zeta(x \mid \mu, \nu) = \frac{1}{\mu}\zeta\left(\frac{x}{\mu} \mid 1, \tau\right), \end{cases} \tag{8}$$

where $\tau = \nu/\mu$ is the lattice ratio. It is known that up to rescaling and unimodular substitution, any lattice ratio may be represented by a unique τ chosen from the following fundamental region [1]:

$$\begin{cases} -\frac{1}{2} < \Re(\tau) \leqslant \frac{1}{2}, \\ \Im(\tau) > 0, \\ |\tau| \geqslant 1, \\ \text{if } |\tau| = 1, \text{ then } \Re(\tau) \geqslant 0. \end{cases} \tag{9}$$

In summary, without loss of generality, we restrict our analysis to the "inhomogeneous" functions, which are obtained from (4), (6) and (7) by fixing $\mu = 1$, $\nu = \tau$, and consider $\Lambda = \Lambda(\tau)$ with τ satisfying (9). For convenience we omit the variables μ, ν below and write, for example, $\wp = \wp(z, \tau)$.

2.2. Plane-wave representations and a 2D geometric series

To facilitate our derivations we define the truncated lattice $\Lambda^K = \{\omega_{m,n} = m + n\tau \mid |m|, |n| < K\} \setminus \{0\}$. We further group lattice points into four overlapping "quadrants"

$$\Lambda^K = \Lambda^K_{(+,\bullet)} \cup \Lambda^K_{(\bullet,+)} \cup \Lambda^K_{(-,\bullet)} \cup \Lambda^K_{(\bullet,-)}$$

defined by

$$\begin{aligned}
\Lambda^K_{(+,\bullet)} &= \{\omega_{m,n} \mid 1 \leqslant m \leqslant K,\ |n| \leqslant m\},\\
\Lambda^K_{(\bullet,+)} &= \{\omega_{m,n} \mid 1 \leqslant n \leqslant K,\ |m| \leqslant n\},\\
\Lambda^K_{(-,\bullet)} &= \{\omega_{m,n} \mid -K \leqslant m \leqslant -1,\ |n| \leqslant -m\},\\
\Lambda^K_{(\bullet,-)} &= \{\omega_{m,n} \mid -K \leqslant n \leqslant -1,\ |m| \leqslant -n\}.
\end{aligned} \tag{10}$$

We recall from the discussion following (4) that for non-integer s the shared boundary between $\Lambda^K_{(-,\bullet)}$ and $\Lambda^K_{(\bullet,-)}$ is identical to the branch cut (see Fig. 1).

We have the following elementary lemma.

Lemma 1. *Assume a complex lattice $\Lambda(\tau)$ and the quadrants defined as in* (10). *An isolated singularity of complex order s, $\Re(s) > 0$ with branch cut defined as in* (5) *may be represented by the following plane-wave integrals, each of which is valid in the appropriate quadrant determined by the location of the point ω*:

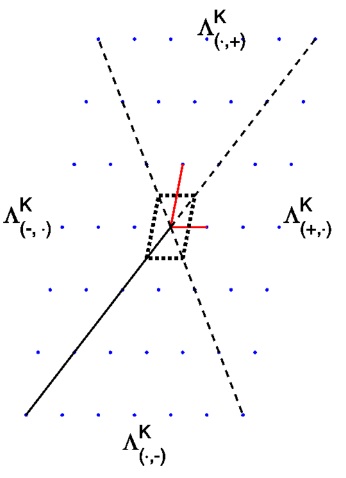

Fig. 1. Partition of Λ^K into subregions. The generators $(1, \tau)$ are shown in red. The central dotted region is the boundary of the fundamental domain. The dashed lines show the divisions into $\Lambda^K_{(\pm,\pm)}$. The solid black line is the branch cut.

$$\frac{1}{\omega^s} = \begin{cases} \frac{1}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \mathrm{e}^{-\lambda\omega}\, \mathrm{d}\lambda, & \omega \in \Lambda^K_{(+,\bullet)}, \\ \frac{\mathrm{e}^{-\mathrm{i}\pi s}}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \mathrm{e}^{\lambda\omega}\, \mathrm{d}\lambda, & \omega \in \Lambda^K_{(-,\bullet)}, \\ \frac{\mathrm{e}^{-\mathrm{i}\pi s/2}}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \mathrm{e}^{\mathrm{i}\lambda\omega}\, \mathrm{d}\lambda, & \omega \in \Lambda^K_{(\bullet,+)}, \\ \frac{\mathrm{e}^{\mathrm{i}\pi s/2}}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \mathrm{e}^{-\mathrm{i}\lambda\omega}\, \mathrm{d}\lambda, & \omega \in \Lambda^K_{(\bullet,-)}. \end{cases} \tag{11}$$

Proof. We have the representation of the Γ function:

$$\Gamma(s) = \int_0^\infty \lambda^{s-1} \mathrm{e}^{-\lambda}\, \mathrm{d}\lambda, \quad \Re(s) > 0.$$

Assume $\omega = \mathrm{i}t$, $t > 0$. As τ satisfies (9), we observe that $-\pi < \theta < -\pi/2$ hence $\arg(\omega^s) = \pi s/2$. With this in mind, rescale the integration variable by t, factor the -1 in the exponential, and multiply and divide by $\exp(\mathrm{i}\pi s/2)$ to obtain

$$\Gamma(s) = t^s \int_0^\infty \lambda^{s-1} \mathrm{e}^{-\lambda t}\, \mathrm{d}\lambda = \mathrm{e}^{-\mathrm{i}\pi s/2} \omega^s \int_0^\infty \lambda^{s-1} \mathrm{e}^{\mathrm{i}\lambda\omega}\, \mathrm{d}\lambda.$$

Dividing both sides by $\Gamma(s)\omega^s$ gives the desired result for $\omega = \mathrm{i}t$. In a similar manner we prove the formula for ω lying on any of the principal coordinate rays emanating from the origin, $\omega \in \pm\mathbb{R}^+, \pm\mathrm{i}\mathbb{R}^+$. The full expressions (11) then follow by analytic continuation into the appropriate quadrants. $\square$

Remark 2. Note that for integer s, the integral expressions may be continued further and are valid in the appropriate half-planes $\pm\Re(\omega) > 0$ and $\pm\Im(\omega) > 0$.

Next we turn to our summation convention (4). From Lemma 1, it is apparent that no single plane-wave expansion formula will be valid for all terms in the summands (4), (6), and (7); terms must be grouped with respect to quadrant. As with the convention of splitting contributions from points $\omega_{-m,-m}$ lying on the cut in (4) equally between branches, we wish to treat each quadrant as symmetrically as possible. We define the symbol ε_{mn} for $m, n \in \mathbb{Z}$ by

$$\varepsilon_{mn} = \begin{cases} \frac{1}{2}, & m = \pm n, \\ 1, & \text{otherwise}. \end{cases} \tag{12}$$

By convention, we sum over the terms in the $\Lambda^K_{(+,\bullet)}$-quadrant as

$$\sum_{\omega \in \Lambda^K_{(+,\bullet)}} f(\omega) = \sum_{m=1}^{K} \sum_{n=-m}^{m} \varepsilon_{mn} f(\omega_{m,n}). \tag{13}$$

The sum over the lattice square Λ^K is the sum of the quadrant sums as in (13); hence the reason for the factor of $1/2$—to avoid double-counting of the contributions from the diagonal terms—is clear. We note that for $s \in \mathbb{N}$, $s > 2$ the numerical values of the sums are independent of any manner of grouping terms. Even so, the forms of the integrands in our integral representations reflect this choice. We have found that the convention (13) yields the most symmetric expressions in appearance (a different grouping for integer s was employed in [8]).

In sums of the form (13) we will substitute the appropriate plane-wave expansion (11) to represent the poles contained in f. This transforms the quadrant sums into geometric series. Concerning the later, we derive the following lemma.

Lemma 3. *For any $p, q \in \mathbb{C}$ and $K \in \mathbb{N}$, the following is an identity*:

$$\sum_{i=1}^{K} \sum_{j=-i}^{i} \varepsilon_{ij} p^i q^j = \frac{1}{2} \frac{p(q^{-1}+2+q)}{(1-p(q+q^{-1})+p^2)} - \frac{1}{2}\left(\frac{1+q}{1-q}\right)\left[\frac{(pq^{-1})^{K+1}}{1-pq^{-1}} - \frac{(pq)^{K+1}}{1-pq}\right]. \tag{14}$$

Proof. The formula follows from iteration of the usual single variable geometric sum, and algebra. □

We record the following corollary for reference.

Corollary 4. *We have the following specializations of Lemma* 3:

$$\sum_{m=1}^{K} \sum_{n=-m}^{m} \varepsilon_{mn} \left(\mathrm{e}^{-\lambda}\right)^m \left(\mathrm{e}^{-\lambda\tau}\right)^n = 2\mathrm{e}^{-\lambda} f_1(\tau,\lambda) - 2\mathrm{e}^{-\lambda(K+1)} f_1^{(K)}(\tau,\lambda)$$

and

$$\sum_{n=1}^{K} \sum_{m=-n}^{n} \varepsilon_{nm} \left(\mathrm{e}^{\mathrm{i}\tau\lambda}\right)^n \left(\mathrm{e}^{\mathrm{i}\lambda}\right)^m = 2\mathrm{e}^{\mathrm{i}\tau\lambda} f_2(\tau,\lambda) - 2\mathrm{e}^{\mathrm{i}\tau\lambda(K+1)} f_2^{(K)}(\tau,\lambda),$$

where the functions f_1, $f_1^{(K)}$, f_2, $f_2^{(K)}$ are:

$$f_1(\tau,\lambda) = \frac{\cosh^2(\tau\lambda/2)}{1-2\mathrm{e}^{-\lambda}\cosh(\tau\lambda)+\mathrm{e}^{-2\lambda}}, \tag{15}$$

$$f_1^{(K)}(\tau,\lambda) = \frac{1}{4}\left(\frac{1+\mathrm{e}^{-\lambda\tau}}{1-\mathrm{e}^{-\lambda\tau}}\right)\left[\frac{\mathrm{e}^{\lambda\tau(K+1)}}{1-\mathrm{e}^{-\lambda(1-\tau)}} - \frac{\mathrm{e}^{-\lambda\tau(K+1)}}{1-\mathrm{e}^{-\lambda(1+\tau)}}\right], \tag{16}$$

$$f_2(\tau,\lambda) = \frac{\cos^2(\lambda/2)}{1-2\mathrm{e}^{\mathrm{i}\tau\lambda}\cos(\lambda)+\mathrm{e}^{2\mathrm{i}\tau\lambda}}, \tag{17}$$

$$f_2^{(K)}(\tau,\lambda) = \frac{1}{4}\left(\frac{1+\mathrm{e}^{\mathrm{i}\lambda}}{1-\mathrm{e}^{\mathrm{i}\lambda}}\right)\left[\frac{\mathrm{e}^{-\mathrm{i}\lambda(K+1)}}{1-\mathrm{e}^{\mathrm{i}\lambda(\tau-1)}} - \frac{\mathrm{e}^{\mathrm{i}\lambda(K+1)}}{1-\mathrm{e}^{\mathrm{i}\lambda(\tau+1)}}\right]. \tag{18}$$

We assume τ is in the fundamental region (9) and make several observations. Concerning real singularities, all of the functions given by (15)–(18) have double poles at the origin, $\lambda = 0$. Since $\Im(\tau) > 0$, neither $f_1(\tau,\lambda)$ nor $f_2(\tau,\lambda)$ have other poles for $\lambda > 0$. For τ strictly imaginary, the denominator $(1-\exp(-\lambda\tau))$ of $f_1^{(K)}(\tau,\lambda)$ will have isolated simple zeros. However, these are balanced by simple zeros of the difference of bracketed terms in (16). Thus $f_1^{(K)}$ has no other singularities for $\lambda > 0$. A similar argument shows that $f_2^{(K)}$ is also finite for $\lambda > 0$. With regards to decay, one has the bounds

$$\left|e^{-\lambda} f_1(\tau,\lambda)\right| < C_1 e^{-\lambda(1-|\Re(\tau)|)} \quad \text{and} \quad \left|e^{i\tau\lambda} f_2(\tau,\lambda)\right| < C_2 e^{i\tau\lambda}$$

for large λ. As $|\Re(\tau)| \leqslant 1/2$ and $\Im(\tau) > 0$, both quantities are exponentially decreasing in λ. Similar reasoning shows that $e^{-\lambda(K+1)} f_1^{(K)}(\tau,\lambda)$ and $e^{i\tau\lambda(K+1)} f_2^{(K)}(\tau,\lambda)$ are exponentially decreasing in λ *and* K.

3. Eisenstein series

As mentioned previously, the summation $\widetilde{E}_s$ for $\Re(s) > 2$ is absolutely convergent. We begin by proving our first integral representation for this case in Theorem 5. As a corollary, the restriction $s = n$, $n \geqslant 3$, gives integral representations for E_n. Further inspection of the integral representation demonstrates the existence of $\widetilde{E}_2$. Elaborating on Theorem 5, we derive an alternative representation for $\widetilde{E}_s$ as a contour integral. As a consequence of this second representation, we prove that $\widetilde{E}_s$ admits an analytic continuation in s as an entire function. Returning to the analysis of $\widetilde{E}_2$, we consider a more general limiting procedure and define $\widetilde{E}_2^{(\alpha)}$ as the limit over increasing "lattice rectangles" with a fixed aspect ratio defined by α. We write $\widetilde{E}_2^{(\alpha)}(\tau) = \widetilde{E}_2(\tau) + \Delta(\alpha,\tau)$ and derive a closed form expression for Δ. As a corollary, we derive the relationship between $\widetilde{E}_2$ and the sum E_2 as defined by Eisenstein.

3.1. Integral representations

For the sums $\widetilde{E}_s$ defined by (4) we prove

Theorem 5. *Given a lattice $\Lambda(\tau)$ with ratio τ chosen from the fundamental region* (9)*, we have the following integral representation for $\widetilde{E}_s$, $\Re(s) > 2$:*

$$\widetilde{E}_s(\tau) = \cos\left(\frac{\pi}{2}s\right)\frac{4}{\Gamma(s)}\int_0^\infty \lambda^{s-1}\left(e^{-is\pi/2}e^{-\lambda} f_1(\tau,\lambda) + e^{i\tau\lambda} f_2(\tau,\lambda)\right)d\lambda, \tag{19}$$

where $f_1(\tau,\lambda)$ and $f_2(\tau,\lambda)$ are given by (15) *and* (17).

Proof. Due to the placement of branch cut (5) and the summation conventions (13), we have the following relations between sums over $\Lambda^K_{(\pm,\bullet)}$ and $\Lambda^K_{(\bullet,\pm)}$:

$$\sum_{\omega\in\Lambda^K_{(-,\bullet)}} \frac{1}{\omega^s} = e^{-is\pi} \sum_{\omega\in\Lambda^K_{(+,\bullet)}} \frac{1}{\omega^s}, \qquad \sum_{\omega\in\Lambda^K_{(\bullet,-)}} \frac{1}{\omega^s} = e^{is\pi} \sum_{\omega\in\Lambda^K_{(\bullet,+)}} \frac{1}{\omega^s}.$$

Therefore, we consider the positive quadrants only and scale the results by an exponential factor. Turning to the quadrant $\Lambda^K_{(+,\bullet)}$, in place of the isolated singularity of degree s, we substitute the appropriate plane-wave expression from (11) to obtain

$$\begin{aligned}
\sum_{\omega\in\Lambda^K_{(\pm,\bullet)}} \frac{1}{(m+n\tau)^s} &= \left(1+\mathrm{e}^{-\mathrm{i}s\pi}\right) \sum_{m=1}^{K} \sum_{n=-m}^{m} \varepsilon_{mn} \frac{1}{(m+n\tau)^s} \\
&= \frac{(1+\mathrm{e}^{-\mathrm{i}s\pi})}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \sum_{m=1}^{K} \sum_{n=-m}^{m} \varepsilon_{mn} \mathrm{e}^{-\lambda(m+n\tau)}\,\mathrm{d}\lambda \\
&= \frac{2(1+\mathrm{e}^{-\mathrm{i}s\pi})}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \left(\mathrm{e}^{-\lambda} f_1(\tau,\lambda) - \mathrm{e}^{-\lambda(K+1)} f_1^{(K)}(\tau,\lambda)\right) \mathrm{d}\lambda,
\end{aligned}$$

where the last line follows from Corollary 4. From the statements following this same corollary, we observe that the two integrands are singular at $\lambda = 0$, and are otherwise finite and exponentially decreasing in K and $\lambda \in \mathbb{R}^+$. In addition, as $\Re(s) > 2$, the singularity at the origin is absolutely integrable. Therefore, one may take the large K limit inside the integral and compute

$$\lim_{K\to\infty} \sum_{\omega\in\Lambda^K_{(\pm,\bullet)}} \frac{1}{(m+n\tau)^s} = \cos\left(\frac{\pi}{2}s\right) \frac{4}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \mathrm{e}^{-\mathrm{i}s\pi/2} \mathrm{e}^{-\lambda} f_1(\tau,\lambda)\,\mathrm{d}\lambda.$$

By a similar analysis, we prove that

$$\lim_{K\to\infty} \sum_{\omega\in\Lambda^K_{(\bullet,\pm)}} \frac{1}{(m+n\tau)^s} = \cos\left(\frac{\pi}{2}s\right) \frac{4}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \mathrm{e}^{\mathrm{i}\tau\lambda} f_2(\tau,\lambda)\,\mathrm{d}\lambda.$$

Adding these two contributions gives the theorem. □

By inspection of (19), we see that the absolute convergence of the Eisenstein series $\widetilde{E}_s$, $\Re(s) > 2$ manifests itself in the behavior of the integrand of (19) near the origin; the factor λ^{s-1} balances the double poles of f_1 and f_2 so as to ensure the product is integrable at $\lambda = 0$. More careful analysis reveals that the formula (19) is finite even for the conditionally convergent case $\widetilde{E}_2$. The Laurent expansions of the integrands about the origin are

$$\begin{aligned}
\lim_{\lambda\to 0} \lambda \mathrm{e}^{-\lambda} \frac{\cosh^2(\tau\lambda/2)}{1-2\mathrm{e}^{-\lambda}\cosh(\tau\lambda)+\mathrm{e}^{-2\lambda}} &= \frac{1}{\lambda(1-\tau^2)} - \frac{(1-2\tau^2)\lambda}{12(1-\tau^2)} + O\left(\lambda^3\right), \\
\lim_{\lambda\to 0} \lambda \mathrm{e}^{\mathrm{i}\lambda\tau} \frac{\cos^2(\tau\lambda/2)}{1-2\mathrm{e}^{\mathrm{i}\tau\lambda}\cos(\tau\lambda)+\mathrm{e}^{\mathrm{i}2\tau\lambda}} &= \frac{1}{\lambda(1-\tau^2)} - \frac{(2-\tau^2)\lambda}{12(1-\tau^2)} + O\left(\lambda^3\right).
\end{aligned}$$

At $s = 2$ the expansions are subtracted hence the integrand of (19) is finite at the origin even in this case. By a similar analysis, one may show that the K-dependent terms also cancel at the origin. We have proved:

Corollary 6. *The summation* (4) *converges in the conditionally convergent case $s = 2$, and its value, $\widetilde{E}_2$, is given by the integral* (19).

In fact, a great deal more may be said. The function

$$F(s,z)=\mathrm{e}^{-\mathrm{i}s\pi/2}\mathrm{e}^{-z}\frac{\cosh^2(\tau z/2)}{1-2\mathrm{e}^{-z}\cosh(\tau z)+\mathrm{e}^{-2z}}+\mathrm{e}^{\mathrm{i}\tau z}\frac{\cos^2(z/2)}{1-2\mathrm{e}^{\mathrm{i}\tau z}\cos(z)+\mathrm{e}^{2\mathrm{i}\tau z}} \tag{20}$$

appearing as a factor in the integrand (19) has a singularity at the origin $z=0$, and additional simple poles in the complex plane at the points

$$z\in P=\left\{\pm\frac{2\pi\mathrm{i}}{1\pm\tau}m,\ \pm\frac{2\pi}{1\pm\tau}n\ \middle|\ m,n\in\mathbb{N}\right\}.$$

We denote the minimum magnitude of all $z\in P$ by ρ. Next, define the contour C which begins at $\infty+\mathrm{i}y$, $y>0$; runs parallel to real axis until it intersects the circle centered at the origin with radius r, where $y<r<\rho$; follows this circle counterclockwise around the origin; and runs back out to $\infty-\mathrm{i}y$, parallel to the real axis. We assume that $y>0$ is small enough such that C encloses only the pole at $z=0$. Finally, for the function z^{s-1}, $s\notin\mathbb{Z}$, situate the branch cut along the positive real axis such that ($\lambda\in\mathbb{R}$)

$$\begin{cases}\lim_{y\to 0}(\lambda+\mathrm{i}y)^{s-1}=\lambda^{s-1},\\ \lim_{y\to 0}(\lambda-\mathrm{i}y)^{s-1}=\mathrm{e}^{2\pi\mathrm{i}(s-1)}\lambda^{s-1}.\end{cases} \tag{21}$$

With these preliminaries established we prove the following theorem.

Theorem 7. *Given a lattice $\Lambda(\tau)$ with ratio τ chosen from the fundamental region* (9)*, we have the following contour integral representation for $\widetilde{E}_s$, $\Re(s)>2$:*

$$\widetilde{E}_s=2\cos\left(\frac{\pi}{2}s\right)\frac{\Gamma(1-s)\mathrm{e}^{-\mathrm{i}s\pi}}{\mathrm{i}\pi}\int_C z^{s-1}F(s,z)\,\mathrm{d}z, \tag{22}$$

where $F(s,z)$ is given by (20).

Proof. Consider the contour integral

$$\int_C z^{s-1}F(s,z)\,\mathrm{d}z.$$

As the integrand is analytic except for the singularity at the origin, we apply contour deformation to shrink the radius of the circle, $r\to 0$, and take the limit as $y\to 0$ for the two components running parallel to the real axis. For $\Re(s)=2+\varepsilon$, we estimate the contribution from the circular arc

$$\lim_{r\to 0}\left|\int_{|z|=r} z^{s-1}F(s,z)\,\mathrm{d}z\right|\leqslant M\lim_{r\to 0}\int_{|z|=r} r^{\varepsilon-1}\,|\mathrm{d}z|=0.$$

Turning to the components parallel to the real axis, from the definition of the branch cut (21), we compute

$$\lim_{y\to 0}\int_{\infty+\mathrm{i}y}^{\varepsilon_r+\mathrm{i}y} z^{s-1}F(s,z)\,\mathrm{d}z=-\int_0^\infty \lambda^{s-1}F(s,\lambda)\,\mathrm{d}\lambda,$$

$$\lim_{y\to 0} \int_{\varepsilon_r - \mathrm{i}y}^{\infty - \mathrm{i}y} z^{s-1} F(s,z)\,\mathrm{d}z = \mathrm{e}^{2\pi\mathrm{i}(s-1)} \int_0^\infty \lambda^{s-1} F(s,\lambda)\,\mathrm{d}\lambda.$$

We recognize the integrals of Theorem 5, make use of the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

and obtain

$$\begin{aligned}\int_C z^{s-1} F(s,z)\,\mathrm{d}z &= \left(\mathrm{e}^{2\pi\mathrm{i}(s-1)} - 1\right) \int_0^\infty \lambda^{s-1} F(s,\lambda)\,\mathrm{d}\lambda = \frac{2\mathrm{i}\mathrm{e}^{\pi\mathrm{i}s}\sin(\pi s)\Gamma(s)}{4\cos(\frac{\pi}{2}s)} \widetilde{E}_s \\ &= \frac{\mathrm{i}\pi}{2\cos(\frac{\pi}{2}s)\mathrm{e}^{-\mathrm{i}s\pi}\Gamma(1-s)} \widetilde{E}_s.\end{aligned}$$

The result (22) follows from algebra. □

Several corollaries follow from Theorem 7. We note here only the most immediate

Corollary 8. *The sums $\widetilde{E}_s$, defined by* (4) *for* $\Re(s) > 2$, *admit an analytic continuation to* $s \in \mathbb{C}$ *as an entire function. This continuation is given by the contour integral representation* (22).

Proof. The contour integral appearing in (22) defines an analytic function of s which is never singular. The same may be said for the cosine factor. Thus the only candidate singularities arise from the factor $\Gamma(1-s)$ which has simple poles for $s \in \mathbb{N}$. From the definition (4) we know that $\widetilde{E}_s$ is finite for $\Re(s) > 2$ (the apparent singularities in (22) in this case are balanced by zeros of the cosine term, the contour integral, or both). Therefore, we need only verify the finite existence of $\widetilde{E}_1$ and $\widetilde{E}_2$. For $s = n = 1$, the pole in the Gamma function is balanced by the simple zero of the cosine factor. Finally, we have argued above that $\widetilde{E}_2$ is finite. □

By inspection of (22), we find that $\widetilde{E}_s$ has simple zeros for $s = 1 - 2j$, $j \geqslant 1$. Thus the continuation respects the symmetric limiting process (4). Preliminary residue computations suggest that $\widetilde{E}_s$ is also zero for $s = -2j$ although we have not carried out this investigation at the time of this writing. For reassurance on this point, however, see [10]. We anticipate further results concerning the evaluation of the contour integral (22) via residue methods and will report on this at a later date. Finally, as mentioned previously, the exact form of the integrands (19) and (22) reflect our summation convention with respect to grouping of summands and placement of the branch-cut. Regarding the latter, similar formulas arise if the branch-cut is situated along any of the lattice diagonals; the effect is to redistribute factors of $\exp(\mathrm{i}\pi s/2)$ between the two functions f_1 and f_2. There are, perhaps, additional treatments of the branch-cut that could yield relatively simple expressions. However, a simple integral expression valid for placement of the cut along an arbitrary ray in the complex plane appears to be intractable.

3.2. Aspect ratio correction

In this section we make an explicit connection between $\widetilde{E}_2$ and Eisenstein's definition of the series.

Given $\alpha \in (0,\infty)$, we define the summation over lattice rectangles with aspect ratio α by

$$\widetilde{E}_2^{(\alpha)} = \lim_{K\to\infty} \sum_{|m|\leqslant\lfloor\alpha K\rfloor} \sum_{|n|\leqslant K} \frac{1}{(m+n\tau)^2}, \tag{23}$$

where $\lfloor x\rfloor$ denotes greatest integer less than or equal to x. Again, although expected, the existence of the limit (23) is not a priori guaranteed but will follow in the course of our analysis.

Clearly, $\widetilde{E}_2^{(1)} = \widetilde{E}_2$ defined in (4). More generally, we write

$$\widetilde{E}_2^{(\alpha)} = \widetilde{E}_2 + \Delta(\alpha,\tau), \tag{24}$$

where the value of $\widetilde{E}_2$ may be computed via the integral expression (19). Proceeding as above we compute a closed form expression for $\Delta(\alpha,\tau)$.

Theorem 9. *For a fixed aspect ratio $\alpha \in (0,\infty)$, the limit $\widetilde{E}_2^{(\alpha)}$ specified in* (23) *exists. Furthermore, when written in the form* (24)*, the α dependence is given by*

$$\Delta(\alpha,\tau) = -\frac{4\mathrm{i}}{\tau}\left(\arctan(\mathrm{i}\tau) - \arctan\left(\frac{\mathrm{i}\tau}{\alpha}\right)\right). \tag{25}$$

Proof. Assume $\alpha \geqslant 1$. We write the limit (23)

$$\widetilde{E}_2^{(\alpha)} = \widetilde{E}_2 + \Delta(\alpha,\tau),$$
$$\Delta(\alpha,\tau) = 2\lim_{K\to\infty} \sum_{m=K+1}^{\lfloor\alpha K\rfloor} \sum_{n=-K}^{n=K} \frac{1}{(m+n\tau)^2}.$$

The contribution from the sum over lattice points $-\lfloor\alpha\cdot K\rfloor \leqslant m \leqslant -K-1$ is accounted for by the factor of two multiplying the sum in the final line. We represent the poles using the $\Lambda^K_{(+,\bullet)}$ plane-wave expansion (11):

$$\begin{aligned}
\Delta(\alpha,\tau) &= 2\lim_{K\to\infty} \sum_{m=K+1}^{\lfloor\alpha K\rfloor} \sum_{n=-K}^{n=K} \int_0^\infty \lambda \mathrm{e}^{-\lambda(m+n\tau)}\,\mathrm{d}\lambda \\
&= 2\lim_{K\to\infty} \int_0^\infty \lambda\left(\frac{\mathrm{e}^{\lambda\tau K} - \mathrm{e}^{-\lambda\tau(K+1)}}{1-\mathrm{e}^{-\lambda\tau}}\right)\left(\frac{\mathrm{e}^{-\lambda(K+1)} - \mathrm{e}^{-\lambda\lfloor\alpha K\rfloor}}{1-\mathrm{e}^{-\lambda}}\right)\mathrm{d}\lambda \\
&= 2\lim_{K\to\infty} \int_0^\infty \frac{\lambda}{K^2}\left(\frac{\mathrm{e}^{\lambda\tau} - \mathrm{e}^{-\lambda\tau(1+1/K)}}{1-\mathrm{e}^{-\lambda\tau/K}}\right)\left(\frac{\mathrm{e}^{-\lambda(1+1/K)} - \mathrm{e}^{-\lambda\lfloor\alpha K\rfloor/K}}{1-\mathrm{e}^{-\lambda/K}}\right)\mathrm{d}\lambda
\end{aligned}$$

$$= \frac{2}{\tau} \int_0^\infty \frac{(e^{\lambda\tau} - e^{-\lambda\tau})(e^{-\lambda} - e^{-\lambda\alpha})}{\lambda} \, d\lambda, \tag{26}$$

where the argument which justifies taking the large K limit inside the integral runs along the same lines as in the proof of Theorem 5. This last integral (26) may be evaluated in closed form using the formula

$$\int_0^\infty e^{-\beta x} \sin(\delta x) \frac{1}{x} \, dx = \arctan\left(\frac{\delta}{\beta}\right), \tag{27}$$

which holds for $\Re(\beta) > |\Im(\delta)|$ (see [5, 3.944.5]). Taking care to write (26) as the difference of two integrals of the form (27), and performing algebra gives the expression for $\Delta(\alpha, \tau)$, $\alpha \geqslant 1$ in (25).

For $\alpha < 1$ the lattice rectangle is such that the longer side is in the τ-direction. As written, Eqs. (23) and (24) suggest that this rectangle is inscribed in a lattice square of size K, and to compute $\Delta(\alpha, \tau)$, one should subtract the extra contributions exterior to the rectangle but interior to the square. The problem with this approach is that, for arbitrary τ and α, it is burdensome to keep track of the quadrants in which these points lie. In lieu of this, for $\alpha < 1$ we rescale the limits in (23),

$$\widetilde{E}_2^{(\alpha)} = \lim_{K \to \infty} \sum_{|m| \leqslant K} \sum_{|n| \leqslant \lfloor K/\alpha \rfloor} \frac{1}{(m + n\tau)^2}.$$

Informally, this is equivalent to inscribing the square in the rectangle and motivates computing the contributions from the points in the difference using the plane-wave formulas appropriate for $\Lambda^K_{(\bullet, \pm)}$. Arguing as above and using the integral identity (27), we compute

$$\begin{aligned} \Delta(\alpha, \tau) &= -2 \lim_{K \to \infty} \sum_{n=K+1}^{\lfloor K/\alpha \rfloor} \sum_{m=-K}^{m=K} \int_0^\infty \lambda e^{i\lambda(m+n\tau)} \, d\lambda \\ &= \frac{2}{\tau} \int_0^\infty \frac{(e^{-i\lambda} - e^{i\lambda})(e^{i\lambda\tau} - e^{i\lambda\tau/\alpha})}{\lambda} \, d\lambda \\ &= -\frac{4i}{\tau} \left(\arctan\left(-\frac{1}{i\tau} \right) - \arctan\left(-\frac{\alpha}{i\tau} \right) \right). \end{aligned} \tag{28}$$

Although perhaps not obvious, the formula (28) is the same as the formula for $\Delta(\alpha, \tau)$ in (25). Using standard trig identities, we have

$$\lim_{z' \to z} \left(\arctan(z) - \arctan\left(-\frac{1}{z'} \right) \right) = \lim_{z' \to z} \arctan\left(\frac{zz' + 1}{z' - z} \right) = \pm \frac{\pi}{2}. \tag{29}$$

Furthermore, for $z = i\tau$ or $z = i\tau/\alpha$ with τ satisfying (9) and $\alpha > 0$, we find that we should choose the minus sign in (29). Using this identity and algebra, we observe that (28) is equal to (25), thus (25) holds for all $\alpha > 0$. $\square$

We use this theorem to find the connection between our summation and Eisenstein's. Starting from Eisenstein's summation convention we obtain

$$\begin{aligned}
E_2 &= \lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M}\frac{1}{(m+n\tau)^2}\\
&= \lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{\alpha\to\infty}\sum_{m=-\lfloor\alpha\cdot N\rfloor}^{m=-\lfloor\alpha\cdot N\rfloor}\frac{1}{(m+n\tau)^2}\\
&= \lim_{\alpha\to\infty}\lim_{N\to\infty}\sum_{n=-N}^{N}\sum_{m=-\lfloor\alpha\cdot N\rfloor}^{m=\lfloor\alpha\cdot N\rfloor}\frac{1}{(m+n\tau)^2}\\
&= \widetilde{E}_2-\frac{4\mathrm{i}}{\tau}\arctan(\mathrm{i}\tau).
\end{aligned}\tag{30}$$

Standard estimates justify commuting the N and α limits between the second and third lines, and we used (25) to compute this limit. As $\tau\neq 0$, (30) shows that, for finite τ, the value of our sum is always different from Eisenstein's. In the limit $\tau\to\mathrm{i}\infty$, $|\Re(\tau)|\leqslant 1/2$, both summations are equal and presumably converge to $2\zeta_R(2)=\pi^2/3$.

In a similar vein, we compute the difference between taking Eisenstein's limit and "its reverse." Arguing as above, we have that

$$\begin{aligned}
\lim_{M\to\infty}\sum_{m=-M}^{M}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{(m+n\tau)^2} &= \widetilde{E}_2+\lim_{\alpha\to 0}\Delta(\alpha,\tau)\\
&= \widetilde{E}_2-\frac{4\mathrm{i}}{\tau}\left(\arctan\left(\frac{\mathrm{i}}{\tau}\right)+\frac{\pi}{2}\right).
\end{aligned}\tag{31}$$

Taking the difference between (30) and (31) and, we obtain

$$\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M}-\lim_{M\to\infty}\sum_{m=-M}^{M}\lim_{N\to\infty}\sum_{n=-N}^{N}\right)\frac{1}{(m+n\tau)^2}=\frac{2\pi\mathrm{i}}{\tau}.$$

For a different proof of this fact see Walker [13].

Finally, in the case of the square lattice ($\tau=\mathrm{i}$) we observe that $f_1(\mathrm{i},\lambda)=f_2(\mathrm{i},\lambda)$. Collecting factors in (19) and performing algebra, we find that

$$\widetilde{E}_s(\mathrm{i})=\mathrm{e}^{-\mathrm{i}s\pi/4}\cos\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{4}s\right)\frac{8}{\Gamma(s)}\int_0^\infty\lambda^{s-1}\mathrm{e}^{-\lambda}f_1(\mathrm{i},\lambda)\,\mathrm{d}\lambda.$$

Corresponding to the added $\pi/2$-symmetry of the square lattice, the product of cosines causes the sum to vanish for n not a multiple of four. In particular $\widetilde{E}_2(\mathrm{i})=0$. Substituting this value into (24), and taking the large α (small α) limits, we obtain

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M}\frac{1}{(m+\mathrm{i}n)^2}=\pi,$$

$$\lim_{M\to\infty}\sum_{m=-M}^{M}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{(m+\mathrm{i}n)^2}=-\pi,$$

well-known identities in the fast multipole community (see, for example, [6]).

4. Weierstrass elliptic functions

Our derivations of integral formulas for Weierstrass' elliptic functions proceed in much the same manner as above. As a preliminary note, the integral representations for $\wp(z,\tau)$ and $\zeta(z,\tau)$ which we derive in Theorem 10 are not valid for all $z\in\mathbb{C}$, but rather have a finite domain of validity. This is a consequence of the way in which we group terms. More precisely, we require that $z\in D(\tau)$ defined by

$$D(\tau)=\bigl\{z \mid \Re(-1\pm z\pm\tau)<0,\ \Im(\tau\pm z)>0\bigr\}. \tag{32}$$

As τ is in the region (9), one may verify that $D(\tau)$ is an open set containing the origin. However, $D(\tau)$ may not contain the fundamental period parallelogram of the lattice, $\Lambda_0=\{\alpha+\beta\tau \mid |\alpha|\leqslant 1/2,\ |\beta|\leqslant 1/2\}$. For example, the standard hexagonal lattice has generators $(1,\tau)=(1,1/2+\mathrm{i}\sqrt{3}/2)$. Thus a corner of Λ_0 is given by the point $z_0=1/2+\tau/2=3/4+\mathrm{i}\sqrt{3}/4$. However, $\Re(-1+z_0+\tau)=1/4>0$, violating the first inequality in (32).

With this aside, we prove the following.

Theorem 10. *Assume $\Lambda=\Lambda(\mu,\nu)$ is an arbitrary complex lattice with generators chosen such that $\tau=\nu/\mu$ is in the fundamental region* (9)*, and that the complex number z is in the domain $D(\tau)$ defined by the inequalities* (32)*. We have the following integral expressions for the inhomogeneous elliptic functions $\wp(z,\tau)$, and $\zeta(z,\tau)$:*

$$\wp(z,\tau)=\frac{1}{z^2}+8\int_0^\infty\lambda\left[\mathrm{e}^{-\lambda}\sinh^2\left(\frac{z\lambda}{2}\right)f_1(\lambda,\tau)+\mathrm{e}^{\mathrm{i}\tau\lambda}\sin^2\left(\frac{z\lambda}{2}\right)f_2(\lambda,\tau)\right]\mathrm{d}\lambda \tag{33}$$

and

$$\zeta(z,\tau)=\frac{1}{z}+\int_0^\infty\bigl[\mathrm{e}^{-\lambda}\bigl(z\lambda-\sinh(z\lambda)\bigr)f_1(\lambda,\tau)-\mathrm{e}^{\mathrm{i}\tau\lambda}\bigl(z\lambda-\sin(z\lambda)\bigr)f_2(\lambda,\tau)\bigr]\,\mathrm{d}\lambda, \tag{34}$$

where the functions f_1, f_2 are defined by (15) *and* (17)*. We evaluate the homogeneous functions, $\wp(x\mid\mu,\nu)$ and $\zeta(x\mid\mu,\nu)$, via the appropriate scaling relations* (8) *and* (33), (34) *with the proviso that $x/\mu=z\in D(\tau)$.*

Proof. In computing the integral representation for the $\wp$ function we will group terms of the sum (6) as in (13). As in the computation of the Eisenstein sums, we wish to combine the contributions from the sums over $\Lambda^K_{(\pm,\bullet)}$. We compute

$$\sum_{\omega\in\Lambda^K_{(-,\bullet)}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)=\sum_{m=1}^{K}\sum_{n=-m}^{m}\varepsilon_{mn}\left(\frac{1}{(z+m+n\tau)^2}-\frac{1}{(m+n\tau)^2}\right),$$

$$\sum_{\omega\in\Lambda^K_{(+,\bullet)}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)=\sum_{m=1}^{K}\sum_{n=-m}^{m}\varepsilon_{mn}\left(\frac{1}{(-z+m+n\tau)^2}-\frac{1}{(m+n\tau)^2}\right).$$

Therefore, the contributions from both quadrants may be expressed as a single sum over $\Lambda^K_{(+,\bullet)}$ of a modified summand. Furthermore, under the assumption $z\in D$, all of the poles in this sum may be expressed using the $\Re(\omega)>0$ plane-wave expansion from (11):

$$\begin{aligned}
&\sum_{\omega\in\Lambda^K_{(\pm,\bullet)}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)\\
&\quad=\sum_{m=1}^{K}\sum_{n=-m}^{m}\varepsilon_{mn}\left(\frac{1}{(-z+m+n\tau)^2}-\frac{2}{(m+n\tau)^2}+\frac{1}{(z+m+n\tau)^2}\right)\\
&\quad=\int_0^\infty\lambda\left(\mathrm{e}^{\lambda z}-2+\mathrm{e}^{-\lambda z}\right)\left(\sum_{m=1}^{K}\sum_{n=-m}^{m}\varepsilon_{mn}\mathrm{e}^{-\lambda(m+n\tau)}\right)\mathrm{d}\lambda\\
&\quad=8\int_0^\infty\lambda\sinh^2\left(\frac{z\lambda}{2}\right)\left(\mathrm{e}^{-\lambda}f_1(\tau,\lambda)-\mathrm{e}^{-\lambda(K+1)}f_1^{(K)}(\tau,\lambda)\right)\mathrm{d}\lambda.
\end{aligned}\tag{35}$$

Arguing as before, we find the large K limit of the K-dependent term to be zero. We compute the contribution from the terms in the quadrants $\Lambda^K_{(\bullet,\pm)}$ in an analogous manner. Adding this result to (35) gives (33).

The derivation of the expression for the ζ function is similar. In brief, the sum over the quadrants $\Lambda^K_{(\pm,\bullet)}$ may again be expressed as a sum over the single quadrant $\Lambda^K_{(+,\bullet)}$ in which we substitute the appropriate plane-wave expansion. Thus,

$$\begin{aligned}
&\sum_{\Lambda^K_{(\pm,\bullet)}}\left(\frac{1}{(z-\omega)}+\frac{1}{\omega}+\frac{z}{\omega^2}\right)\\
&\quad=\sum_{m=1}^{K}\sum_{n=-m}^{m}\varepsilon_{mn}\left(-\frac{1}{-z+m+n\tau}+\frac{2z}{(m+n\tau)^2}+\frac{1}{z+m+n\tau}\right)\\
&\quad=\int_0^\infty\left(\mathrm{e}^{\lambda z}+2z\lambda-\mathrm{e}^{-\lambda z}\right)\left(\sum_{m=1}^{K}\sum_{n=-m}^{m}\varepsilon_{mn}\mathrm{e}^{-\lambda(m+n\tau)}\right)\mathrm{d}\lambda\\
&\quad=4\int_0^\infty\left(z\lambda-\sinh(z\lambda)\right)\left(\mathrm{e}^{-\lambda}f_1(\lambda,\tau)-\mathrm{e}^{-\lambda(K+1)}f_1^{(K)}(\lambda,\tau)\right)\mathrm{d}\lambda.
\end{aligned}\tag{36}$$

As before the K-dependent term goes to zero in the limit. The analogous sums over $\Lambda^K_{(\bullet,\pm)}$ give the other half of the expression (34). □

Remark 11. As an alternative to the above derivation of Theorem 10, we recall that the Eisenstein series appear as coefficients in the Laurent expansion for Weierstrass' $\wp$ function,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)\\ &= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \sum_{n=3}^{\infty}(n-1)z^{n-2}\frac{1}{\omega^n}\\ &= \frac{1}{z^2} + \sum_{n=3}^{\infty}(n-1)E_n z^{n-2}.\end{aligned}$$

Substituting the integral representations (19) for the E_n, the Taylor series may be summed explicitly inside the integrand. The formula (33) above follows after algebraic simplification. Furthermore, the expression (34) for the ζ function follows from anti-differentiation of (33).

Remark 12. As with the series $\widetilde{E}_2$, and its dependence on aspect ratios derived in Theorem 9, the slowly decaying terms of the sums defining $\wp$ and ζ manifest themselves at the origin in the integral representations (Eqs. (6), (7), and (33), (34), respectively). In the integral representations, we observe that Weierstrass' "correction" terms are arranged in such a way as to create third order zeros at $\lambda = 0$, which appropriately balance the second order poles from f_1 and f_2.

5. Conclusion

We conclude with a brief discussion of our results in relation to previous research in this field. To the best of our knowledge, there is no analog to the integral expressions for the $\wp$ and ζ functions (33) and (34). The possibility of developing numerical routines for evaluation of these functions based on these representations deserves further study. We observe that the integrands are not extremely oscillatory, and decay exponentially. Thus N-point Gauss–Laguerre quadrature rules will converge rapidly in N. As one drawback, there is the perhaps awkward domain of validity in z. However, it may be that symmetries of the $\wp$ and ζ functions imply that it is sufficient to evaluate them over domains that are smaller than the fundamental period parallelogram. Furthermore, at least for the $\wp$ function, there exists the following closed-form Fourier expansion [12]:

$$\begin{aligned}\wp(z,\tau) = -2\left(\frac{1}{6} + \sum_{n=1}^{\infty}\frac{1}{\sin^2(n\pi\tau)}\right)\\ + \frac{\pi^2}{\sin^2(\pi z)} - 8\pi^2\left(\sum_{n=1}^{\infty}\frac{n\mathrm{e}^{2\pi\mathrm{i}\tau n}}{1-\mathrm{e}^{2\pi\mathrm{i}\tau n}}\cos(2\pi nz)\right).\end{aligned} \tag{37}$$

Both summands in (37) are exponentially decreasing and the sums converge rapidly—stiff competition from a numerical perspective. Nevertheless, we have not fully explored the relative merits of this approach over the plane-wave representation (33). In addition, the integral representations may have further analytic implications.

Turning to the representations for the Eisenstein series, the existence of $\widetilde{E}_2$ (Corollary 6) is not unexpected. In addition to the original finiteness proofs given by Eisenstein, many years prior to this present work, Walker derived the remarkable formula for the conditionally convergent series (see [12])

$$\lim_{K\to\infty}\sum_{0<m^2+n^2\leqslant K^2}\frac{1}{(m+n\tau)^2}=\frac{-2\pi}{1-\mathrm{i}\tau}-4\pi\mathrm{i}\frac{\eta'(\tau)}{\eta(\tau)},$$

where the Dedekind $\eta(\tau)$-function with $\Im(\tau)>0$ is defined by

$$\eta(\tau)=\mathrm{e}^{\pi\mathrm{i}\tau/12}\prod_{n=1}^{\infty}\bigl(1-\mathrm{e}^{2\pi\mathrm{i}\tau n}\bigr).$$

We also note that a different treatment, initiated by Hecke, has become a standard approach to resolving convergence and transformation properties of E_2 [11].

As indicated by Theorem 7, our expressions are quite general, and have broad implications. Riemann demonstrated both the functional equation satisfied by $\zeta_R(s)$, and the evaluation of $\zeta_R(-2n+1)$ (and, via the functional equation, $\zeta_R(2n)$) in terms of Bernoulli numbers using the "version" of Theorem 7 appropriate for his zeta function. Similarly, we anticipate that a residue argument will give the evaluation of $\widetilde{E}_n=E_n$ in terms of multiple Bernoulli numbers, see [2] and [9] for related results pertaining to multiple zeta-functions. For an alternative treatment of Eisenstein series for negative even integers using Hecke convergence factors see the recent work of Pribitkin [10]. The functional equation satisfied by the continuation of $\widetilde{E}_s$ is more elusive. We are currently pursuing this and hope to report our results in the future.

Finally, there is a possibility that representations of the form (19) may exist for certain Dirichlet series

$$G(s,\chi)=\sum_{\omega\in\Lambda\setminus\{0\}}\frac{\chi(\omega)}{|\omega|^{2s}},\qquad \chi(\omega)=\exp\bigl(\mathrm{i}(m\mu\alpha+n\nu\beta)\bigr)$$

for $\alpha,\beta\in\mathbb{R}$. (These are called "Kronecker series" in [14, Chapter VIII].) A detailed discussion of the convergence of these series is given in [3]. We note that Laplace–Mellin techniques have been employed frequently in the analysis of such series. The approach up until now has been to think of

$$|\omega_{m,n}|^2=|m\mu+n\nu|^2=Q(m,n)$$

as defining a positive definite quadratic form taking m and n as arguments. Treating this form as "indivisible," one may use the $\Re(\omega)>0$ plane-wave formula in Lemma 1 and obtain the integral representation

$$\sum_{\omega\in\Lambda\setminus\{0\}}\frac{\chi(\omega)}{|\omega|^{2s}}=\frac{1}{\Gamma(s)}\int_0^{\infty}\lambda^{s-1}\sum_{\omega\in\Lambda\setminus\{0\}}\chi(\omega)\mathrm{e}^{-\lambda Q(m,n)}\,\mathrm{d}\lambda.$$

The analysis then proceeds via θ functions.

Our approach would be different. “Plane-wave-like” representations exist for the function $f(x, y) = \sqrt{x^2 + y^2}$. Formally, one may take the true plane-wave expressions for $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ derived in [4], and set $z = 0$. The result is a 2D integral as opposed to the Eisenstein case analyzed above where one complex dimension (two real) collapses into a 1D integral. However, the critical element of this representation is that the exponential function in the “plane-wave” representation is linear in m and n. As a consequence, again the summation under the integrand becomes a 2D geometric series. We are considering this as a possible direction for future research.

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