Definitions of "total" estimators of common time-domain variances

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Abstract

This paper summarizes the definitions and properties of frequency and time variance estimators based on the "total" approach. These definitions and properties are not available in one document, yet this statistical approach produces variance estimates with the best confidence, as indicated by equivalent degrees of freedom (edf), for the time-domain sample variances most frequently used to analyze the stability of oscillators. There are differences in the total approach as applied to each variance because each variance reports noise levels for a specific range of power-law noise processes. Development of these total definitions has been motivated by the goal of realizing the highest edf and the lowest bias across this range while keeping the algorithms relatively simple. The relevant variances are [1-4]:

- Allan variance, $\sigma^2_y(\tau)$
- Modified Allan variance, Mod-$\sigma^2_y(\tau)$
- Time variance, $\sigma^2_x(\tau)$
- Hadamard variance, $_H\sigma^2_y(\tau)$

The Hadamard variance is used in managing clocks of the Global Positioning System (GPS).

Specifically we will give precise definitions for sample variances that use the "total" approach and whose corresponding names are:

- Total variance
- Modified Total variance
- Time Total variance
- Hadamard Total variance

The square root of a sample variance is usually reported; hence its designation is "deviation" rather than "variance." In addition to total definitions, we will give corresponding bias and edf formulae.

Introduction

Random fluctuations in frequency in precision oscillators are reasonably modeled by a spectral density function $S_y(f)$ that follows broadband power-law $h_\alpha f^\alpha$ behavior. These random fluctuations can be represented as the sum of seven independent noise processes, thus:

$$S_y(f) = \sum_{\alpha=-4}^{+2} h_\alpha f^\alpha \quad \text{for } 0 < f < f_h,$$

$$S_y(f) = 0 \quad \text{for } f > f_h,$$

where each h_α is a constant and α takes integer values 2, 1, 0, -1, -2, -3, -4 (corresponding respectively to white phase modulation or WHPM, flicker PM or FLPM, white frequency modulation or WHFM, flicker FM or FLFM, random-walk FM or RWFM, flicker-walk FM or FWFM, and random-run FM or RRFM). Note that this power-law range has been extended to $\alpha = -4$ because oscillators with high levels of drift need to have "noise on drift" characterized. f_h is the high-frequency equivalent cut-off of a low-pass filter.

Time statistics that are frequency variances in the tau domain, where τ = averaging time, have a τ dependence given by τ^μ for power-law processes. Two variances efficiently distinguish noises in the range $-4 \leq \alpha \leq +2$ as follows:

$$\text{Mod-}\sigma^2_y(\tau) \quad -2 \leq \alpha \leq +2,$$
$$_H\sigma^2_y(\tau) \quad -4 \leq \alpha \leq 0,$$

and a simple straight-line (log-log scale) mapping between τ domain and f domain is $\mu = -\alpha-1$. The Allan variance ($\sigma^2_y(\tau)$) covers $-2 \leq \alpha \leq +2$ but does not distinguish $\alpha = +1$ from $\alpha = +2$, so its effective range, $-2 \leq \alpha \leq 0$, is primarily for distinguishing FM noises. Finally, Time variance is defined as $\sigma^2_x(\tau) = (\tau^2/3) \cdot \text{Mod-}\sigma^2_y(\tau)$.

The "Total" Approach

"Total" statistics improve the confidence of the analysis and characterization of instabilities in oscillators and synchronization systems [5-11]. The total variance approach periodically extends a data sequence beyond its normal measurement duration in such a way that a particular time statistic is expected to have the same value with extended data as without. For those statistics that estimate components of broadband noise processes, the approach can significantly reduce the spread or uncertainty in the result as represented by an increase in equivalent degrees of freedom. The edf formulae are a convenient, empirical or "fitted" approximation with an observed error below 10% of numerically computed exact values derived from a Monte-Carlo simulation method using the coefficients of Tables 1, 2, and 3 to follow. For computing biases, a proven power-law detection scheme is given in ref. [4].

Total Variance

Total variance is a descriptive statistic that serves as an excellent estimator of the Allan variance [8]. We are given sequential time-error Δt values denoted as a data set $\{x_n\}$, $n = 1$ to N_x, where N_x represents the total number of points in the data set (Figure 1). We first create an extended, virtual data sequence, $\{x^{\#}\}$. This is accomplished by performing an inverted, even reflection (Figure 2). Using a utility index j, for $j = 1$ to N_x let

$$x_j^{\#} = x_j$$

$$x_{1-j}^{\#} = 2x_1 - x_{1+j}$$

$$x_{N_x+j}^{\#} = 2x_{N_x} - x_{N_x-j} .$$

We compute the total variance using the extended virtual data sequence.

$$Total\text{-}\sigma_y^2(\tau) =$$

$$\frac{1}{2(m\tau_0)^2(N_x - 2)} \sum_{n=2}^{N_x-1} \left(x_{n-m}^{\#} - 2x_n^{\#} + x_{n+m}^{\#} \right)^2,$$

where $m = \tau/\tau_0$.

There are two important differences in the computation of the Total variance when compared to the following other types of total variances. First, since we have not removed a slope from the data, the reflection **must** be inverted to avoid a discontinuity at the extensions. Second, we have used the **entire** data set for one extension and subsequent computation of the Total variance. One overall extension is sufficient for the relatively short range of power laws (WHFM, FLFM, and RWFM).

Bias and Equivalent Degrees of Freedom

For a data run of duration $T = \tau_0(N_x-1)$, where τ_0 = sampling interval, the normalized bias and edf for Total variance are given by

$$nbias(\tau) = \frac{E\{Total - \sigma_y^2(\tau,T)\}}{\sigma_y^2(\tau)} - 1 = -a\frac{\tau}{T},$$

$$edf(\tau) = edf\left[Total - \sigma_y^2(\tau,T)\right] \approx b\frac{T}{\tau} - c,$$

where $E\{\cdot\}$ is expectation of $\{\cdot\}$, $0 < \tau \leq T/2$ and a, b, c are given in Table 1.

Table 1. Coefficients for computing normalized bias and edf for Total variance.

Noise	α	a	b	c
White FM	0	0	1.500	0
Flicker FM	-1	0.481	1.168	0.222
Random Walk FM	-2	0.750	0.927	0.358

Modified Total Variance

The Modified Total variance is used to distinguish white phase modulation (WHPM) and flicker phase modulation (FLPM). We are given a data set $\{x_n\}$, $n = 1$ to N_x, where N_x represents the total number of points in the data set. We choose a subsequence $\{x_i\}_n$ of $\{x_n\}$, $n \leq i \leq n+3m$, consisting of $3m$ data points, where $m = \tau/\tau_0$ (Figure 3). Working only on this subsequence, we remove a linear trend (or slope) from the data (Figure 4):

$$^{\circ}x_i = x_i - c_1 i , \qquad (1)$$

where c_1 is a frequency offset removed to minimize

$$\sum_{i=n}^{n+3m-1} \left(^{\circ}x_i - \overline{^{\circ}x_i} \right)^2 \qquad (2)$$

to satisfy a least-squared-error criterion for the subsequence. In practice, it is sufficient to remove a background slope computed by averaging the first and last halves of the subsequence divided by half the interval.

Now extend the subsequence $\{^{\circ}x_i\}_n$ at both ends by an uninverted, even reflection (Figure 5). For $i = n+1$ to $n+3m$, let

$$^{\circ}x_i^{\#} = {^{\circ}x_{3m+1-i}}$$
$$^{\circ}x_{3m+i}^{\#} = {^{\circ}x_i} \qquad (3)$$
$$^{\circ}x_{6m+i}^{\#} = {^{\circ}x_{3m+1-i}}$$

to create

$$\{^{\circ}x_i^{\#}\}.$$

In the above equations, we are creating and re-numbering the extended subsequence in one step.

We now compute the Modified Total variance on the extended subsequence. Define $^{\circ}z_i^{\#}$ as

$$°z_i^{\#}(m) = \overline{°x_i^{\#}}(m) - 2\overline{°x_{i+m}^{\#}}(m) + \overline{°x_{i+2m}^{\#}}(m), \quad (4)$$

where the overbar means an average over m points starting at the indexes i, i+m, and i+2m.

Maximum-overlap estimator of TotalMod-$\sigma_y^2(\tau)$ is a simple average of its subestimates given by:

$$TotalMod - \sigma_y^2(\tau) =$$

$$\frac{1}{2(m\tau_0)^2(N_{x\,max} - 3m + 1)} \sum_{n=1}^{N_{x\,max}-3m+1} \left(\frac{1}{6m} \sum_{i=n-3m}^{n+3m-1} \left(°z_i^{\#}(m)\right)^2 \right), \quad (5)$$

for $1 \le m \le \lfloor N_x/3 \rfloor$, where $\lfloor c \rfloor$ means the integer part of c.

Bias and Equivalent Degrees of Freedom

The normalized bias and edf for Modified Total variance are given by

$$nbias(\tau) = \left[\frac{E\{TotalMod - \sigma_y^2(\tau,T)\}}{E\{Mod - \sigma_y^2(\tau,T)\}} - 1 \right] = a,$$

$$edf(\tau) = edf\left[TotalMod - \sigma_y^2(\tau,T)\right] = \frac{T/\tau}{b_0 + b_1\tau/T},$$

where $E\{\cdot\}$ is expectation of $\{\cdot\}$, $0 < \tau \le T/3$, $\tau \ge 16\tau_0$, and a, b_0, and b_1 are given in Table 2.

Table 2. Coefficients for computing normalized bias and edf for Modified Total variance.

Noise	α	a	b_0	b_1
White PM	+2	-0.005	0.559	1.004
Flicker PM	+1	-0.149	0.868	1.140
Random Walk PM	0	-0.229	0.938	1.696
Flicker Walk PM	-1	-0.283	0.974	2.554
Random Run PM	-2	-0.321	1.276	3.149

Time Total Variance

Time Total variance, as with the original Time variance, is derived directly from the Modified Total variance. We perform exactly the same removal of a linear trend and extension on the chosen subsequence as we did with the Modified Total variance (equations 1-3 above). Modified Total

$$Total - \sigma_x^2(\tau) = \frac{\tau^2}{3} TotalMod - \sigma_y^2(\tau).$$

variance is computed (equations 4-5), and Total Time variance is finally derived:

This definition supersedes an earlier definition given in ref. [7]. Bias and equivalent degrees of freedom for Time Total variance are derived as in Modified Total variance.

Hadamard Total Variance

At the expense of efficient use of data, the main advantage to the Hadamard variance over the Allan variance is its insensitivity to linear frequency drift. This is also true with the Hadamard Total variance over the Total variance. The method of computation is identical to that of the Modified Total variance, except here we are working with frequency data instead of time data.

$$°y_i = y_i - c_1 i,$$

We choose a subsequence $\{y_i\}_n$ of $\{y_n\}$, consisting of $3m$ points, where $m = \tau/\tau_0$. Working only on this subsequence, we remove a linear trend (or slope) from the data:

where c_1 is a frequency offset removed to minimize

$$\sum_{i=n}^{n+3m-1} \left(°y_i - \overline{°y_i}\right)^2$$

to satisfy a least-squared-error criterion for the subsequence.

Extend the subsequence $\{°y_i\}_n$ by an uninverted, even reflection:

$$°y_i^{\#} = °y_{3m+1-i}$$

$$°y_{3m+i}^{\#} = °y_i$$

$$°y_{6m+i}^{\#} = °y_{3m+1-i}$$

to create

$$\left\{°y_i^{\#}\right\}.$$

In the above equations, we are again creating and re-numbering the extended subsequence in one step.

We now have a tripled range of $n\text{-}3m \le i \le n\text{+}6m\text{-}1$ with $9m$ points; in other words, the extended subsequence is now

$$\left\{ {}^{\circ}y_i^{\#} \right\}_n = \left\{ {}^{\circ}y_i^{\#}, i = n - 3m, \ldots, n + 6m - 1 \right\}.$$

Define:

$$Total -_H \sigma_y^2 \left(m, \tau_0, N_{y\max} \right) =$$

$$\frac{1}{6\left(N_{y\max} - 3m + 1\right)} \sum_{n=1}^{N_{y\max}-3m+1} \left(\frac{1}{6m} \sum_{i=n-3m}^{n+3m-1} \left({}^{\circ}H_i^{\#}(m) \right)^2 \right)$$

for

$$1 \le m \le \left\lfloor \frac{N_{y\max}}{3} \right\rfloor$$

and

$${}^{\circ}H_i^{\#}(m) = \overline{{}^{\circ}y_i^{\#}}(m) - 2\overline{{}^{\circ}y_{i+m}^{\#}}(m) + \overline{{}^{\circ}y_{i+2m}^{\#}}(m)$$

is calculated from the extended subsequence

$$\left\{ {}^{\circ}y_i^{\#} \right\}.$$

The method of computing the Hadamard Total variance directly from time-error data rather than frequency-error data is given in ref. [4].

We can somewhat decrease the computation time for the Hadamard Total variance by using a shortcut approach. Because of symmetries in the values of subestimates as a function of index n, we need to calculate ${}^{\circ}y_i^{\#}$ for only the range $n\text{-}k \le i \le n\text{+}k\text{+}3m\text{-}1$, and ${}^{\circ}H_i^{\#}(m)$ only for $n\text{-}k \le i \le n\text{+}k$. The shortcut that follows can also be applied to the Modified and Time Total variances by substituting ${}^{\circ}z_i^{\#}(m)$ for ${}^{\circ}H_i^{\#}(m)$.

For even m values,

$$\sum_{i=n-3m}^{n+3m-1} \left({}^{\circ}H_i^{\#}(m) \right)^2 =$$

$$2\sum_{i=n-k+1}^{n+k-1} \left({}^{\circ}H_i^{\#}(m) \right)^2 + \left({}^{\circ}H_{n-k}^{\#}(m) \right)^2 + \left({}^{\circ}H_{n+k}^{\#}(m) \right)^2.$$

For odd m values,

$$\sum_{i=n-3m}^{n+3m-1} \left({}^{\circ}H_i^{\#}(m) \right)^2 = 2\sum_{i=n-k}^{n+k} \left({}^{\circ}H_i^{\#}(m) \right)^2,$$

where $k = \lfloor 3m/2 \rfloor$.

Bias and Equivalent Degrees of Freedom

Like the Modified Total variance, the normalized bias and edf for the Hadamard Total variance are given by

$$nbias(\tau) = \left[\frac{E\left\{Total-_H \sigma_y^2(\tau,T)\right\}}{E\left\{_H \sigma_y^2(\tau,T)\right\}} - 1 \right] = a,$$

$$edf(\tau) = edf\left[Total-_H \sigma_y^2(\tau,T)\right] = \frac{T/\tau}{b_0 + b_1\tau/T},$$

where $E\{\cdot\}$ is expectation of $\{\cdot\}$, $0 < \tau \le T/3$, $\tau \ge 16\tau_0$, and a, b_0, and b_1 are given in Table 3.

Table 3. Coefficients for computing the normalized bias and edf for the Hadamard Total variance.

Noise	α	a	b_0	b_1
White FM	0	-0.005	0.559	1.004
Flicker FM	-1	-0.149	0.868	1.140
Random Walk FM	-2	-0.229	0.938	1.696
Flicker Walk FM	-3	-0.283	0.974	2.554
Random Run FM	-4	-0.321	1.276	3.149

Table 3 has the same values as Table 2 but with 2 added to each value of α. This is because of the coincidence that the GPS Hadamard Total variance that acts on fractional frequency fluctuations is identical to the Modified Total variance acting on time fluctuations.

Conclusion

Using the total approach on standard frequency and time variances results in a significant increase in equivalent degrees of freedom with only a modest bias. At the longest averaging times, edf is increased by a factor of 2 to 4, which means that the corresponding confidence interval is reduced by 70% to 50% as compared to classical approaches. Thus, the often crucial task of determining noise levels and types at long averaging times is

substantially improved. We have summarized common total formulae and presented edf and normalized bias tables and formulae.
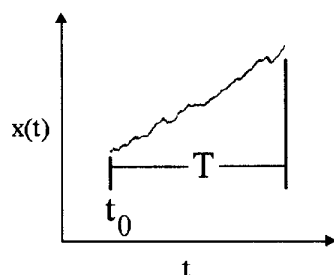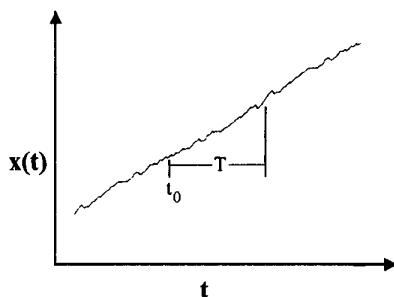

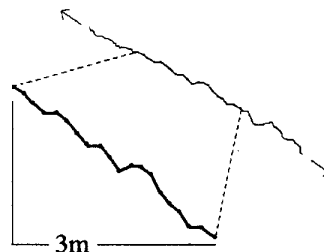**Figure 3.** Procedure for TotalMod-$\sigma_y(\tau)$, Total-$\sigma_x(\tau)$, and Total- $_H\sigma_y(\tau)$ starts with original data subsequence.
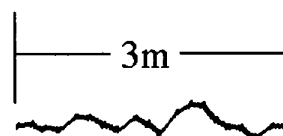

Figure 1. Original data stream, $\{x_n\}$.

Figure 4. Original data subsequence with slope removed.
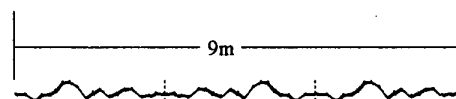
Figure 5. Extended data subsequence using uninverted, even reflection.

Figure 2. Inverted, even reflection of original whole data run for computing Total-$\sigma_y(\tau)$.

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