WEPM 5-1

Bounds on Least-Squares Four-Parameter Sine-Fit Errors

Due to Harmonic Distortion and Noise

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Abstract Least-squares sine-fit algorithms are used extensively in signal processing applications. The parameter estimates produced by such algorithms are subject to both random and systematic errors when the record of input samples consists of a fundamental sine wave corrupted by harmonic distortion or noise. The errors occur because, in general, such sine-fits will incorporate a portion of the harmonic distortion or noise into their estimate of the fundamental. Bounds are developed for these errors for least-squares four-parameter (amplitude, frequency, phase, and offset) sine-fit algorithms. The errors are functions of the number of periods in the record, the number of samples in the record, the harmonic order, and fundamental and harmonic amplitudes and phases. The bounds do not apply to cases in which harmonic components become aliased.

Introduction

Sine-fit routines are used extensively during characterization of analog-to-digital converters (ADCs) and digital oscilloscopes [1-3]. These sine fit algorithms estimate the four parameters (amplitude, frequency, phase, and offset) of the sine wave that best fits a given finite length record of discrete samples, which are assumed to be samples of a sine wave, possibly corrupted by noise and distortion. Examples of such sine fit algorithms are found in IEEE Std. 1057 [1]. Because the records are finite in length (i.e., limited number of samples and number of periods), random additive noise and noise produced by timing jitter cause the parameter estimates themselves to be random variables with an associated variance. In addition, harmonic distortion can cause the parameter estimates to be biased with respect to the true, steady-state parameters. This occurs because truncated sinusoids of different frequencies are in general not strictly orthogonal.

A four parameter sine function is linear in only three of the four parameters; it is nonlinear with respect to frequency. Therefore, sine-fit algorithms, started from different initial parameter estimates, may converge on different local minima and different parameter estimates. In the work described here, it is assumed that the sine fit algorithm has converged on the *global* least squares solution.

The goal of this work was to develop bounds for the errors in parameter estimates returned by four-parameter, least squares sine fit algorithms, due to noise, jitter, or harmonic distortion of the sampled signal.

Errors Due to Harmonic Distortion

Fig. 1 illustrates the problem. A four-parameter, least-squares sine fit was performed on a 1000-point record of a sinusoid of unit amplitude plus 2nd harmonic distortion with an rms amplitude of 0.1429, sampled over 2.2 periods. The fitted sine wave however, has an

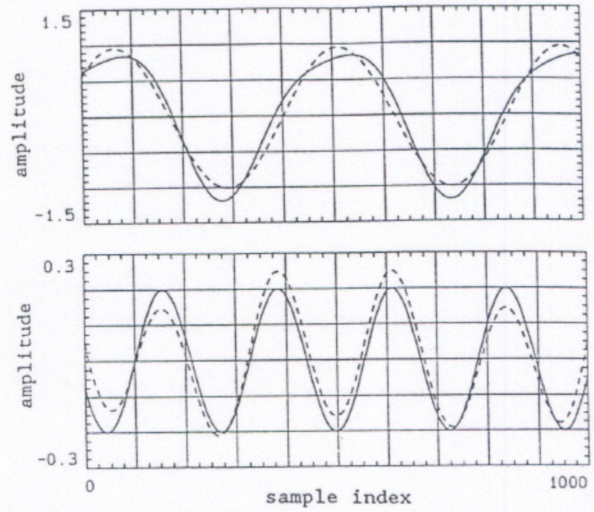

Fig. 1 Illustration of sine fit errors. Top plot shows a sine wave plus 2nd harmonic distortion (solid) and the best-fit sine wave (dashed) obtained with a 4-parameter least squares algorithm. Bottom plot shows the harmonic distortion (solid) compared to the fit residue (dashed).

amplitude of 0.9803, and its frequency is such that 2.237 periods subtend the record. The rms value of the residuals of the fit is 0.1357. Therefore, the amplitude and frequency estimates are each in error by 2%, and the residuals are reduced by 5.0%.

Efforts to derive a closed-form expression for such errors in terms of the amplitude, phase and order of harmonic distortion, and the parameters of the fundamental sine wave, were unsuccessful. Bounds on the errors could be found by performing a brute force search, varying the parameters of the fundamental and harmonic components, and performing a full four parameter least squares sine fit. This approach has two shortcomings. First, as surmised above, the results could be dependent on the algorithm and the initial conditions chosen to perform the sine fit. The second shortcoming and perhaps the more serious from a practical standpoint, is the very long computation time that such a search would require. Because of these problems, an alternative approach was chosen, based on estimating the functional relationship using a linear approximation. This provided a better understanding of the error mechanisms, and made the task of computing bounds on the maximum errors more tractable. In addition, within the attendant approximations of the linear model, it is possible to combine errors by superposition, e.g., the errors due to the combination of two harmonics can be estimated by combining their errors computed individually. The bounds computed from the linearized model can be spot-checked using a Monte Carlo approach with a full four parameter algorithm.

Linearization

The linearization proceeds as follows. Consider an M-point record of a uniformly sampled sinusoidal waveform with added harmonic distortion, given by

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$$y[n] = y_f[n] + y_h[n] = A_0 \sin(\omega_0 nT) + B_0 \cos(\omega_0 nT) + C_0 + y_h(nT) \quad (1)$$

$$n = 0,1,2,\ldots(M-1)$$

where T is the sample interval
f designates the fundamental component
h designates the harmonic distortion
0 designates the parameters of the fundamental component.

Note that any sine wave of arbitrary amplitude, offset, phase, and frequency can be expressed by $y_f[n]$ above.

Assume that the sine fit algorithm returns an estimated sine wave, $y_e[n]$, plus residue given by

$$y[n] = y_e[n] + r[n] = A \sin(\omega nT) + B \cos(\omega nT) + C + r[nT] \quad (2)$$

where $r[nT]$ is the residue of the fit.

The fitted sine wave given by (2) can be represented as a Taylor series expansion about the fundamental sine wave in (1):

$$y[n] = y_f[n] + (dy_f[n]/dA_0)\,\Delta A + (dy_f[n]/dB_0)\,\Delta B + (dy_f[n]/dC_0)\,\Delta C + (dy_f[n]/d\omega_0)\,\Delta\omega + HOT[n] + r[nT] \quad (3)$$

where HOT[n] represents the second and higher order terms of the expansion, and

$$\Delta A = A - A_0,\ \Delta B = B - B_0,\ \Delta C = C - C_0,\ \text{and}\ \Delta\omega = \omega - \omega_0.$$

Combining (3) with (1) and rearranging gives

$$y_h[n] = y[n] - y_f[n] = (dy_f[n]/dA_0)\,\Delta A + (dy_f[n]/dB_0)\,\Delta B + (dy_f[n]/dC_0)\,\Delta C + (dy_f[n]/d\omega_0)\,\Delta\omega + HOT[n] + r[nT] \quad (4)$$

or in matrix notation,

$$\mathbf{y_h} = \mathbf{D}\ \mathbf{x} + \epsilon \quad (5)$$

where D =

$$\begin{bmatrix} dy_f[0]/dA_0 & dy_f[0]/dB_0 & dy_f[0]/dC_0 & dy_f[0]/d\omega_0 \\ dy_f[1]/dA_0 & dy_f[1]/dB_0 & dy_f[1]/dC_0 & dy_f[1]/d\omega_0 \\ \vdots & & & \vdots \\ dy_f[M-1]/dA_0 & dy_f[M-1]/dB_0 & dy_f[M-1]/dC_0 & dy_f[M-1]/d\omega_0 \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} HOT[0] + r[0] \\ HOT[1] + r[1] \\ \vdots \\ HOT[M-1] + r[M-1] \end{bmatrix} \quad \text{and} \quad \mathbf{x} = [\Delta A\ \Delta B\ \Delta C\ \Delta\omega]^T$$

The least squares estimate of x is given by

$$\hat{x} = (D^T D)^{-1} D^T y_h \quad (6)$$

This estimate minimizes $\|\epsilon\|$, which minimizes $\|r\|$ when the higher order terms, HOT[n], are negligibly small (where $\|*\|$ designates the 2-norm of *). Therefore, to a first order approximation, (6) gives the vector of parameter error values, x, that produces the least squares sine fit to the data record given by (1).

Search for Bounds

Even though (6) gives an analytical expression (to first order) for the errors caused by harmonic distortion, it is of limited use to the practitioner because it requires that matrix D be generated and the normal equations solved for each situation. In addition, the error depends dramatically on the parameters of the signal and distortion, which are not necessarily known *a priori*. It would be preferable to have simple expressions that bound the errors for conditions that would likely be known or could be assumed, *a priori*. Examples of the complex dependencies inherent in (6) are given in figs. 2 and 3. Fig. 2 illustrates how the error in the estimate of amplitude depends on the phases of the harmonic and fundamental components, for a specific harmonic order (2nd) and actual number of fundamental periods contained in the record (2.2). Also, as the harmonic order and actual number of periods change, the locations of the maxima (with respect to the fundamental and harmonic phases) also change, necessitating an extensive search if bounds on the maxima are to be found. Fig. 3 is a plot of the maximum errors (for all fundamental and harmonic phases) in the estimate of the number of fundamental periods, $\hat{p}$, in the record, versus p, for (3rd) harmonic distortion of unit amplitude relative to the fundamental. Note that p is proportional to frequency.

Estimates for upper bounds on the parameter errors were generated using the following procedure based on (6): (i) accumulate the error estimates for many different sample records: different numbers of periods, harmonic orders, fundamental phases, and harmonic phases; (ii) search the accumulated parameter error estimates for maxima; and finally (iv) fit these maxima by regression to produce bound expressions.

Simulations were run for lower harmonics (h=2,3,4,5, 7,10), because these produced the greatest errors in the sine-fit algorithm, and because these are commonly the most significant distortion components in the outputs of ADCs and digital oscilloscopes. The tests were run for data records of M=200, 1000, 2000, and 4000 samples. The effect of each harmonic component (e.g., 2nd, 3rd) was determined separately.

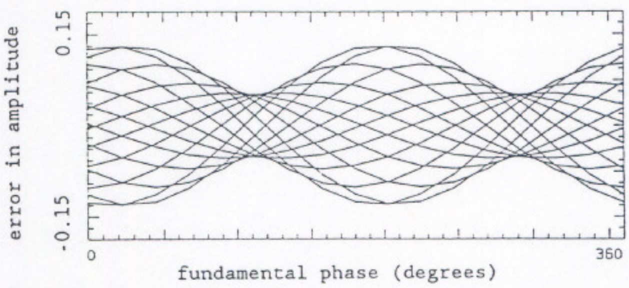

Fig. 2 Example of complex dependency of amplitude estimate on the phases of the harmonic (2nd) and fundamental components. Multiple curves correspond to different fundamental phases.

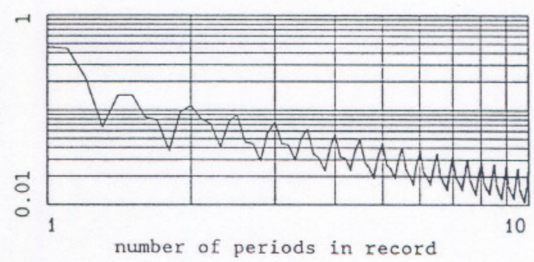

Fig. 3 Plot of the maximum errors (for all fundamental and harmonic phases) in the period estimate versus the number of periods in the data record, for (3rd) harmonic distortion of unit amplitude relative to fundamental.

Results

Exponential regression produced an excellent fit of the dependence of the maxima on the number of periods in the record, and on the order of the distorting harmonic.

The resulting estimated bounds on the parameter errors follow:

Let Δp the error in the estimated number of fundamental periods in the record

ΔA_f the error in the estimated fundamental amplitude $= ((A+\Delta A)^2+(B+\Delta B)^2)^{1/2}-(A^2+B^2)^{1/2}$

$\Delta\phi$ the error in the estimated fundamental phase, in degrees $= \tan^{-1}((B+\Delta B)/(A+\Delta A)) - \tan^{-1}(B/A)$

Δoff the error in the estimated DC offset of the signal.

M number of samples in the record

p the number of fundamental sine wave periods in the record $= (\omega MT)/(2\pi)$

h the order of the distorting harmonic (positive integer)

A_f the amplitude of the input fundamental

A_h the amplitude of the input distorting harmonic

Then

$$\max|\Delta p| = \frac{0.90}{ph^{1.2}}\left(\frac{A_h}{A_f}\right), \quad \text{for } p\geq 2.0,\ M>2ph \tag{7}$$

$$\max\left|\frac{\Delta A_f}{A_f}\right| = \frac{1.00}{ph^{1.25}}\left(\frac{A_h}{A_f}\right), \quad \text{for } p\geq 2.0,\ M>2ph \tag{8}$$

$$\max|\Delta\phi| = \frac{180°}{ph^{1.25}}\left(\frac{A_h}{A_f}\right), \quad \text{for } p\geq 2.0,\ M>2ph \tag{9}$$

$$\max\left|\frac{\Delta off}{A_f}\right| = \frac{0.61}{p^{1.2}h^{1.1}}\left(\frac{A_h}{A_f}\right), \quad \text{for } p\geq 2.0,\ M>2ph \tag{10}$$

Expressions (7)-(10) are graphed in fig. 4 as a function of the ratio (A_h/A_f), for a number of combinations of harmonic order and number of periods of the fundamental component contained in the record.

Note that the error bounds are for $p\geq 2.0$. For two or more periods, the estimated error magnitude maxima fit the peaks very well. By contrast, for $p<2.0$, the error maxima exceeded these bounds (sometimes reaching as much as ten times higher), particularly for the lower order harmonics (2nd, 3rd).

Note also that the bounds are for $M>2ph$. In other words, the bounds apply to the case where the harmonic is being sampled above the Nyquist rate. If $M<2ph$ then the harmonics alias into neighboring bands, and these bounds break down, because the aliased harmonic frequencies can be near or equal to the fundamental frequency, in which case much or all of the aliased harmonic power is incorporated into the fundamental estimate. For $M>2ph$, the dependence of the bounds on M was negligible.

The reasons for the non-integer exponents of p and h are not clear; for quick but conservative approximations the exponents of p and h can be rounded to unity.

The experiments showed that for $p\geq 2.0$ and over the range of (A_h/A_f) for which the linear approximation holds, the effect of the fit errors on effective bits estimation [1-3] was fairly small, producing errors of 0.1 effective bits or less.

Region of Validity of Linear Model

A cautionary note is that as the ratio of the harmonic to fundamental amplitude increases, the first order approximation of the sine-fit errors given here will become invalid, as higher order terms become important. The region of validity of these bounds has been evaluated using a full four-parameter least-squares sine-fit algorithm in a Monte Carlo sampling approach.

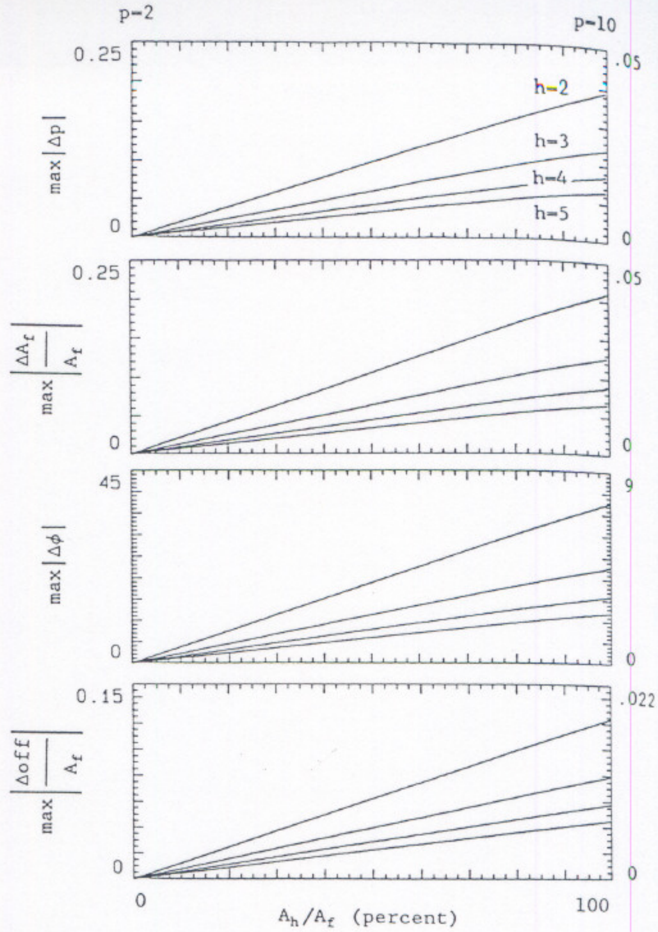

Fig. 4 Plots of error bounds. Use left and right vertical scales, respectively, for p=2 and p=10.

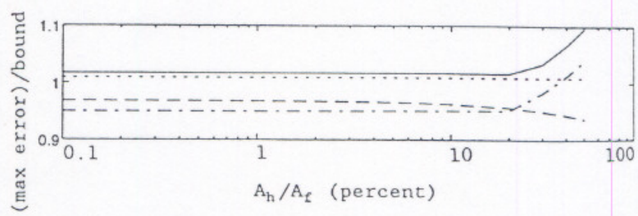

Fig. 5 Results of Monte Carlo tests of bounds given in (7)-(10). The harmonic order is 2, and the number of trials at each level of distortion was 1000. (—— = Δp, $\cdots$ = ΔA_f, $-\cdot-$ = $\Delta\phi$, and $---$ = Δoff).

The results are presented in fig. 5. For 15 values of harmonic distortion, the maximum normalized error is plotted, determined from 1000 trials in which the fundamental and harmonic phases and the number of periods (2-10) were all chosen at random for each trial. It can be seen from the plot that none of the four bounds is exceeded by more than 4% for distortion up to 30%. For this plot, the second harmonic was used. Higher harmonic orders give similar results.

Errors Due to Noise and Jitter

To compute the variance-covariance of the parameter estimates when noise and jitter are present, we again use a linearization of the problem, assuming that, at convergence, four parameter estimates are available (A_c, B_c, C_c, and ω_c). Adding noise, Eq. (5) becomes

$$y_h = D\,x + b + e \qquad (11)$$

where

- D — the same as in (5), but with parameters A, B, and ω evaluated at convergence
- b — the nonrandom component of the residue, and is assumed orthogonal to D, so that $D^T b = 0$
- e — random variable, independent but not necessarily identically distributed, with zero mean
- c — designates the value at convergence
- x — $[\Delta A\ \Delta B\ \Delta C\ \Delta\omega]^T$

Following the approach outlined in [4], it can be shown that

$$E[\hat{x}] = x \qquad (12)$$

where E[*] designates the expectation of *, and $\hat{x}$ is the least-squares estimate of x. Furthermore, it can be shown that the variance-covariance matrix of $\hat{x}$ is given by

$$\textstyle\sum(\hat{x}) = \sigma^2\ (D^T D)^{-1} D^T W W^T D (D^T D)^{-1} \qquad (13)$$

where $\sum(\hat{x})$ the variance-covariance matrix of $\hat{x}$
W a diagonal weighting matrix such that $e = We'$ and e' is iid, $(0,\sigma^2)$.

Noise

In the case of random noise, the elements of e are assumed to be identically distributed, so $W = I$ (where I is the identity matrix), and (13) reduces to

$$\textstyle\sum(\hat{x})_{noise} = \sigma^2\ (D^T D)^{-1} \qquad (14)$$

where σ^2 is the noise variance.

The parameter variance, σ_p^2, of the least-squares fit is given by the diagonal elements of the matrices in (13) and (14):

$$\sigma_p^2 = [\sigma_A^2\ \sigma_B^2\ \sigma_C^2\ \sigma_\omega^2]^T = \mathrm{diag}\{*\} \qquad (15)$$

where {*} is the corresponding matrix in (13) or (14). The parameter covariances are given by the off-diagonal components.

Plots of σ_A^2/σ^2 and $\sigma_\omega^2 A_f^2/\omega^2\sigma^2$ are given in fig. 6, versus the phase of the fundamental component and the number of periods, respectively. For these plots, M, the number of samples in the record, was 100. The values of σ_A^2/σ^2 and $\sigma_\omega^2 A_f^2/\omega^2\sigma^2$ are inversely proportional to M. To determine the proportional variance of the estimate of parameter A (i.e., σ_A^2/A_f^2) from fig. 6a, find the normalized A-variance corresponding to the appropriate phase and multiply by the expected proportional variance of the noise (σ^2/A_f^2) times 100/M. To determine the proportional variance in ω (i.e., σ_ω^2/ω^2) from fig. 6b, find the normalized ω-variance corresponding to the appropriate number of periods, and again multiply by σ^2/A_f^2 times 100/M.

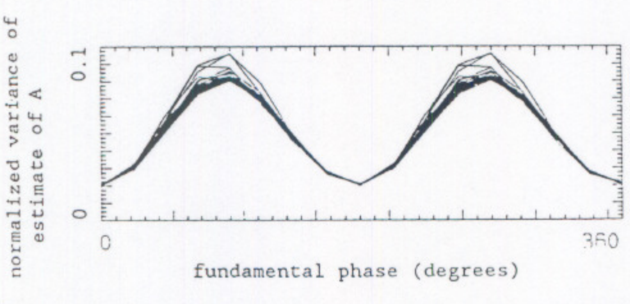

Fig. 6-a Plots of normalized variance of parameter A (σ_A^2/σ^2) vs. phase, for 2-10 periods.

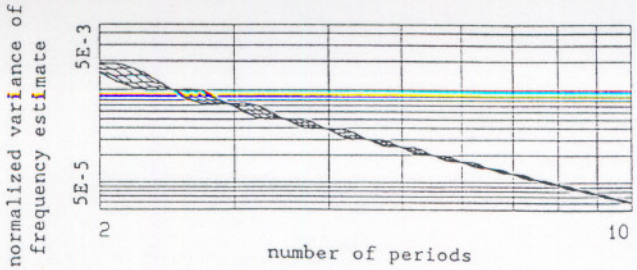

Fig. 6-b Plots of normalized variance of frequency, ω, ($\sigma_\omega^2 A_f^2/\omega^2\sigma^2$) vs. no. of periods, for different phases.

Jitter

In the case of jitter, the noise produced by jitter having a constant variance of σ_j^2 is distributed (to first order) according to the time derivative of the input signal, so that the (i,j)th element of the weighting matrix used in (13) is given by

$$w_{ij}\begin{cases} dy_f[i]/dt = \omega_c A_c T\cos(\omega_c iT) - \omega_c B_c T\sin(\omega_c iT) & i=j=n \\ 0 & i\neq j \end{cases} \qquad (16)$$

and σ^2 in (13) is replaced by the jitter variance, σ_j^2.

For (16), several simplifying assumptions have been made: it is assumed that (a) terms involving higher order derivatives are negligibly small, (b) the harmonic distortion is small enough that its contribution to the derivative is negligible, (c) that the mean of the noise produced by the jitter (see (11)) is zero, although this is not strictly true [5], and (d) that the jitter is independent from sample to sample.

Conclusions

Four-parameter, least-squares sine wave curve fits produce parameter estimates that are subject to both random and systematic errors when the input samples consist of a fundamental sine wave corrupted by harmonic distortion or noise. In the case of harmonic distortion, the errors are bounded by simple expressions that are approximately inversely proportional to both harmonic order and number of periods in the record, and directly proportional to the ratio of harmonic distortion to fundamental amplitude. These bounds hold for data records containing at least two periods of the fundamental and for relatively large signal to noise ratios, but are not valid for cases in which harmonic components become aliased. For signals corrupted by noise or jitter, the variances of the parameter estimates (normalized by the variance of the corrupting noise) are functions of the number of periods and the fundamental phase.

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