

# SPIN SQUEEZING APPLIED TO FREQUENCY STANDARDS\*

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## ABSTRACT

In experiments on  $N$  atoms initially prepared in uncorrelated states, the precision of a transition frequency measurement (limited only by the quantum fluctuations in the measurement) improves as  $N^{-1/2}$ . We show that the precision of a frequency measurement can improve as  $N^{-1}$ , if the atoms are prepared in particular quantum mechanically correlated states. These correlated states can be interpreted as squeezed spin states in analogy with the squeezing that has been discussed with reference to the electromagnetic field. We discuss states which provide the maximum possible precision in a transition frequency measurement and briefly describe possible methods for generating correlated states.

## 1. Spin Squeezing and Ramsey Spectroscopy

In this manuscript we consider the spectroscopy of  $N$  two-level atoms, where transitions are driven by classical radiation and detected by measuring changes in the state populations. We assume that the relaxation of the atomic states can be neglected. The use of correlated states to improve the signal-to-noise ratio in this type of spectroscopy has been discussed recently under the subject of spin squeezing.<sup>1-3</sup> In particular, Refs. 1 and 2 discuss spin squeezing for the Ramsey technique of separated oscillatory fields (SOF). After reviewing this discussion below, we discuss in Sec. 2 the best squeezed spin states for  $N=2$  atoms. In Sec. 3 we give a state that can be used in Ramsey spectroscopy to provide a frequency imprecision of  $(NT)^{-1}$  where  $T$  is the time of a single measurement. From a time-energy uncertainty relation, we show that this is the best precision possible. Finally in the last section we briefly describe possible methods for generating correlated or squeezed spin states of ions in a trap.

Let  $|e\rangle$  denote the excited state of a two-level atom and  $|g\rangle$  the ground state, and consider the case where  $N$  atoms are uncorrelated and each atom is initially prepared in its ground state  $|g\rangle$ . Because any two-level system interacting with classical radiation

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is equivalent to a spin- $1/2$  magnetic dipole in a magnetic field,<sup>4</sup> the initial state of the composite system for this case is equivalent to the  $|J=N/2, M_J=-N/2\rangle$  state of a  $J=N/2$  spin. This initial ( $t=0$ ) state has  $\langle J_z \rangle_0 = -N/2$ ,  $\langle J_x \rangle_0 = \langle J_y \rangle_0 = 0$ , and  $\Delta J_x(0) = \Delta J_y(0) = N^{1/2}/2$ . (Here  $\langle A \rangle_t$  denotes the expectation value of an operator  $A$  at time  $t$  and  $\Delta A(t) = \langle \Delta A^2 \rangle_t^{1/2}$ , where  $\Delta A^2 = A^2 - \langle A \rangle^2$ .) The Hamiltonian for the equivalent spin system is  $H = -\bar{\mu} \cdot \bar{B}$  where  $\bar{\mu} = \mu_0 \bar{J}$  is the magnetic moment of the composite system and  $\bar{B}$  is the applied field. Here  $\bar{B} = B_0 \hat{z} + \bar{B}_1$ , where  $B_0$  is a time independent field set by the frequency of the transition ( $\omega_0 = -\mu_0 B_0 / \hbar$ , where we assume  $\mu_0 < 0$ ), and  $\bar{B}_1$  is an applied oscillating field used to perform spectroscopy on the atoms. We assume  $\bar{B}_1$  is perpendicular to, and is rotating about the  $z$  axis according to  $\bar{B}_1 = B_1 [-\hat{x} \sin \omega t + \hat{y} \cos \omega t]$ . In the Ramsey SOF technique,<sup>5</sup>  $B_1$  is applied (that is, nonzero) for two periods of length  $t_{\pi/2} = \pi / (2\Omega_R)$  where  $\Omega_R = |\mu_0 B_1| / \hbar$  is the Rabi frequency, separated by a period of length  $T$  during which  $B_1 = 0$ . Throughout this manuscript we assume  $\Omega_R \gg |\omega_0 - \omega|$  and  $T \gg t_{\pi/2}$  which results in a simple expression for the Ramsey lineshape. In a frame of reference rotating with  $\bar{B}_1$ , the first pulse rotates the spin vector about  $\bar{B}_1$  (the  $y$  axis in the rotating frame) by  $90^\circ$ . The spin vector then precesses about the  $z$  axis during the field free period, acquiring an angle  $(\omega_0 - \omega)T$  relative to the  $x$  axis in the rotating frame. This angle could be read out by measuring, for example,  $J_x$  in the rotating frame. Experimentally this is done by the second  $\pi/2$  pulse, which rotates the spin vector by  $90^\circ$  about the  $\bar{B}_1$  axis, and then measuring the number of atoms in  $|e\rangle$ , or equivalently  $J_z$ . We obtain

$$\langle J_z \rangle_{t_f} = (N/2) \cos(\omega_0 - \omega)T \quad (1)$$

where  $t_f = 2t_{\pi/2} + T$ .

Measurements of  $J_z$  along with Eq. (1) can be used to estimate the frequency  $\omega_0$ . Because of the statistical nature of quantum mechanics, the number of particles which make a transition to  $|e\rangle$  from measurement to measurement will fluctuate by  $\Delta J_z(t_f)$ .<sup>6</sup> This produces an uncertainty in the estimate of  $\omega_0$  of  $|\Delta\omega| = \Delta J_z(t_f) / |\partial \langle J_z \rangle_{t_f} / \partial \omega|$ . We obtain

$$|\Delta\omega| = \Delta J_y(0) / [T |\langle J_z \rangle_0|] = N^{-1/2} T^{-1} \quad (2)$$

independent of  $\omega$ . (Experimental measurements are usually made with  $\omega \sim \omega_0 \pm \pi / (2T)$  where Eq. (1) has its steepest slope. This minimizes the effect of any added technical noise.) The fundamental noise limit of Eq. (2) has been achieved with small numbers of ions in a trap.<sup>6</sup> Qualitatively, the dependence of  $|\Delta\omega|$  on  $\Delta J_y(0)$  in Eq. (2) can be understood from the picture that the Ramsey SOF technique measures the accumulated angle  $(\omega_0 - \omega)T$  of a spin in the  $x$ - $y$  plane which is precessing about the  $z$  axis. In particular, spectroscopy with the Ramsey SOF technique is equivalent to determining the rotation of a spin state (after the first  $\pi/2$  pulse) about the  $z$  axis. The precision of this

measurement is limited by the uncertainty of the spin normal to the mean spin vector and in the x-y plane, which here is  $\Delta J_y(0)$ . The idea of spin squeezing is to start with a state  $\langle \vec{J} \rangle = \hat{z} \langle J_z \rangle_0$  such that  $\Delta J_y(0) < N^{1/2}/2$ . Such a state can be used to improve the frequency resolution in Ramsey spectroscopy if the squeezing parameter  $\xi_R \equiv (2J)^{1/2} \Delta J_y(0) / |\langle J_z \rangle_0| < 1$ .<sup>7</sup> The parameter  $\xi_R$  is a measure of the improvement in Ramsey spectroscopy over the case where the initial state consists of uncorrelated atoms. States with  $\xi_R < 1$  are in general entangled; they cannot be written as a product of individual atomic wave functions. These states are multiparticle versions of the correlated two-particle states discussed in the Einstein-Podolsky-Rosen experiments. Spin squeezing has also been discussed in the context of interferometry.<sup>8-11</sup> The discussion there parallels the discussion here because there is a one-to-one correspondence between the Ramsey SOF technique and a Mach-Zehnder interferometer.

## 2. Two-Atom Squeezed States

For simplicity, consider  $N=2$  ( $J=1$ ) and let the mean spin vector be parallel to the z-axis,  $\langle \vec{J} \rangle = \langle J_z \rangle \hat{z}$ . In this case it is straightforward to show that the state with the best squeezing (lowest value of  $\xi_R$ ) is

$$|\psi_M\rangle = (2\cosh(2\theta))^{-1/2} \{ e^{-\theta} |J=1, m_J=1\rangle + e^{\theta} |1, -1\rangle \}, \quad (3)$$

where  $\theta$  is determined by  $\langle J_z \rangle = -\tanh(2\theta)$ . As discussed above, except for the  $\theta \rightarrow \pm \infty$  limits, this is a correlated state of the two atoms; it cannot be written as the product of individual atomic states. In addition,  $\Delta J_y^2 = 1/2(1 - \text{sech}(2\theta))$ ,  $\Delta J_x^2 = 1/2(1 + \text{sech}(2\theta))$ , and  $\xi_R = \sqrt{2\text{sech}\theta} / (\cosh(2\theta))^{1/2}$ . The smallest (that is, best) values of  $\xi_R$  are obtained for small  $\theta$ . As  $\theta \rightarrow 0$ ,  $\xi_R \rightarrow 1/\sqrt{2}$ . However, as  $\theta \rightarrow 0$ , the mean spin vector  $\langle \vec{J} \rangle = \langle J_z \rangle \hat{z}$  and the SOF signal also approach 0. The next section indicates how this problem can be solved by detecting an operator different than  $J_z$ .

## 3. Optimal Frequency Measurements

Some squeezed states for  $N > 2$  are discussed in Refs. 2 and 3. Reference 3 shows that certain "intelligent" states, discussed in detail by Rashid,<sup>12</sup> provide a squeezing  $\xi_R \rightarrow \sqrt{2N^{-1/2}}$  as  $N \rightarrow \infty$ . By detecting an operator different than  $J_z$ , we can find a state with a squeezing  $\xi_R = N^{-1/2}$ . In order to motivate the choice of this state, recall that the angle  $(\omega_0 - \omega)T$  which is measured in the Ramsey method is, in the rotating frame, just the phase factor  $e^{-i(\omega_0 - \omega)T}$  that the excited state  $|e\rangle$  acquires relative to the ground state  $|g\rangle$  during the free precession period  $T$ . Consequently it may be possible to improve the precision of Ramsey spectroscopy by making a coherent superposition (after the first  $\pi/2$  pulse) of two energy eigenstates whose energies differ by more than  $\hbar(\omega_0 - \omega)$ . For  $N$

two-level atoms the eigenstates  $|N/2, -N/2\rangle$  and  $|N/2, N/2\rangle$  provide the largest energy difference, with an accumulated phase difference of  $N(\omega_0 - \omega)T$  over the free precession period. It appears that the state

$$|\psi_M\rangle = \{ |N/2, N/2\rangle + |N/2, -N/2\rangle \} / \sqrt{2} \quad (4)$$

may therefore provide a frequency imprecision of  $(NT)^{-1}$ , which is  $N^{1/2}$  smaller than that provided by  $N$  independent atoms. However, because  $\langle \psi_M | \vec{J} | \psi_M \rangle = 0$ , some operator other than  $J_z$  with higher-order tensor components must be detected.

Consider the operator  $\tilde{O} = \prod_{i=1}^N \sigma_{z_i}$  which is a product of the z-component Pauli spin matrices for the atoms. For  $J=N/2$  this operator is diagonal in the  $|J, M_J\rangle$  basis with eigenvalues  $(-1)^{J-M_J}$ . (This follows by expressing  $|J, M_J\rangle$  as a sum over all states which are products of individual spin up or spin down states with  $J-M_J$  spins down.) Quantum jump detection can be used to measure  $\tilde{O}$  (or  $J_z$ ) without added noise.<sup>6</sup> The result of measuring  $x$  atoms in the spin up state is assigned the value  $(-1)^{N-x}$ . Suppose at the end of the first  $\pi/2$  pulse the state  $|\psi_M\rangle$  is created. After the second  $\pi/2$  pulse we detect  $\langle \tilde{O} \rangle_{t_f} \equiv \langle \psi_f | \tilde{O} | \psi_f \rangle$  where

$$|\psi_f\rangle = e^{-i\frac{\pi}{2}J_z} e^{-i(\omega_0 - \omega)TJ_z} |\psi_M\rangle. \quad (5)$$

Explicit computation gives  $\langle \tilde{O} \rangle_{t_f} = (-1)^N \cos[N(\omega_0 - \omega)T]$  and, because  $\tilde{O}^2 = 1$ ,  $\langle \Delta \tilde{O}^2 \rangle_{t_f} = \sin^2[N(\omega_0 - \omega)T]$ . By detecting the operator  $\tilde{O}$ , we can therefore use the state  $|\psi_M\rangle$  in spectroscopy with a frequency imprecision

$$|\Delta\omega| = \Delta \tilde{O}(t_f) / |\partial \langle \tilde{O} \rangle_{t_f} / \partial \omega| = (NT)^{-1} \quad (6)$$

independent of  $\omega$ .

The Ramsey method measures  $\omega_0$  by measuring the phase  $\omega_0 T$  (we now use the laboratory frame of reference) which state  $|e\rangle$  acquires relative to  $|g\rangle$  during the free precession period; that is,  $\omega_0$  is measured by observing the free time evolution of the system. With the state  $|\psi_M\rangle$  a frequency imprecision of  $(NT)^{-1}$  is obtained. We show that this is in fact the best precision that can be obtained on  $N$  identical two-level atoms undergoing free time evolution. This follows from an application of the time-energy uncertainty principle

$$\delta t^2 \langle \Delta H^2 \rangle \geq \hbar^2 / 4, \quad (7)$$

where  $\langle \Delta H^2 \rangle$  is the variance of the Hamiltonian and  $\delta t^2$  is the variance in estimating time, derived from measurements of an operator of the system. (In many texts  $\delta t$  is interpreted

as a characteristic evolution time of the system and Eq. (7) then relates the uncertainty in the system's energy to this characteristic evolution time. However  $\delta t$  can also be interpreted as the uncertainty in determining time from measurements of an operator of the system. For example, measurements of an operator  $A$  can be used to determine time with an uncertainty  $\Delta A/|d\langle A \rangle/dt|$ . See Ref. 13 for a simple proof of the time-uncertainty relation and Refs. 14-16 for a more rigorous discussion.) For the system of  $N$  identical two-level particles, Eq. (7) can be expressed in terms of dimensionless quantities

$$\delta\phi^2 \langle \Delta h^2 \rangle \geq 1/4 \quad (8)$$

where  $\phi = \omega_0 t$  and

$$h = \sum_{i=1}^N \left\{ \frac{1}{2} |e\rangle_{ii} \langle e| - \frac{1}{2} |g\rangle_{ii} \langle g| \right\}. \quad (9)$$

We are here considering the full  $2^N$ -dimensional Hilbert space and not just the  $J=N/2$  subspace discussed earlier. We can establish an upper limit  $\langle \Delta h^2 \rangle \leq N^2/4$  from  $\langle \Delta h^2 \rangle = \langle h^2 \rangle - \langle h \rangle^2 \leq \langle h^2 \rangle$  and  $\langle h^2 \rangle \leq N^2/4$ . The last inequality follows because the maximum eigenvalue of  $h^2$  is  $N^2/4$ . From these inequalities and Eq. (8) we have

$$\delta\phi^2 \geq \frac{1}{4 \langle \Delta h^2 \rangle} \geq \frac{1}{N^2}. \quad (10)$$

For a free time evolution of duration  $T$ , Eq. (10) implies that

$$\delta\omega_0 \geq \frac{1}{NT}, \quad (11)$$

where  $\delta\omega_0$  is the uncertainty in determining  $\omega_0$  from measurements at the end of the evolution period. Therefore the optimum precision in measuring  $\omega_0$  can be obtained using Ramsey spectroscopy with  $|\psi_M\rangle$  and detecting the operator  $\tilde{O}$  as described above.

We can apply the above results to particle interferometry. In particular, for a Mach-Zehnder interferometer employing  $n$  input particles for each measurement, the desired state corresponding to Eq. (4) (that is, the state after the first beam splitter) is  $|\psi\rangle = \{ |n\rangle_a |0\rangle_b + |0\rangle_a |n\rangle_b \} / \sqrt{2}$  where  $a$  and  $b$  denote the two modes after the first beam splitter.<sup>8,9</sup> In this case, the operator  $\tilde{O}$  can be detected by measuring the number of particles  $n(b')$  detected in the  $b'$  output mode of the second beam splitter. The result of such a measurement is assigned the value  $(-1)^{n(b')}$ . Equation (6) implies that the phase sensitivity of the interferometer is given by  $\Delta\phi = n^{-1}$ , and Eq. (11) shows that this is the maximum sensitivity possible.

#### 4. Methods for Making Squeezed States

In this section we briefly discuss several ideas for making squeezed spin states with ions in a trap. (There are also ideas for making squeezed spin states with neutral atoms. One possibility replaces the center-of-mass mode of the ion cloud discussed below with an electromagnetic cavity mode.<sup>17)</sup> Ion traps provide an isolated environment for the internal energy states of trapped ions. At first sight, this isolation appears to make correlating the internal states of the trapped ions difficult. However, with cooling, the translational states of the trapped ions can be strongly interacting. (For example, with laser cooling, ions in a linear rf trap freeze into a 1-D string.<sup>18,19)</sup> In the schemes discussed so far for generating correlated states of trapped ions, the internal states of the ions are coupled to a particular (shared) motional degree of freedom. In Refs. 1 and 2, quasi-static inhomogeneous fields and stimulated Raman transitions were discussed as possible methods for coupling ground state hyperfine levels of an ion with the center-of-mass (COM) mode of a trapped ion cloud. The ions were assumed to be prepared in either one of the  $|N/2, \pm N/2\rangle$  states and all ions were assumed to see the same coupling with the COM mode. (For the stimulated Raman interaction, this will be satisfied if the laser beam waists and confocal parameters are large compared to the ion cloud.) In this case, in the interaction picture and the rotating wave approximation, the interaction  $H_1$  between the internal states and the COM mode can take the form of a Jaynes-Cummings-type interaction

$$H_1 = -\hbar\Omega(J_+a + J_-a^\dagger) \quad (12)$$

where  $a^\dagger$  ( $a$ ) is the raising (lowering) operator for the COM mode,  $J_+$  ( $J_-$ ) is the raising (lowering) operator for the  $J=N/2$  ladder of atomic states, and  $\Omega$  is the strength of the interaction. Two different schemes were investigated. In one, the COM mode is initially assumed to be a coherent state and in the other a squeezed state. The interaction of Eq. (12) was then applied for a specific time and the resulting squeezing of the spins calculated. In general, this method results in a mixed state of the ions' internal states and the COM mode, and a squeezing less than the optimal value discussed in the last section. By preparing appropriate (probably nonclassical) states of the COM mode, we may be able to prepare to prepare the state  $|\psi_M\rangle$ . For example, with  $N=2$  the coupling in Eq. (12) can be used to evolve the  $|n=1\rangle|J=1, M_J=-1\rangle$  state into the  $|n=0\rangle|J=1, M_J=0\rangle$  state. For  $N=2$ , the internal state differs from  $|\psi_M\rangle$  by a rotation.

More recently, Cirac and Zoller<sup>20</sup> have discussed the use of a string of  $N$  ions in a linear rf trap for quantum computation and described a method for preparing general quantum states of this system. Their method is similar to that discussed in the previous paragraph but differs in that it uses a well focussed laser beam to couple the internal states of *individual* ions with a mode of the ion string. The mode is initially prepared

in  $|n=0\rangle$ . The laser beam is then used to address an individual ion and correlate, or entangle, the internal state of the ion with the mode. The laser beam can then be used to address a different ion and change the correlation between the ion and the mode to a correlation between two different ions. Reference 20 lays out the steps needed to prepare the state  $|\psi_M\rangle$ . Here we discuss the simple case of  $N=2$ . We assume lasers can be focussed and used to drive transitions on an individual ion without perturbing the other ion. (If  $|g\rangle$  and  $|e\rangle$  are hyperfine levels of an atomic ground state, lasers can be used to drive a stimulated Raman transition between the states.<sup>21</sup>) Suppose that the 2 ions are prepared in the initial state  $|g\rangle_1|g\rangle_2|n=0\rangle$  where the indices refer to the internal states of ions 1 and 2 and  $|n=0\rangle$  is the  $n=0$  state of a mode of the 2-ion string. First a  $\pi/2$ -pulse on the blue sideband of ion 1 takes the  $|g\rangle_1|g\rangle_2|n=0\rangle$  state into the  $|g\rangle_2\{|g\rangle_1|n=0\rangle+|e\rangle_1|n=1\rangle\}/\sqrt{2}$  state. This is then followed by a  $\pi$  pulse on the red sideband of ion 2. This leaves the  $|g\rangle_1|g\rangle_2|n=0\rangle$  state unperturbed, but takes the  $|e\rangle_1|g\rangle_2|n=1\rangle$  state into the  $|e\rangle_1|e\rangle_2|n=0\rangle$  state. In this manner  $|\psi_M\rangle$  can be prepared for  $N=2$ .

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