Coupled Thermal-Elastic Response of Structures to Fires

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1 Introduction

There has been a resurgence of interest in the response of building structures to fires over the past several years. This interest was greatly enhanced by the attack on, and subsequent collapse of, the World Trade Center (WTC) towers. Traditional methods of modeling this behavior are based on computing the thermal response of an un-deformed structure and performing structural analysis in a sequential manner [1]. This procedure can lead to significant errors in the thermally induced structural response. While the applications of interest clearly involve highly non-linear calculations, the starting point for most fire scenarios is almost always an undamaged building at room temperature. Since virtually all buildings are designed to keep the stresses well below the elastic limit and the deflections of the load bearing structure reasonably small, the starting point for simulations of fire induced damage *must* lie within the domain of *linear* elasticity. Moreover, the difficulties that arise are evident before the temperature rise is large enough to affect the elastic or thermal properties of most structural materials. Under these circumstances the thermally induced stresses are also linear, and the temperature fields can be described by the heat conduction equation for the material(s) of interest.

The facts described above justify an analysis of the coupling between the temperature and thermally induced stresses based on the *linear* thermo-elastic equations. The temporal dependence of the stresses is the focus of this analysis. A popular technique for solving the thermo-elastic equations is to first compute (or assume) the time dependent temperature distribution in the load bearing structure. Then, given this information, the temperature distributions are "frozen" at a succession of discretely chosen times and an equilibrium solution is sought for the state of stress at each chosen time. The fact that the temperature is changing continuously and that this continuous change *must* affect the stresses is ignored in this approach. This technique is justified by noting that the elastic wave propagation speed is so fast compared with the time scales of interest in thermo-elastic phenomena induced by fires that a quasi-steady analysis is justified.

The analyses that follow are intended to show that computational techniques that "freeze" the temperature at a given time and compute an equilibrium stress distribution may not be consistent with the dynamical equations of thermo-elasticity, even if the elastic wave propagation speed is taken to be infinite. The next section demonstrates how the general solutions to the equations of thermo-elasticity couple the the time scale for the evolution of the displacements to that of the temperature field. In particular, it is shown that the solutions for the displacements cannot obey an equilibrium equation unless the temperature field is independent of time. Following this, formal solutions for a half-space loaded thermally are derived. Again, it is clear that part of the solution for the stresses and displacements are inherently time dependent.

2 The Thermo-Elastic Equations

The starting point for the analysis are the linear themo-elastic equations which relate the displacements u_i and stresses τ_{ij} to each other and to the temperature T in the elastic medium. They can be written in the form [2]:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{1}$$

$$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \alpha \left(3\lambda + 2\mu\right) \left(T - T_o\right) \delta_{ij} \tag{2}$$

Here, ρ is the density of the material, F_i is the body force per unit volume, μ and λ the Lamé constants, α the coefficient of thermal expansion, and T the temperature. The temperature T_o is the reference temperature of the material in its unstressed state before the fire. That temperature is taken to be uniform here, which is a reasonable simplification given the temperature rise associated with a building fire. These equations are supplemented by suitable boundary conditions that express the connections to other portions of the structure and external loads. The temperature field evolves according to the heat conduction equation.

$$\rho C_p \frac{\partial \left(T - T_o\right)}{\partial t} = k \frac{\partial^2 \left(T - T_o\right)}{\partial x_k^2} \tag{3}$$

The coupling between the stresses and the temperature field comes from the effect of the temperature gradients on the volumetric expansion ϕ. The body force plays no role in this and will henceforth be ignored. Taking the divergence of the thermo-elastic evolution equation (1) yields:

$$\phi \equiv \frac{\partial u_k}{\partial x_k} \tag{4}$$

$$\rho \frac{\partial^2 \phi}{\partial t^2} = \left(\lambda + 2\mu\right) \nabla^2 \phi - \alpha \left(3\lambda + 2\mu\right) \nabla^2 \left(T - T_o\right) \tag{5}$$

The next step is to simplify these equations by eliminating the fast wave motion associated with the irrotational waves. It is easy to see from equation (5) that these waves propagate with a speed $c^2 = (\lambda + 2\mu)/\rho$. For typical steels, $c \sim O(10^3)\, m/s$. Moreover, the thermal diffusivity $k/(\rho C_p) \sim O(10^{-5})\, m^2/s$. We now introduce dimensionless thermal and mechanics variables respectively as follows:

$$T - T_o = \left(q_o L/k\right) \Theta \left(y_i\, , \tau\right) \qquad x_i = L\, y_i \qquad t = t_o\, \tau \qquad L = \sqrt{\frac{k\, t_o}{\rho\, C_p}} \tag{6}$$

$$\phi = \beta \Phi \left(y_i,\, \tau\right) \qquad u_i = \beta L v_i \qquad \beta = \alpha\, q_o \sqrt{\frac{t_o}{k\, \rho\, C_p}} \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right) \qquad \epsilon^2 = \frac{k}{C_p t_o} \frac{1}{\left(\lambda + 2\mu\right)} \tag{7}$$

The non-dimensional variables are chosen so that the full time dependent heat conduction equation is retained, and the temperature rise is related to a heat flux to a bounding surface q_o. The coupling parameter β scales the thermally induced deformations to the temperature rise, and ϵ is the ratio of the speed of a thermal "front" to the irrotational wave speed in the elastic medium. The time scale t_o is arbitrary, and its choice sets both the diffusion controlled length scale and the magnitude of ϵ. Given the approximate values of wave speed and thermal diffusivity shown above, it is clear that $\epsilon \ll 1$ for any time scale of interest in a fire scenario.

The dimensionless evolution equations for the normalized dilation Φ and temperature rise Θ take the following form:

$$\frac{\partial \Theta}{\partial \tau} = \nabla^2 \Theta \tag{8}$$

$$\epsilon^2 \frac{\partial^2 \Phi}{\partial \tau^2} = \nabla^2 \Phi - \nabla^2 \Theta \tag{9}$$

$$\Phi = \nabla \cdot \vec{v} \tag{10}$$

It is now easy to show the structure of the solution for the irrotational portion of the deformation. Ignoring terms of order ϵ^2, it is clear that Φ *must* have the form:

$$\Phi = \Theta + \Phi^* \qquad \nabla^2 \Phi^* = 0 \tag{11}$$

This clearly shows that the dilation has two parts; a harmonic contribution Φ^* that can be regarded as a local equilibrium solution corresponding to the instantaneous boundary conditions, and a part that is directly proportional to the local temperature rise. This part of the solution *never* is in equilibrium, unless the temperature is in steady state.

This result can be seen even more clearly by noting that since the dimensionless displacement is a vector field, it can always be decomposed into an irrotational and a solenoidal part.

$$\vec{v} = \nabla \Psi + \nabla \times \vec{A} \tag{12}$$

Then, since $\nabla \cdot \vec{v} = \Phi = \nabla^2 \Psi$, the scalar potential function Ψ satisfies the equation:

$$\nabla^2 \Psi = \Theta + \Phi^* \tag{13}$$

The harmonic function Φ^* can be eliminated to obtain an explicit relation between the irrotational component of the displacement and the temperature field in the following form:

$$\nabla^2 \nabla^2 \Psi = \frac{\partial \Theta}{\partial \tau} \tag{14}$$

Here, the fact that Θ satisfies the heat conduction equation has been used. Alternatively, the solution for Ψ can be decomposed into an evolution equation and an equilibrium equation as follows:

$$\Psi = \Psi^* + \Psi_1 \tag{15}$$

$$\nabla^2 \nabla^2 \Psi^* = 0 \qquad \left(\frac{\partial}{\partial \tau} - \nabla^2 \right) \nabla^2 \Psi_1 = 0 \tag{16}$$

Both equation (14) and (16) show that only part of the solution for the thermally induced displacement can correspond to a local equilibrium state if the temperature varies with time. Any solution procedure that works by “freezing” the temperature in time and using an equilibrium equation to satisfy the boundary conditions *must* miss at least part of the solution to the equations of thermo-elasticity.

3 Heated Elastic Half-space

In order to make some of these ideas more precise, consider the idealized problem of an elastic half-space heated at the surface by a prescribed heat flux that in general depends on space and time. The variables are made dimensionless as before, with the coordinate normal to the surface pointing *into* the solid denoted by z, and the coordinates parallel to the surface denoted by $\vec{r} \equiv (x, y)$. The *dimensional* heat flux to the surface at $z = 0$ is given by $q_z = q_o \, Q\,(x, y, \tau)$. Furthermore, body forces are ignored and there are no mechanical forces acting on the surface $z = 0$. Thus, the only reason any stresses and deformations are set up in the solid is because of the thermal loading.

Under these circumstances, the boundary conditions at the surface become:

$$\tau_{z\,i} = 0, \qquad i \equiv x, y, z \qquad \frac{\partial \Theta}{\partial z} = -Q\,(x, y, \tau) \tag{17}$$

The stress free boundary conditions can be conveniently rewritten in terms of displacements by introducing the parallel displacement vector $\vec{V} \equiv (u, v)$, the perpendicular displacement w, and the parallel gradient operator ∇_h defined by:

$$\nabla_h \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \tag{18}$$

The requirement that the two shear components of the stress vanish at the surface can then be written in the form:

$$\nabla_h w + \frac{\partial}{\partial z} \vec{V} = 0 \qquad @z = 0 \tag{19}$$

Similarly, the normal stress will vanish at the surface provided that:

$$\lambda \Phi + 2\mu \frac{\partial w}{\partial z} = (\lambda + 2\mu)\,\Theta \qquad @z = 0 \tag{20}$$

In the present notation, Φ can be written in the form:

$$\Phi = \nabla_h \cdot \vec{V} + \frac{\partial w}{\partial z} = \Theta + \Phi^* \tag{21}$$

The heat flux is assumed to be applied to the surface over a finite area. Thus, the temperature rise together with all displacements must vanish as $z \to \infty$ and $\vec{r} \to \infty$.

The equations that must be solved can now be rewritten as follows: The heat conduction equation takes the form:

$$\frac{\partial \Theta}{\partial \tau} = \left(\nabla_h^2 + \frac{\partial^2}{\partial z^2} \right) \Theta \tag{22}$$

Using the general results obtained in the previous section, the equilibrium equations for the parallel components of the displacement become:

$$\left(\nabla_h^2 + \frac{\partial^2}{\partial z^2}\right)\vec{V} + \left(\frac{\mu+\lambda}{\mu}\right)\nabla_h \Phi^* = \nabla_h \Theta \tag{23}$$

The auxiliary function Φ^* defined in equation (11) satisfies the Laplace equation, written in the present notation as:

$$\left(\nabla_h^2 + \frac{\partial^2}{\partial z^2}\right)\Phi^* = 0 \tag{24}$$

The system of equations that must be solved thus consists of equation (22), which determines Θ, equation (23), which determines $\vec{V}$, equation (24), which determines Φ^*, and equation (21), which determines w.

The solutions can be obtained using transform methods. Let the Fourier-Laplace transform of an arbitrary function $f\left(\vec{r},\, z,\, \tau\right)$ be defined as follows:

$$\bar{f}\left(\vec{k},\, z,\, p\right) = \int_{-\infty}^{\infty} d^2\vec{r} \int_0^{\infty} d\tau \exp\left(-i\vec{k}\cdot\vec{r} - p\tau\right) f\left(\vec{r},\, z,\, \tau\right) \tag{25}$$

The Fourier-Laplace transform of the solutions satisfying boundary conditions at infinity can then readily be found to be:

$$\bar{\Theta} = \bar{Q}\left(\vec{k},\, p\right)\frac{\exp\left(-\sqrt{p+k^2}\, z\right)}{\sqrt{p+k^2}} \tag{26}$$

$$\overline{\Phi^*} = \bar{A}\left(\vec{k},\, p\right)\exp\left(-k\, z\right) \tag{27}$$

$$\overline{\vec{V}} = i\vec{k}\,\bar{\Psi}\left(\vec{k},\, p\right) \tag{28}$$

$$\bar{\Psi} = \bar{Q}\frac{\exp\left(-\sqrt{p+k^2}\, z\right)}{p\sqrt{p+k^2}} + \bar{B}\left(\vec{k},\, p\right)\exp\left(-k\, z\right) + \left(\frac{\mu+\lambda}{\mu}\right)\bar{A}\left(\vec{k},\, p\right)\frac{z}{2k}\exp\left(-k\, z\right) \tag{29}$$

$$\bar{w} = -\frac{\bar{Q}}{p}\exp\left(-\sqrt{p+k^2}\, z\right) - k\bar{B}\exp\left(-k\, z\right) - \left(\frac{3\mu+\lambda+(\mu+\lambda)\, k\, z}{2\, k\, \mu}\right)\bar{A}\exp\left(-k\, z\right) \tag{30}$$

The unknown functions $\bar{A}\left(\vec{k},\, p\right)$ and $\bar{B}\left(\vec{k},\, p\right)$ are determined by the surface boundary conditions given in equations (19) and (20). The results are:

$$\bar{A} = \left(\frac{\mu}{\mu+\lambda}\right)\frac{2k\,\bar{Q}}{p}\left(1 - \frac{k}{\sqrt{p+k^2}}\right) \tag{31}$$

$$\bar{B} = \frac{\bar{Q}}{(\mu+\lambda)\, p}\left(\frac{\mu}{\sqrt{p+k^2}} - \frac{2\mu+\lambda}{k}\right) \tag{32}$$

Since the primary interest in this solution is the extent to which the time dependence is imbedded in the result, attention is focused on the vertical displacement at the surface. This part of the solution can be readily obtained and interpreted in the light of the general formulation discussed in the previous section. Physically, it represents the "bulge" in the surface that would appear in the vicinity of the heated area. The recipe for the bulge can be readily compared with that for the temperature rise at the surface induced by the heat transfer. Since it is well known that the temperature distribution *must* be treated as a transient phenomenon, the similarities and contrasts between these two results will lend insight into the importance of a coupled transient analysis of the stresses and displacements induced by the heat transfer.

First consider the surface temperature distribution. Using the convolution theorem, the solution can be written in the form:

$$\Theta\left(\vec{r},\, \tau,\, 0\right) = \int_0^{\tau} d\tau_o \int_{-\infty}^{\infty} d^2\vec{r}_o\, Q\left(\vec{r}_o,\, \tau_o\right) G\left(\vec{r} - \vec{r}_o,\, \tau - \tau_o\right) \tag{33}$$

$$G\left(\vec{r},\, \tau\right) = \frac{1}{4\left(\pi\tau\right)^{3/2}}\exp\left(-\eta^2/4\right) \qquad \eta = r/\sqrt{\tau} \tag{34}$$

In order to simplify the analysis, consider the special case where $Q(\vec{r},\,\tau)$ is concentrated at a point $\vec{r}=\vec{R}(\tau)$ with a strength $Q_T(\tau)$. While this is *not* a realistic representation of the spatial distribution of the heat flux induced by an individual fire to a large floor area, it does pick up two key features of such a fire. First, the overall strength of the fire (measured by its overall heat release rate) will change with time. Second, the fire will migrate from place to place as its fuel is consumed and the availability of oxygen changes with time. Under these circumstances the surface temperature distribution simplifies to:

$$\Theta(\vec{r},\,\tau,\,0)=\int_0^\tau d\tau_o\frac{Q_T(\tau_o)}{4\left(\pi\left(\tau-\tau_o\right)\right)^{3/2}}\exp\left(-\eta_T^2/4\right)\qquad\eta_T=\left|\vec{r}-\vec{R}(\tau_o)\right|/\sqrt{(\tau-\tau_o)}\tag{35}$$

The most relevant parts of this result in the present context are the dependence of the solution on the previous history of the surface heat flux distribution, and the fact that the Greens function G has a structure determined primarily by the similarity variable η_T which itself is inherently time dependent. Note that the singularity in the integrand at $\vec{r}=\vec{R}(\tau_o)\,,\tau=\tau_o$ is an artifact of injecting a finite heat flux into a point. The results make perfectly good sense for points away from the current location of the heat flux source.

The solution for the displacement normal to the surface can be found in an analogous manner. However, since the inversion process is somewhat more complex here a few details are provided. The Fourier-Laplace transform of the surface displacement takes the form:

$$\bar{w}\left(\vec{k},\,p,\,0\right)=\left(\frac{2\mu+\lambda}{\mu+\lambda}\right)\frac{\bar{Q}}{p}\left(\frac{k}{\sqrt{p+k^2}}-1\right)\tag{36}$$

The solution for $w(\vec{r},\,\tau,\,0)$ takes the same form as equation (33) except that the Greens function $G(\vec{r},\,\tau)$ is replaced by a new kernel function $K(\vec{r},\,\tau)$ defined as:

$$K=\left(\frac{2\mu+\lambda}{\mu+\lambda}\right)\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d^2\vec{k}\exp\left(i\vec{k}\cdot\vec{r}\right)\frac{1}{2\pi i}\oint dp\exp(p\tau)\frac{1}{p}\left(\frac{k}{\sqrt{p+k^2}}-1\right)\tag{37}$$

Carrying out the inversion of the Laplace transform first, and noting that the resulting expression depends only on $k\equiv|\vec{k}|$, the Fourier inversion integral can be reduced to:

$$K=-\left(\frac{2\mu+\lambda}{\mu+\lambda}\right)\frac{1}{2\pi\tau}\int_0^\infty d\xi\,\xi\,\mathrm{erfc}(\xi)\;J_0(\xi\,\eta)\qquad\eta=r/\sqrt{\tau}\tag{38}$$

Here, J_0 denotes the Bessel function of the first kind of order zero. The final integral can be evaluated with the aid of *Mathematica* to yield:

$$K=-\left(\frac{2\mu+\lambda}{\mu+\lambda}\right)\frac{1}{8\pi\tau}\exp\left(-\eta^2/8\right)\left(I_o\left(\eta^2/8\right)-I_1\left(\eta^2/8\right)\right)\tag{39}$$

The quantities I_0 and I_1 are the modified Bessel functions of the first kind of order zero and one respectively. The minus sign in front of the (positive) expression for K arises from the definition of positive w pointing *into* the material. Thus, the thermal expansion induces a bulge out of the plane of the surface, so that w must be negative. Finally, if the heat flux to the surface is concentrated at a point as described above, the solution for the normal surface displacement becomes:

$$w=\int_0^\tau d\tau_o\,Q_T(\tau_o)\,K(\tau-\tau_o,\,\eta_T)\qquad\eta_T=\left|\vec{r}-\vec{R}(\tau_o)\right|/\sqrt{(\tau-\tau_o)}\tag{40}$$

4 Discussion

Clearly, the only significant differences between the solutions for Θ and w at the surface are in the mathematical structure of the kernel functions G and K. They are plotted in figure 1 in normalized form, so that the value at $\eta=0$ for each function is one. The thermal function G has an exponential decay, consistent with the inherently transient diffusion of heat from the point source at the surface. The normal displacement kernel K however, decays algebraically with large η with $K\sim\eta^{-3}$. This is a consequence of the fact that the displacements have both equilibrium and transient components. On the other hand, the spatial dependence of both functions appears in the inherently transient independent variable $\eta=r/\sqrt{\tau}$, which describes a diffusion controlled process. Moreover, the full solutions for both the temperature and the displacement are dependent on the previous history of each quantity.

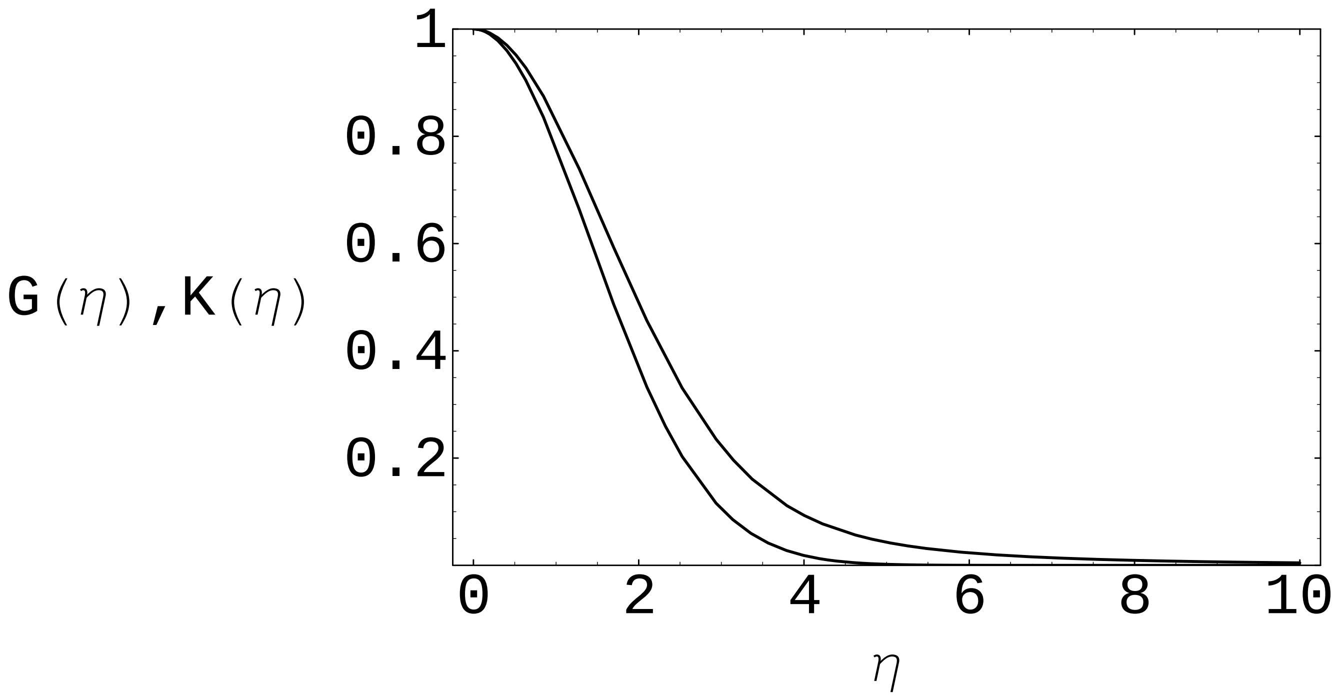

Figure 1: **Kernel functions normalized by their value at $\eta = 0$. The quantities actually plotted are $G(\eta, \tau)/G(0, \tau)$ and $K(\eta, \tau)/K(0, \tau)$. The Gaussian function G decays more rapidly than K which decays $\sim \eta^{-3}$.**

How can such a close similarity in the solutions that emerge from the above analysis be reconciled with the approach typically employed in the analysis of structures immersed in fires? Given a simulation of the fire dynamics, an elaborate procedure that couples the temperature and radiation fields in the gas to the temperature distribution in the load bearing structure can be devised [1]. The procedure involves, among other things, using commercial software packages (e.g. ANSYS) to calculate the transient temperature distribution through realistic representations of the structure and any relevant insulation. When this process is completed, a small set of times are chosen, and the spatial distribution of temperature at each of these times is interpolated into the finite element representation to be used for the structural analysis. This representation is invariably different from that employed for the thermal analysis. This process inevitably *must* introduce errors. Each temperature distribution is then considered fixed in time, and the body forces induced by the thermal expansion are computed. The software is then used to compute an equilibrium solution corresponding to the body forces, together with any external and gravity loads on the structure. If such a solution is found, the problem is then repeated at the next chosen time, until either the time sequence has been completed or no equilibrium solution is found. The absence of any need for a previous time history of the state of stress is disturbing. The calculation of the state of stress at intermediate times prior to an estimated “collapse” is a purely mathematical interpolation towards the final state. No knowledge of the state of stress at intermediate times is required in such a procedure. Indeed, no information about either the thermal or stress states between specified time intervals is supplied. This does not seem consistent with the equations of linear thermo-elasticity. It is probably not consistent with the non-linear equations that are being solved either.

References

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[2] Sokolnikoff, I.A., *Mathematical Theory of Elasticity*, Second Edition, McGraw Hill, New York, pp. 358-367, (1956).